Due by 4:00 PM on Monday, April 5

Consultation about **any** aspect of the exam with persons other than the instructor is expressly prohibited. The use of internet sources is also prohibited. You may use your class notes and Curtis. If you use any math books other than Curtis, please be sure to cite them. Try to give complete, clear and succinct explanations. If you cannot solve a problem as stated (or if you can prove something better), I will be glad to consider modified problems for modified credit.

Throughout these problems, $\mathbb{F}$ denotes an arbitrary field.

1. Let V be a finite dimensional vector space over $\mathbb{F}$ and let $T \in L(V,V)$, $k \in \mathbb{Z}^+$, and a polynomial φ be given. Assume that the minimal polynomial m for T satisfies

$$m = \varphi^k.$$

   (a) Show that $\mathcal{R}(\varphi(T)) \subset \mathcal{N}((\varphi(T))^{k-1})$.

   (b) Give an example to show that the inclusion in (a) can be proper.

   (c) Show that $\mathcal{R}(\varphi(T))$ is T-invariant and show that the minimal polynomial for the restriction of T to $\mathcal{R}(\varphi(T))$ is given by

$$m_r = \varphi^{k-1}.$$

2. Let V be a vector space over $\mathbb{F}$ (not necessarily finite dimensional). Let $T \in L(V,V)$ be given and assume that $T^3 = T$. Let

$$V_0 = \{x \in V : \ Tx = 0\}, \ V_1 = \{x \in V : \ Tx = x\}, \ V_{-1} = \{x \in V : \ Tx = -x\}.$$

   (a) Show that $V = V_0 \oplus V_1 \oplus V_{-1}$.

   (b) Find polynomials $f_1, f_{-1} : \ \mathbb{F} \to \mathbb{F}$ such that $V_1 = f_1(T), \ V_{-1} = f_{-1}(T)$.

3. Let $m, n \in \mathbb{Z}^+$ with $m < n$ be given and let V be a finite dimensional vector space over $\mathbb{F}$ with $\dim V = n$. Let $x_1^*, \ x_2^*, \ldots, x_m^* \in V^*$ be given. Show that there exists $x \in V \backslash \{0\}$ such that

$$x_1^*(x) = x_2^*(x) = \ldots = x_m^*(x) = 0.$$

4. Assume that $\mathbb{F}$ is algebraically closed and let $m, n \in \mathbb{Z}^+$ be given. Let U, V be finite dimensional vector spaces over $\mathbb{F}$ with $\dim U = m$, $\dim V = n$. Let $S \in L(U,U),\ T \in L(V,V)$ be given.

   (a) Show that

   $$\det(S \otimes T) = (\det S)^m (\det T)^n.$$

   (b) Do you think the result of part (a) remains valid if $\mathbb{F}$ is not algebraically closed? Explain.

5. Let V be a vector space over $\mathbb{C}$ (not necessarily finite dimensional) and assume that $< \cdot, \cdot >: V \times V \to \mathbb{C}$ satisfies

   (i) $< x, y >= \overline{< y, x >}$

   (ii) $< x + y, z >=< x, z > + < y, z >$

   (iii) $< \alpha x, y >= \alpha < x, y >$

   (iv) $< x, x >\geq 0$
   for all $x, y, z \in V,\ \alpha \in \mathbb{C}$.

   Let $W = \{x \in V :< x, x >= 0\}$.

   (a) Show that W is a subspace of V.

   (b) Let $Y = V/W$. Show that the mapping $(\cdot, \cdot) : Y \times Y \to \mathbb{C}$ defined by

   $$([x], [x]) =< x, x > \quad \forall x \in X$$

   is an inner product on Y.

6. Let V be a finite dimensional complex inner product space and let $B \in L(V,V)$ be given. Assume that B is skew, i.e. $B^* = -B$.

   (a) Show that $B - \mathbb{1}$ is invertible.

   (b) Show that $(B + \mathbb{1})(B - \mathbb{1})^{-1}$ is unitary.

7. Let V be a finite dimensional real or complex inner product space and let $A \in L(V,V)$ be given.

   (a) Show that $tr(A^*A) \in \mathbb{R},\ tr(A^*A) \geq 0$, and that $tr(A^*A) = 0$ if and only if $A = 0$.

   (b) Let $B \in L(V,V)$ be given and assume that $AA^* = A^*A$ and $AB = BA$. Show that $A^*B = BA^*$.