Department of Mathematical Sciences
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**21-476** **Ordinary Differential Equations** **Fall 2003**

## II. Preliminaries

Let $n$ be a positive integer. We denote by $\mathbb{R}^n$ the set of all $n$-tuples of real numbers $x = (x_1, x_2, \dots x_n)$ with the usual notions of addition and scalar multiplication. We use the same symbol 0 to denote the real number zero as well as the zero element of $\mathbb{R}^n$ when there is no danger of confusion.

By a *norm* on $\mathbb{R}^n$ we mean a function $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$ satisfying

(i) $\|x\| > 0 \qquad \text{for all } x \in \mathbb{R}^n \backslash \{0\},$

(ii) $\|\alpha x\| = |\alpha|\|x\| \qquad \text{for all } x \in \mathbb{R}^n,\ \alpha \in \mathbb{R},$

(iii) $\|x + y\| \leq \|x\| + \|y\| \qquad \text{for all } x, y \in \mathbb{R}^n.$

Property (iii) is called the *triangle inequality*. An important consequence of this property is that if $a$ and $b$ are real numbers with $a < b$ and $g : [a, b] \to \mathbb{R}^n$ is continuous then

$$\|\int_a^b g(t)dt\| \leq \int_a^b \|g(t)\|dt. \tag{2.1}$$

All norms on $\mathbb{R}^n$ are *equivalent* in the sense that if $\|\cdot\|$ and $|||\cdot|||$ are norms then there exist constants $m, M > 0$ such that

$$m\|x\| \leq |||x||| \leq M\|x\| \qquad \text{for all } x \in \mathbb{R}^n. \tag{2.2}$$

For each $p \in [1, \infty)$ the function $\|\cdot\|_p : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\|x\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p} \quad \text{for all } x \in \mathbb{R}^n \tag{2.3}$$

is a norm. In addition, the function $\|\cdot\|_\infty : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\|x\|_\infty = \max\{|x_i| : i = 1, 2 \dots n\} \tag{2.4}$$

is also a norm. Observe that

$$\|x\|_\infty \le \|x\|_1 \le n\|x\|_\infty \qquad \text{for all } x \in \mathbb{R}^n. \tag{2.5}$$

The case $p = 2$ is especially important because $\|\cdot\|_2$ is associated with an inner product. Recall that the *dot product* or *inner product* of $x, y \in \mathbb{R}^n$ is defined by

$$x \cdot y = \sum_{i=1}^{n} x_i y_i, \tag{2.6}$$

so that

$$\|x\|_2 = \sqrt{x \cdot x} \qquad \text{for all } x \in \mathbb{R}^n. \tag{2.7}$$

The *Cauchy-Schwarz inequality*, which says that

$$|x \cdot y| \le \|x\|_2 \; \|y\|_2 \qquad \text{for all } x, y \in \mathbb{R}^n, \tag{2.8}$$

will play an important role in our analysis of differential equations.

The norm $\|\cdot\|_2$ is called the *Euclidean norm*. An especially useful feature of this norm is that if $I$ is an interval, $g : I \to \mathbb{R}^n$ is differentiable then the function $t \mapsto \|g(t)\|_2^2$ is differentiable on $I$ and

$$\frac{d}{dt}\left(\|g(t)\|_2^2\right) = 2g(t) \cdot \dot{g}(t) \qquad \text{for all } t \in I. \tag{2.9}$$

For each $\delta > 0$ and $x \in \mathbb{R}^n$, we put

$$B_\delta(x) = \left\{ y \in \mathbb{R}^n : \|y - x\|_2 < \delta \right\}. \tag{2.10}$$

Let $D$ be a subset of $\mathbb{R}^n$. A point $x_0 \in D$ is said to be an *interior point* of $D$ if there exists $\delta > 0$ such that $B_\delta(x_0) \subset D$. The set of all interior points of $D$ is called the *interior* of $D$ and is denoted by $\operatorname{int}(D)$. We say that $D$ is *open* if $\operatorname{int}(D) = D$. We say that $D$ is *closed* if $\mathbb{R}^n \backslash D$ is open.

A point $x_0 \in \mathbb{R}^n$ is called a *boundary point* of $D$ if

$$\forall \delta > 0, \; B_\delta(x_0) \cap D \ne \phi \qquad \text{and} \qquad B_\delta(x_0) \cap (\mathbb{R}^n \backslash D) \ne \phi, \tag{2.11}$$

i.e. for every $\delta > 0$, $B_\delta(x_0)$ contains points that belong to $D$ as well as points that do not belong to $D$. The set of all boundary points of $D$ is called the *boundary* of $D$ and is denoted by $\partial D$. It is not too difficult to show that $D$ is closed if and only if $\partial D \subset D$. We say that $D$ is *bounded* if there exists $M \in \mathbb{R}$ such that

$$\|x\|_2 \leq M \text{ for all } x \in D. \tag{2.12}$$

**Remark 2.1**: In view of the equivalence of norms on $\mathbb{R}^n$, the notions of interior, boundary, open set, closed set, bounded set do not change if $\|\cdot\|_2$ is replaced by any other norm in (2.10).

We say that $D$ is *convex* if

$$tx + (1-t)y \in D \qquad \text{for all } x, y \in D,\ t \in [0,1], \tag{2.13}$$

i.e., $D$ contains the line segment joining each pair of points in $D$. The following result will be very useful.

**Brouwer's Fixed-Point Theorem**: *Let $D$ be a nonempty, closed, bounded, convex subset of $\mathbb{R}^n$ and assume that $f : D \to \mathbb{R}^n$ is continuous. If $f(x) \in D$ for every $x \in D$ then there is at least one point $x^* \in D$ such that $f(x^*) = x^*$.*

Let $m$ be a positive integer. Then $\mathbb{R}^m \times \mathbb{R}^n$ can be identified with $\mathbb{R}^{m+n}$.

**Remark 2.2**: Let $S$ be a subset of $\mathbb{R}^m$ and $T$ be a subset of $\mathbb{R}^n$.

(i) If both $S$ and $T$ are open, then $S \times T$ is open.

(ii) If both $S$ and $T$ are closed, then $S \times T$ is closed.

(iii) If both $S$ and $T$ are bounded, then $S \times T$ is bounded.

(iv) If both $S$ and $T$ are convex, then $S \times T$ is convex.