Department of Mathematical Sciences
Carnegie Mellon University

21-476 **Ordinary Differential Equations** **Fall 2003**

Take-Home Midterm Exam

Due by 4:30 P.M. on Friday, October 31

1. Determine as much as you can about solutions of the first-order scalar equation

$$\dot{x}(t) = -x(t)^3 + \sin t. \tag{1}$$

2. Consider the autonomous system

$$\begin{cases} \dot{x}_1 = x_2 - x_2^3 \\ \dot{x}_2 = x_1. \end{cases} \tag{2}$$

   (a) Sketch the phase portrait for (2).

   (b) What can you deduce about solutions of (2) from the phase portrait?

3. Assume that $g : \mathbb{R}^4 \to \mathbb{R}$ is continuously differentiable and satisfies $g(t, y, z, 0) = 0$ for all $t, y, z \in \mathbb{R}$. Show that every solution of third-order scalar equation

$$u^{(3)}(t) = g(t, u(t), \dot{u}(t), \ddot{u}(t)) \tag{3}$$

is either convex or concave, i.e. show that if u is a solution of (3) then either $\ddot{u}(t) \geq 0$ for all $t \in \mathrm{Dom}(u)$ or $\ddot{u}(t) \leq 0$ for all $t \in \mathrm{Dom}(u)$.

4. Let $g : \mathbb{R}^n \to \mathbb{R}^n$, $h : \mathbb{R} \to \mathbb{R}^n$, $x_0 \in \mathbb{R}^n$, and $\delta > 0$ be given. Assume that g and h are continuous, h is bounded, and that g satisfies $g(0) = 0$ and $(g(z) - g(y)) \cdot (z - y) \leq 0$ for all $y, z \in \mathbb{R}^n$. Consider the initial value problem

$$\dot{x}(t) = g(x(t)) + h(t);\ x(0) = x_0. \tag{4}$$

   (a) Show that if x and x^* are solutions of (4) on $[0, \delta)$ then $x(t) = x^*(t)$ for all $t \in [0, \delta)$.

(b) Show that if x is a noncontinuable solution of (4) then $[0, \infty) \subset \mathrm{Dom}(x)$.

5. Show that the second-order scalar equation

$$\ddot{u}(t) + 2\dot{u}(t) + 2u(t) + (u(t) + \dot{u}(t))^3 = \cos t \tag{5}$$

has a 2π-periodic solution. (Suggestion: Rewrite (5) as a system by letting $x_1 = u,\ x_2 = u + \dot{u}$.)

6. Let $a, b > 0$ be given. The autonomous system

$$\begin{cases} \dot{x}_1 = -ax_1 + bx_1x_2 \\ \dot{x}_2 = -bx_1x_2 \\ \dot{x}_3 = ax_1 \end{cases} \tag{6}$$

provides a simple model for the spread of a disease. Here x_1 represents the number of infected individuals, x_2 represents the number of susceptible individuals, and x_3 represents the number of immune individuals. Determine as much as you can about solutions of (6).

7. (Extra Credit) Let $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, $\lambda, \delta > 0$, and $x_0 \in \mathbb{R}^n$ be given. Assume that f is continuous and satisfies

$$||f(t, y) - f(t, z)||_2 \leq \frac{\lambda}{t}||y - z||_2 \qquad \text{for all } t \in (0, \delta), y, z \in \mathbb{R}^n,$$

and consider the initial value problem

$$\dot{x}(t) = f(t, x(t));\ x(0) = x_0.$$

(a) Assume $\lambda < 1$. Show that if x and x^* are solutions of (7) on $[0, \delta)$ then $x(t) = x^*(t)$ for all $t \in [0, \delta)$.

(b) What is the situation regarding forward uniqueness for (7) if $\lambda \geq 1$?