October 17, 2001

Midterm Exam

50 minutes; open notes

Part I (Short Answer)

1. (16 points) Find all cluster points of $\{x_n\}_{n=1}^{\infty}$ and determine $\limsup_{n\to\infty} x_n$ and $\liminf_{n\to\infty} x_n$ if

   (a) $x_n = \dfrac{3n}{n+1}\sin\left(\dfrac{n\pi}{2}\right)$ for all $n \in \mathbb{N}$.

   (b) $x_n = \dfrac{n^2}{n^2+1} + 2((-1)^n + 1)$ for all $n \in \mathbb{N}$.

2. (8 points) Give an example of an unbounded sequence $\{x_n\}_{n=1}^{\infty}$ that has exactly one cluster point $l \in \mathbb{R}$.

3. (8 points) Give an example of a closed set $T \subset \mathbb{R}$ such that $T \neq cl(\text{int}(T))$.

4. (16 points) Find the interior and the closure of each of the following sets.

   (a) $S = (0,1) \cup \{2,3\}$

   (b) $T = \left\{\dfrac{1}{n} : n \in \mathbb{N}\right\} \cup (2,3]$

5. (8 points) Consider the sequence defined recursively by,

$$x_1 = 3,\ x_{n+1} = \frac{2}{3}x_n + \frac{8}{3x_n^2} \quad \text{for all } n \in \mathbb{N}.$$

   You may take it for granted that $\{x_n\}_{n=1}^{\infty}$ is convergent and that $x_n \geq 1$ for all $n \in \mathbb{N}$. Find $\lim_{n\to\infty} x_n$.

6. (8 points) Give an example of an infinite set $S \subset \mathbb{R}$ such that every subset of S is closed.

Part II (36 points) Do any **two** of the following four problems.

7. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence satisfying $x_n \geq n$ for all $n \in \mathbb{N}$ and define the sequence $\{y_n\}_{n=1}^{\infty}$ by

$$y_n = \frac{5x_n}{2x_n + 1} \quad \text{for all } n \in \mathbb{N}.$$

Use the definition of limit to show that $y_n \to \dfrac{5}{2}$ as $n \to \infty$.

8. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence satisfying

$$0 \leq a_n \leq 1 \quad \text{for all } n \in \mathbb{N}$$

and let $\alpha > 0$ be given. Show that the sequence $\{x_n\}_{n=1}^{\infty}$ defined recursively by

$$x_1 = \alpha, \quad x_{n+1} = a_n x_n \quad \text{for all } n \in \mathbb{N}$$

is convergent.

9. Let S and T be subsets of $\mathbb{R}$. Show that

$$\operatorname{int}(S \cap T) = \operatorname{int}(S) \cap \operatorname{int}(T)$$

10. Let A and B be subsets of $\mathbb{R}$ and put

$$S = \{y + z : y \in A,\ z \in B\}.$$

Show that if B is open then S is open.