21-371 **Functions of a Complex Variables** **Fall 2005**

Test 2 (Take Home)
Due at 4:30 P.M., Friday, November 18

Do not discuss any aspects of this exam with anyone other than the instructor until after 4:30 PM on November 18, 2005. You are permitted to use your class notes, the textbook, and your basic calculus textbook. You are not permitted to use other sources such as the internet or other mathematics books.

1. (10 points) Define $u : \mathbb{R}^2 \to \mathbb{R}$ by

$$u(x, y) = e^{3x} \sin 3y - 2y.$$

   Find a function $v : \mathbb{R}^2 \to \mathbb{R}$ such that the function $f : \mathbb{C} \to \mathbb{C}$ defined by

$$f(x + iy) = u(x, y) + iv(x, y)$$

   is analytic on $\mathbb{C}$, or explain why no such function exists.

2. (10 points) Give an example of a continuous function $f : \mathbb{C} \to \mathbb{C}$ such that f is differentiable at exactly one point, or explain why no such function exists.

3. (20 points) Find the radius of convergence of each of the following power series.

   (a) $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{\sqrt{n!}} (z - 3)^n$

   (b) $\displaystyle\sum_{n=0}^{\infty} \frac{n^n}{3^n n!} z^n$

4. (15 points) Let $f(z) = \displaystyle\sum_{n=0}^{\infty} \frac{n^2 z^n}{2^n}$. Find the radius of convergence R of the series and find a formula, in closed form, for $f(z)$ that is valid for all $z \in \mathbb{C}$ with $|z| < R$.

5. (20 points) Use Cauchy's Formula to evaluate each of the following real integrals.

   (a) $\displaystyle\int_0^{2\pi} \frac{d\theta}{4 + \cos\theta}$

   (b) $\displaystyle\int_0^{\infty} \frac{x^2}{x^4 + 1} dx$

6. (15 points) Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and assume that the series has radius of convergence $R > 0$. For each $r \in \mathbb{R}$ with $0 < r < R$, let

$$M(r) = \max\{|f(z)| : z \in \mathbb{C},\ |z| = r\}.$$

Show that

$$|a_n| \le \frac{M(r)}{r^n},\ n = 0, 1, 2, 3, \ldots r \in (0, R).$$

7. (15 points) Assume that $f : \mathbb{C} \to \mathbb{C}$ is entire (i.e., analytic on $\mathbb{C}$) and let $N \in \mathbb{N}^+$, $M \in \mathbb{R}$ be given. Assume that

$$|f(z)| \le M(1 + |z|)^N \quad \text{for all } z \in \mathbb{C}.$$

Show that f is a polynomial of degree $\le N$.

8. (20 points) For each $\alpha \in \mathbb{R}$, put $P_\alpha = \{z \in \mathbb{C} : Re(z) > \alpha\}$ and let $CB_\alpha[0, \infty)$ denote the set of all continuous functions $f : [0, \infty) \to \mathbb{R}$ for which there is a constant K (depending on f) such that

$$|f(t)| \le K e^{\alpha t} \quad \text{for all } t \ge 0.$$

Given $\alpha \in \mathbb{R}$ and $f \in CB_\alpha[0, \infty)$ we define the Laplace Transform $\hat{f} : P_\alpha \to \mathbb{C}$ of f by

$$\hat{f}(z) = \int_0^\infty e^{-zt} f(t)\,dt, \quad z \in P_\alpha. \tag{1}$$

Under reasonable conditions, a function $f \in CB_\alpha[0, \infty)$ can be recovered from its Laplace transform via the formula

$$f(t) = \frac{1}{2\pi i} \lim_{b \to \infty} \int_{\gamma_{a,b}} e^{zt} \hat{f}(z)\,dz, t \ge 0, \tag{2}$$

where a is any fixed real number with $a > \alpha$ and $\gamma_{a,b}$ is the line segment from $a - bi$ to $a + bi$, where b is real.

(a) Let $\alpha \in \mathbb{R}$ and $f \in CB_\alpha[0, \infty)$ be given. Show that $\hat{f}$ is analytic on P_α.

(b) Given that the function $\hat{f} : P_0 \to \mathbb{C}$ defined by

$$\hat{f}(z) = \frac{1}{z^2 + 1} \quad \text{for all } z \in \mathbb{C}$$

is the Laplace transform of a function $f \in CB_0[0, \infty)$, use (2) to find an explicit formula for $f(t), t \geq 0$.

(c) Remark: Let $\alpha \in \mathbb{R}$ be given. For each $f \in CB_\alpha[0, \infty)$, the function f can be recovered from $\hat{f}$ via the formulas

$$f(t) = F''(t), F(t) = \frac{1}{2\pi i} \lim_{b \to \infty} \int_{\gamma_{a,b}} \frac{e^{zt} \hat{f}(z)}{z^2} dz,$$

where a is any fixed real number with $a > \max\{\alpha, 0\}$. (You do not need to prove the remark.)