Summary of Day 25

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June 25, 2014

1 Objectives

- Continue exploring ideas of linear transformations.
- Define range and kernel of a linear transformation.
- Expand ideas of functions to ideas of linear transformations.
- Define a homomorphism and isomorphism.

2 Summary

- Here's some non-examples:
  1. $T : M_{22} \to \mathbb{R}$ by $T(A) = \det(A)$
  2. $T : \mathbb{R} \to \mathbb{R}$ by $T(x) = 2^x$
  3. $T : \mathbb{R} \to \mathbb{R}$ by $T(x) = x + 1$

- There are some transformations that are useful:
  - If V and W are vector spaces then **zero transformation** from V to W, defined by:
  $$T(\mathbf{v}) = \mathbf{0}$$
  is a linear transformation.
  - If V is a vector space the **identity transformation** is a linear operator on V defined by
  $$T(\mathbf{v}) = \mathbf{v}$$
  - If W is a subspace of V then the **inclusion transformation** is a transformation from W to V defined by
  $$T(\mathbf{v}) = \mathbf{v}$$
- Linear transformations (as in the case of $\mathbb{R}^n$) are completely determined by a basis.

  Theorem Suppose V and W are vector spaces. Let B be a basis for V, and let $T' : B \to W$ be a function from B to W. Then there exists a unique linear transformation $T : V \to W$.

  Proof.

$\square$

Theorem Suppose V and W are vector spaces. Let $T : V \to W$ be a linear transformation, and S a spanning set for V. Then $T(S) = \{\, T(\mathbf{v}) \mid \mathbf{v} \in S \,\}$ spans the range.

Proof.

$\square$

- As with functions (and linear transformations) we can define the **composition** of the maps $T : U \to V$ and $S : V \to W$ as follows:

$$(S \circ T)(\mathbf{u}) = S(T(\mathbf{u}))$$

  where $\mathbf{u} \in U$.

- We can also define what it means for a linear transformation to be invertible. We say $T : U \to V$ linear transformation is invertible if there is some map (which, incidentally, will be a linear transformation), which we call $T^{-1} : V \to U$ such that:

$$T \circ T^{-1} = I_V \qquad T^{-1} \circ T = I_U$$

  Remark Note that we only talked about invertibility with square matrices before; that is, when the domain and codomain were the same. We do not have this restriction anymore. That said, we could have imposed this definition for matrix multiplication as well, it's just that only square matrices would have been invertible (which will be a consequent of our future studies).

  Inverses, as before, are unique if they exist.

  Example Consider the following map $T : \mathbb{R}^3 \to P^2$:

$$T \begin{pmatrix} a \\ b \\ c \end{pmatrix} = a + bx + cx^2$$

  This map is invertible. It's inverse is given by:

$$T^{-1}(a + bx + cx^2) = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

- There were two subspaces that we saw when we viewed a matrix as a linear transformation: it's range and it's null space. In the context of arbitrary vector spaces, we will call the null space the **kernel**, and denote it $\ker(T)$. The range of T will be denoted $\mathrm{range}(T)$. They have the same definitions:

  If $T : U \to V$ then
  $$\ker(T) = \{\, \mathbf{u} \in U \mid T\mathbf{u} = \mathbf{0} \,\}$$
  and
  $$\mathrm{range}(T) = \{\, \mathbf{v} \in V \mid \exists \mathbf{u}.\ T\mathbf{u} = \mathbf{v} \,\}$$

  Example Consider the map $T : \mathbb{R}^2 \to P_3$ defined by
  $$T\begin{pmatrix} a \\ b \end{pmatrix} = a + bx + bx^2$$

  Example Consider the map $T : P^3 \to P^2$ defined by:
  $$T(ax^2 + bx + c) = 2ax + b$$

- As before, we have the following property:

  Theorem If $T : U \to V$ is a linear transformation:

  1. $\ker(T)$ is a subspace of U.
  2. $\mathrm{range}(T)$ is a subspace of V

  The proof is the same as it was.

- As they are subspaces we can make the same definitions for the rank and the nullity.

  The **rank** of a linear transformation $T : U \to V$ is the dimension of it's range; we call it $\mathrm{rank}(T)$. Its **nullity** is the dimension of its kernel; we denote it $\mathrm{nullity}(T)$.

  Example Define the map $T : P_1 \to \mathbb{R}$ by
  $$T(p(x)) = \int_0^1 p(x)\,dx$$

- We had a very strong theorem that came with many consequences for our study of transformations on $\mathbb{R}^n$: The Rank Theorem. This theorem carries over for all finite dimensional vector spaces:

Theorem If $T : U \to V$ is a linear transformation between finite dimensional vector spaces then:

$$\text{rank}(T) + \text{nullity}(T) = \dim U$$

Proof.

$\square$

- If you took concepts, you can recall there are two properties a function can have that are of particular interest: one-to-one and onto.

  A function $f : A \to B$ is **onto** (or **surjective**) if for every $b \in B$ there is some $a \in A$ such that $f(a) = b$. That is: everything in the codomain is hit by the function.

  A function $f : A \to B$ is **one-to-one** (or (**injective**) if for every $a_1, a_2 \in A$ we have $f(a_1) = f(a_2)$ then $a_1 = a_2$. That is, things only get hit once.

  Example The maps above: which were onto, one-to-one, both, or neither?

- These are of interest for linear transformations because they tell us about aspects of the spaces $\ker(T)$ and $\text{range}(T)$.

Theorem If $T : U \to V$ is a linear transformation then

- T is onto if and only if $\text{range}(T) = V$.
- T is 1-1 if and only if $\ker(T) = \{0\}$.

Proof.

$\square$

Theorem If T is a linear transformation on a finite dimensional vector space then it is one to one if and only if it is onto.