21-342 **Linear Algebra II** **Spring 2004**

Assignment 5

Due on Monday, April 25

1. Let $n \in \mathbb{Z}$ be given. A town of population of population n wishes to form clubs according to the following rules.

   (i) No two clubs can have identical membership.

   (ii) Every club must have an even number of members.

   (iii) Each club must have an odd number of members in common with every other club.

   What can you say about the maximum number of clubs that can be formed?

2.* Let $m, n \in \mathbb{Z}$ be given. A town of population n has m red clubs $R_1, R_2, \ldots, R_m$ and m blue clubs $B_1, B_2, \ldots, B_m$. Assume that

   (i) No two red clubs have identical membership.

   (ii) No two blue clubs have identical membership.

   (iii) $|R_i \cap B_i|$ is odd for every $i \in \{1, 2, \ldots, m\}$

   (iv) $|R_i \cap B_j|$ is even for every $i, j \in \{1, 2, \ldots, m\}$ with $i \neq j$.

   Show that $m \leq n$.

3. Let $m \in \mathbb{Z}^+$ and $A \in \mathbb{R}^{m \times m}$ be given. Assume that A satisfies

$$A_{ii} > \sum_{\substack{j=1 \\ i \neq j}}^{m} |A_{ij}| \quad \text{for all } i \in \{1, 2, \ldots m\}.$$

   (Such a matrix is said to be diagonally dominant.) Show that $\det A \neq 0$.

4.* Let $n \in \mathbb{Z}^+$ and $\epsilon, \alpha, \beta > 0$ with $\alpha \neq \beta$ be given. Let $S \subset \mathbb{R}^n$ such that

$$||x - y||_2 \in (\alpha - \epsilon, \alpha + \epsilon) \cup (\beta - \epsilon, \beta + \epsilon) \quad \forall x, y \in S \text{ with } x \neq y.$$

   Show that if ϵ is sufficiently small then $|S| \leq \frac{1}{2}(n+1)(n+4)$.

5.* Let $n \in \mathbb{Z}^+$ and $\alpha, \beta, \gamma > 0$ be given. Let $S \subset \mathbb{R}^n$ and assume that $||x - y||_2 \in \{\alpha, \beta, \gamma\} \quad \forall x, y \in S$ with $x \neq y$.

   Find an upper bound for $|S|$.

6.* Let $n, m \in \mathbb{Z}^+$ be given. Let $A_1, A_2, \ldots A_m$ be subsets of $\{1, 2, \ldots n\}$ such that the pairwise symmetric differences $A_i \Delta A_j (i \neq j)$ have only two sizes.

(a) Show that $m \leq \frac{n}{2}(n+3)$
(Suggestion: Consider "incidence vectors" whose components are $-1, 1$.)

(b) Can you improve the bound in part (a)?