Department of Mathematics
CARNEGIE MELLON UNIVERSITY

21-259 Calculus in 3-D
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Review Problems for Test 2

1. Consider the function f defined by $f(x, y, z) = xy + y^2 z$.

   a) Find $\vec{\nabla} f$.

   b) Find the directional derivative of f at the point $(2, 3, -1)$ in the direction of the vector $\vec{v} = \langle 2, 4, -3 \rangle$.

2. Suppose that f is a smooth function with the following property: At the point $(0, 0, 0)$, f decreases most rapidly in the direction of $\langle 6, -1, 2 \rangle$ and the rate of change of f in this direction is -5. Find $\vec{\nabla} f(0, 0, 0)$.

3. Find the tangent plane and normal line to the surface $z = x^2 + 4y^2$ at the point $P_0 = (-1, 2, 17)$.

4. Let $\vec{u} = \frac{1}{\sqrt{2}} \langle 1, 1 \rangle$ and $\vec{v} = \frac{1}{\sqrt{2}} \langle 1, -1 \rangle$.

   Assume that f_x and f_y are continuous. If $D_{\vec{u}} f(2, 1) = 2\sqrt{2}$ and $D_{\vec{v}} f(2, 1) = -8\sqrt{2}$, find $\vec{\nabla} f(2, 1)$.

5. Find all points on the surface $x^2 + 4y^2 - z^2 = 4$ at which the tangent plane is parallel to the plane $x + 2y + z = 6$.

6. Suppose that $w = x^3 y + y^2 - 4z^3$, $x = rs$, $y = (r^2 - s^2)^3$, and $z = 2r + s$. Use the chain rule to compute $\frac{\partial w}{\partial s}$. (Express your answer in terms of r and s alone. Algebraic simplication is not necessary.)

7. Suppose that $w = f\left(\frac{y}{y - x}\right)$ where f is continuously differentiable. Show that
$$x \frac{\partial w}{\partial x} + y \frac{\partial w}{\partial y} = 0, \; x \neq y.$$

8. Assume that u and v have continuous second order partial derivatives and that
$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\ \frac{\partial u}{\partial y}=\frac{-\partial v}{\partial x}.$$
Show that
$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \text{ and } \frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0.$$

9. Assume that $w=f(u)$, where $u=x^2+y^2$ and f is a smooth function. Find an expression for $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}$ in terms of u, $\dfrac{dw}{du}$, and $\dfrac{d^2 w}{du^2}$.

10. Find all local maxima, local minima, and saddle points for the function f defined by

    a) $f(x,y)=x^2+xy+y^2+x-y+2$,

    b) $f(x,y)=3x^2-3xy^2+y^3+3y^2$.

11. Let R be region described by
$$0\le y\le\sqrt{16-x^2},\ 0\le x\le 4$$
Find the absolute maximum and the absolute minimum values of
$$f(x,y)=x^2+y^2-4x-2y+1$$
on R.

12. Use Lagrange multipliers to find all possible maxima and minima of f subject to the given constraint.

    a) $f(x,y,z)=\frac{2}{3}x^3+y-3z$;
    $x^2+y^2+z^2=10$

    b) $f(x,y)=xy$; $9x^2+y^2=4$

13. Write an equivalent double integral with the order of integration reversed.

a) $\displaystyle\int_0^1 \int_y^{\sqrt{y}} f(x,y)\,dxdy$

b) $\displaystyle\int_0^2 \int_1^{e^{2x}} f(x,y)\,dydx$

c) $\displaystyle\int_0^2 \int_{2x}^4 f(x,y)\,dydx$

14. Evaluate each of the following.

a) $\displaystyle\int_0^1 \int_0^{x^3} 4x^2y\,dydx$

b) $\displaystyle\int_0^1 \int_x^1 3x\sqrt{x^2+y^2}\,dydx$