21-341 **Linear Algebra I** **Fall 2003**

Test 2 (Take Home) Corrected version
Due by 4:30 PM on Friday, November 7

Do **not** consult with anyone other than the instructor concerning any aspects of the test. You may use the book and your class notes. You are discouraged from using any other sources such as the internet or other mathematics books. If you do use any such sources, you must cite them. Please write up your solutions as completely and carefully as you can. Problems 1-10 are worth 10 points each. Problem 11 is for extra credit.

1. Let $\mathbb{F} = \mathbb{R}$ and consider the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 4 & 3 & 2 & 1 \end{pmatrix}.$$

   Determine whether or not A is invertible. If A is invertible, find A^{-1}.

2. Let $\mathbb{F}$ be a field and $n \in \mathbb{Z}^+$ be given. Denote by $\mathbb{F}^{n\times n}$ the set of all $n \times n$ matrices with entries from $\mathbb{F}$. We define the *trace* of a matrix $A \in \mathbb{F}^{n\times n}$ to be the element of $\mathbb{F}$ obtained by summing the entries on the main diagonal;

$$tr(A) = \sum_{i=1}^{n} A_{ii},$$

   where A_{ij} denotes the i, j entry of A. Show that

$$tr(AB) = tr(BA) \quad \forall A, B \in \mathbb{F}^{n\times n}.$$

3. Let $\mathbb{F}$ be a field and $T \in L(\mathbb{F}^4, \mathbb{F}^2)$ be given.

   Assume that

$$\mathcal{N}(T) = \{< x_1, x_2, x_3, x_4 > \in \mathbb{F}^4 : \; x_1 = x_4 = 0\}.$$

   Show that T is surjective.

4. Let $\mathbb{F}$ be a field and V be a vector space over $\mathbb{F}$. Let $T \in L(V, V)$ be given and assume that $T^2 = T$. Show that $\mathcal{N}(T) \cap \mathcal{R}(T) = \{0\}$ and that $V = \mathcal{N}(T) + \mathcal{R}(T)$.

5. Let $\mathbb{F}$ be a field and V, W be finite dimensional vector spaces over $\mathbb{F}$. Let U_1 be a subspace of V and U_2 be a subspace of W such that $\dim V = \dim U_1 + \dim U_2$. Show that there exists $T \in L(V, W)$ such that

$$\mathcal{N}(T) = U_1 \text{ and } \mathcal{R}(T) = U_2.$$

6. Let $\mathbb{F} = \mathbb{R}$ and

$$V = \{f \in P_3(\mathbb{R}) : \ f(0) = 0\}.$$

Let $f_1(x) = x, \ f_2(x) = x^2, \ f_3(x) = x^3$ for all $x \in \mathbb{R}$. You may take it for granted that f_1, f_2, f_3 is a basis for V. Consider the linear mapping $T \in L(V, V)$ defined by

$$(Tf)(x) = f(x) + xf''(x) - \int_0^x \frac{f(t)}{t} dt \quad \forall f \in V.$$

(You may take it for granted that T is linear and maps V into V.)

   (a) Find the matrix for T relative to the basis f_1, f_2, f_3.
   (b) Describe $\mathcal{N}(T)$ and $\mathcal{R}(T)$ as completely as you can.
   (c) Is T invertible? Explain.

7. Let V be a real vector space with inner product $(\cdot, \cdot) : V \times V \to \mathbb{R}$ and associated norm $|| \cdot ||$ (i.e., $||u|| = \sqrt{(u, u)}$ for all $u \in V$). Let $e_1, e_2, \ldots, e_m$ be an orthonormal list in V. (Note: It is not assumed that span $(e_1, e_2, \ldots, e_m) = V$.) Show that

$$||v||^2 \geq \sum_{i=1}^{m} (v, e_i)^2 \quad \forall v \in V.$$

8. Let V be a real vector space with inner product $(\cdot, \cdot) : \ V \times V \to \mathbb{R}$ and associated norm $|| \cdot ||$. Let $T \in L(V, V)$ and $u, v \in V$ be given and assume that $Tu = u, \ Tv = -v, \ ||Tx|| = ||x||$ for all $x \in V$. Show that $(u, v) = 0$.

9. Let V be a finite dimensional real vector space with inner product $(\cdot, \cdot) : \ V \times V \to \mathbb{R}$ and associated norm $||\cdot||$. Let $\alpha > 0$ and $T \in L(V, V)$ be given. Assume that $(Tx, x) \geq \alpha ||x||^2$ for all $x \in V$. Show that T is invertible.

10. Let $\mathbb{F} = \mathbb{Q}$ and V, W be vector spaces over $\mathbb{Q}$. Let $T : V \to W$ be given and assume that

$$T(x + y) = T(x) + T(y) \quad \forall x, y \in V.$$

Show that T is linear, i.e. show that $T(\lambda x) = \lambda T(x)$ for all $x \in V, \ \lambda \in \mathbb{Q}$.

11. (Extra Credit) Prove or Disprove: Let V, W be real vector spaces and let $T : V \to W$ be given. Assume that

$$T(x + y) = T(x) + T(y) \quad \forall x, y \in V.$$

Then T is linear.