Supplementary Problems for Assignment 1

1. Let $n \in \mathbb{Z}^+$ be given and let $\mathbb{Z}_n = \{0, 1, 2, \ldots n-1\}$ equipped with addition and multiplication modulo n.

   (a) Show that if $\mathbb{Z}_n$ is a field then n is prime.

   (b) Show that if n is prime then $\mathbb{Z}_n$ is a field. (Here you need to turn in only proof of the existence of additive and multiplicative inverses.)

2. Let $\mathbb{F}$ be a field. Show that the characteristic of $\mathbb{F}$ is either zero or prime.

3. Show that if $\mathbb{F}$ is a finite field then the characteristic of $\mathbb{F}$ is not zero.

4. Give an example of a field having exactly four elements.

5. Show that $\mathbb{Q}$ has exactly one positive half.

6. Show that $\mathbb{R}$ has exactly one positive half.

7. Show that $\mathbb{C}$ does not have a positive half.

8. Let $\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} | a, b \in \mathbb{Q}\}$ together with the usual addition and multiplication on $\mathbb{R}$. You may take it for granted that $\mathbb{Q}(\sqrt{2})$ is a field. Find two different positive halves for $\mathbb{Q}(\sqrt{2})$.