Review Problems for Test 1

1. Let $z = 1 - i,\ w = 3 + 4i$. Compute each of the following. Express your answers in the form $x + iy$ with $x, y \in \mathbb{R}$.

   a) $z + 2w$ b) $|z\overline{w}|$ c) $\text{Arg}(z)$

   d) $\text{Re}(zw^2)$ e) $\text{Im}\left(\frac{z}{w}\right)$.

2. Find the fourth roots of $-8 + i8\sqrt{3}$.

3. Determine whether or not the inequality

$$|z + w| \geq \frac{1}{2}(|z| + |w|)\left|\frac{z}{|z|} + \frac{w}{|w|}\right|$$

   holds for all $z, w \in \mathbb{C}\backslash\{0\}$.

4. Describe geometrically the locus of the set of all points $z \in \mathbb{C}$ satisfying each equation.

   a) $|z - i| = 2$ b) $|z - 1 + i| = |z|$

   c) $|z - 1 + i| = 2|z|$ d) $\text{Re}((2 - 3i)z + 4i) = 0$

5. For each of the following sets S, answer each of the following questions.

   a) Find $\text{int}(S)$. b) Find $bdry(S)$

   c) Is S open? d) Is S closed?

   e) Is S convex? f) Is the point at infinity in the interior of S?

   (i) $S = \{z \in \mathbb{C} : \text{Im}(z) \geq (\text{Re}(z))^4\}$

   (ii) $S = \left\{\frac{1+i}{n} :\ n = 1, 2, 3, \ldots\right\}$

   (iii) $S = \{z \in \mathbb{C} :\ |z| + |z - i| > 3\}$

   (iv) $S = \{z \in \mathbb{C} :\ \text{Im}(iz - 3) = 1\}$

6. Consider the function $f : \mathbb{C} \to \mathbb{C}$ defined by

$$f(z) = \sqrt{|z|}e^{i\psi}$$

   where ψ is the unique angle such that $\psi \in \left[\frac{-3\pi}{2}, \frac{\pi}{2}\right)$ and $\sin\psi = \sin\left(\frac{\theta}{2}\right)$, $\cos\psi = \cos\left(\frac{\theta}{2}\right)$, where $\theta = \text{Arg}(z)$. Here $\sqrt{\ }$ denotes the usual square root function on $[0, \infty)$.

   a) Compute $f(4), f(-4), f(4i)$, and $f(-4i)$.

b) Where is f continuous?

7. Determine the limit or explain why it does not exist.

   $\lim_{z\to\infty} \frac{2}{1+|\text{Im}(z)|}$ b) $\lim_{z\to -e} \text{Log}(z)$

   c) $\lim_{z\to 0} \frac{\text{Re}(z^2)}{|z|^2}$ d) $\lim_{z\to 0} \frac{z\text{Re}(z)}{|z|}$

   e) $\lim_{z\to i} \frac{z^3+i}{z-i}$

8. Describe the behavior of $\sin(n(1+i))$ as $n\to\infty$, $n\in\mathbb{Z}^+$.

9. Show that $\sin z = \cos\left(\frac{\pi}{2}-z\right)$ for all $z\in\mathbb{C}$.

10. compute all possible values for each of the following

    a) $1^{\sqrt{2}}$ b) $(-2)^{\sqrt{2}}$ c) 2^i d) 1^{-i}

    e $\left(\frac{1-i}{\sqrt{2}}^{1+i}\right)$

11. Investigate the convergence of the following series.

    a) $\sum_{n=1}^{\infty} \frac{n}{(2i)^n}$ b) $\sum_{n=1}^{\infty} e^{in}$

    c) $\sum_{n=1}^{\infty} \frac{1}{(\text{Log}(i+n))^n}$ d) $\sum_{n=1}^{\infty} \frac{(1+3i)^n}{4^n}$

    e) $\sum_{n=1}^{\infty} \frac{i^n}{\sqrt{n}}$ f) $\sum_{n=1}^{\infty} \frac{(3-4i)^n}{5n}$

12. Evaluate $\int_\gamma \frac{dz}{z^2+1}$ where γ is the circle of radius 3 centered at the origin and traversed counterclockwise exactly once.

13. Evaluate $\int_\gamma (3z^5+15z^2-1)dz$, where γ is the top half of the ellipse described by $\frac{x^2}{4}+y^2=1$ starting at 2 and ending at -2.

14. Consider the function $f:\mathbb{C}\to\mathbb{C}$ defined by

    $$f(x+iy) = x^2+y^3+iy^2$$

    for all $x,y\in\mathbb{R}$. Evaluate

    $$\int_\gamma f(z)dz$$

    where γ is the triangle with vertices 0,1, and $1+i$ traversed counterclockwise exactly once.

15. Evaluate $\int_\gamma |z|^2 dz$, where γ is the right half of the circle described by $|z| = 3$ starting at $-3i$ and ending at $3i$.