21-342 **Linear Algebra II** **Spring 2004**

Assignment 1

Due on Monday, January 26

1. Let $\mathbb{F} = \mathbb{R}$ and $V = P(\mathbb{R})$, the set of all real polynomials. Define $T \in L(V, V)$ by

$$(Tf)(x) = xf'(x) + f(x) \quad \forall x \in \mathbb{R},\ f \in V,$$

where f' is the derivative of f. Find the eigenvalues and eigenvectors for T.

2.* Let $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Find the eigenvalues and eigenvectors for A

(a) if $\mathbb{F} = \mathbb{R}$.

(b) if $\mathbb{F} = \mathbb{C}$.

(c) if $\mathbb{F} = \mathbb{Z}_5$.

(d) if $\mathbb{F} = \mathbb{Z}_3$.

3. Let $\mathbb{F} = \mathbb{C}$ and

$$A = \begin{pmatrix} 0 & 0 & 3 \\ 0 & 2 & 0 \\ -3 & 0 & 0 \end{pmatrix}$$

Find the eigenvalues and eigenvectors for A.

4.* Let $\mathbb{F} = \mathbb{C}$ and

$$A = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

(a) Find the minimal polynomial for A.

(b) Find the characteristic polynomial for A.

5.* Let $\mathbb{F}$ be a field and V be a vector space over $\mathbb{F}$ with $\dim V = n$. Let $S, T \in L(V, V)$ be given and assume that S, T each have n distinct eigenvalues. Show that $ST = TS$ if and only if S and T have the same eigenvectors.

6.* Let $\mathbb{F}$ be a field and V be a finite dimensional vector space over $\mathbb{F}$ with $\dim V = n$. Let $S, T \in L(V, V)$ be given. Show that rank $(ST) \geq$ rank (S) + rank$(T) - n$.

7. Prove or Disprove: Let $\mathbb{F} = \mathbb{C}$ and $T \in L(\mathbb{C}^5, \mathbb{C}^3)$ be given. Assume that $< 1, 0, i, 0, 2 >$, $< -1, 1, 0, 1, i >$ is a basis for $W(T)$. Then T is surjective.

8. Let $\mathbb{F} = \mathbb{R}$ and $V = \mathbb{R}^3$ equipped with the standard inner product $(x, y) = \sum_{i=1}^{3} x_i y_i$ and let $u_1 = \frac{1}{\sqrt{2}} < 1, 0, 1 >, u_2 =< 0, -1, 0 >$, $u_3 = \frac{1}{\sqrt{2}} < 1, 0, -1 >$. Let $T \in L(V, V)$ be given and assume that $Tu_1 =< 1, 1, 1 >$, $Tu_2 =< 0, 0, 1 >$, $Tu_3 =< 1, 1, 0 >$. Find the matrix for T relative to the basis u_1, u_2, u_3.

9.* Let $n \in \mathbb{Z}^+$ be given, let $p_1, p_2, \ldots, p_n$ be distinct prime numbers and let $P = \{p_1, p_2, \ldots, p_n\}$. Let $a_1, a_2, \ldots a_{n+1}$ be positive integers such that for each $i \in \{1, 2, \ldots, n + 1\}$ the prime factorization of a_i involves only primes from the set P. Use a linear algebra argument to show that there is a nonempty set $I \subset \{1, 2, \ldots, n + 1\}$ such that $\prod_{i \in I} a_i$ is a perfect square.

(Comment: The field $\mathbb{Z}_2$ may be useful.)

*Problems marked with an asterisk should be written up and handed in.