21-259 **Calculus in 3-D** **Spring 2003**

Review Problems for Test 3

1. Evaluate each of the following.

   (a) $\int_0^1 \int_0^y (x^2 + y^2)\ dxdy$

   (b) $\int_{-2}^2 \int_0^{\sqrt{4-x^2}} e^{x^2+y^2}\ dydx$

2. Let D be the region in the first quadrant that is bounded by $y = \sqrt{1-x^2}$, $y = x$, and the x-axis. Evaluate

$$\int\int_D (1 + \sqrt{x^2+y^2})\ dA$$

3. Find the volume of the region enclosed by $y = x^2,\ y = 1,\ z = 0$, and $z = 2$.

4. Find the volume of the region that is inside the sphere $x^2 + y^2 + z^2 = 4$, but outside the cylinder $x^2 + y^2 = 1$.

5. Let E be the region in the first octant that is bounded by the coordinate planes, the plane $x + y = 1$ and the plane $z = 3$. Evaluate

$$\int\int\int_E (1 + 2z) dV.$$

6. Let E be the region in the first octant that is bounded by the coordinate planes and the sphere $x^2 + y^2 + z^2 = 1$. Evaluate

$$\int\int\int_E (x^2 + y^2 + z^2)^{3/2} dV.$$

7. Evaluate each of the following line integrals.

   (a) $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F}(x, y, z) = y\ \vec{i} - x\ \vec{j} + (x^2 + y^2 + z^2)\vec{k}$ and C is described by $\vec{r}(t) = \sin \vec{i} + \cos t\ \vec{j} + t\ \vec{k}, 0 \leq t \leq \pi$.

   (b) $\int_C \left(\vec{\nabla} f\right) \cdot d\vec{r}$, where $f(x, y, z) = xyz$ and C is described by $\vec{r}(t) = (1 + t^4)\vec{i} + t^3\ \vec{j} + e^t\ \vec{k}, 0 \leq t \leq 1$.

   (c) $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F}(x, y, z) = (yz + x)\vec{i} + xz\ \vec{j} + xy\ \vec{k}$, and C is described by $\vec{r}(t)t = t\ \vec{i} + t^2\ \vec{j} + t\ \vec{k}, 0 \leq t \leq 1$.

8. Evaluate each of the following line integrals.

(a) $\displaystyle\int_C ydx + xydy$, where C is the segment of $y = x^3$ from $(0, 0)$ to $(1, 1)$.

(b) $\displaystyle\int_C - ydx + (x + y^2)dy$, where C is the segment of $x^2 + y^2 = 1$ from (1,0) to (0,1).

9. Find a function $f(x, y)$ such that $\vec{\nabla} f(x, y) = (2xy + x)\vec{\imath} + (x^2 + y^3)\vec{\jmath}$.

10. Evaluate

$$\int_C (e^y + xy)dx + xe^y dy,$$

where C is the segment of $y = x^2$ from (0,0) to (1,1).

11. Let C be a piecewise smooth simple closed curve in the $x - y$ plane and let D be the region enclosed by C. Given that

$$\int_D\int dA = \pi, \int_D\int ydA = 3, \int_C x^2dy = 2,$$

evaluate

$$\int_C (xy + 2y + 1)dx + (y^2 + 3xy + 5x)dy.$$

12. Use Green's Theorem to evaluate $\displaystyle\int_C y^3dx - x^3dy$ where C is the circle $x^2 + y^2 = 1$.

In Problems 11-13, C is oriented counterclockwise.

13. Let C be a piecewise smooth simple closed curve in the $x-y$ plane and let D be the region enclosed by C. Show that if $u(x, y)$ is a smooth function satisfying $u_{xx} + u_{yy} = 0$ in D then

$$\int_D\int \left(u_x^2 + u_y^2\right) dA = \int_C - uu_ydx + uu_xdy.$$

14. Let S be the portion of the cone $z = 4 - \sqrt{x^2 - y^2}$ for which $1 \le x^2 + y^2 \le 4$.

(a) Find the surface area of S.

(b) Evaluate $\displaystyle\int_S\int \left(z + 2\sqrt{x^2 + y^2}\right) dS$

15. Let S be the hemisphere $z = \sqrt{9 - x^2 - y^2}$ and let $\vec{n}$ be the unit normal to S that points upward. Evaluate

$$\int\int_S \vec{F} \cdot \vec{n} dS,$$

where $\vec{F}(x, y, z) = x\ \vec{i} + y\ \vec{j} + z\ \vec{k}$.