Summary of Day 24

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1 Objectives

- Generalize the notion of basis and dimension to abstract vector spaces.

2 Summary

- Moving on, we'd like to generalize the notion of dimensions as well. This is where we reach a snag. We can't do it right away just by making a definition, because remember that dimension only make sense if we know that all bases have the same size. If all bases didn't have the same size, we'd be in trouble.

  We're trouble with this for the following reason though:

  Theorem $\{1, x, x^2, \ldots\}$ is a basis for the space of polynomials.

  Proof. We already established it is linearly independent, and it clearly spans the space. □

  So, what does it mean for two infinite sets to have the same size? If you've taken concepts, you know that we can invoke cardinality here. The space of polynomial has a countable basis, and therefore it's dimension should be countably infinite (if we can make sense out of dimension). The point however is that this is a difficult theorem for spaces with infinite bases.

  The theorem easily carries over if there's a finite basis:

  Theorem if V is a vector space with basis $\mathbf{v}_1, \ldots, \mathbf{v}_n$ then set of $< n$ vectors cannot span the space, and any set of $> n$ vectors is linearly independent. Therefore, all bases have the same size.

  Proof. The same proof as in $\mathbb{R}^n$. □

- A vector space which has a finite basis is **finite dimensional**. Otherwise, it is **infinite dimensional**

  Example All the coordinate spaces are finite dimensional. P_n is as well (what is its dimension?). The space of polynomials is infinite dimensional (have we proved it?). The space of real valued function, differentiable function, etc is infinite dimensional.

  Remark The theorem about dimension does carry over with respect to cardinality, but it is beyond the scope of this course. All bases for the space of polynomials, therefore, are countable. All bases for the space of real valued functions have a much higher cardinality (which is uncountable).

  Theorem If V is a finite dimensional vector space of dimension n then

  1. Any linearly independent set in V contains at most n vectors.
  2. Any spanning set for V contains at least n vectors.
  3. Any linearly independent set which exactly n vectors is a basis.
  4. Any spanning set for V consisting of exactly n vectors is a basis.
  5. Any linear independent set can be extended to be a basis.

6. Any spanning set can be reduced to be a basis.

<u>**Example**</u> We have already established that the dimension of P_2 is 3. Consider these vectors of P_3:

$$\left\{x^2 - x + 1, x + 10, 3\right\}$$

This is a basis. Why?

<u>**Example**</u> Consider this set of vectors from P_2:

$$\{1 + x, 1 - x\}$$

We first claim they are linearly independent.

Then we want to extend it to a basis for all of P_2.

<u>**Theorem**</u> If W is a subspace of a finite dimensional vector space V then:

1. W is finite dimensional, and $\dim(W) \leq \dim(V)$
2. $\dim W = \dim V$ if and only if $V = W$.

- We will now turn out attention to generalizing the notion of linear transformations to abstrary vector spaces.

  $T : V \to W$ is a **linear transformation** from a vector space V to a vector space W (over the same field F) such that for all $\mathbf{u}, \mathbf{v} \in V$ and all scalars c:

  1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$
  2. $T(c\mathbf{u}) = cT(\mathbf{u})$

  <u>**Theorem**</u> $T : V \to W$ if and only if

  $$T(c_1\mathbf{v_1} + \cdots + c_k\mathbf{v_k}) = c_1T(\mathbf{v_1}) + \cdots + c_kT(\mathbf{v_k})$$

- Let's explore some examples of linear transformations.

  <u>**Example**</u> A transformation by matrix multiplication is a linear transformation, and is the one we primarily studied in this course.

  <u>**Example**</u> Consider the space of $n \times n$ matrices. Then the map $T(M) = M^T$ is a linear transformation.

  <u>**Example**</u> Consider the space of differentiable function, $\mathcal{D}$. Let $D : \mathcal{D} \to \mathcal{F}$ be the differential operator; that is $\mathcal{D}(f) = f'$.

Example Let $\mathcal{C}[a,b]$ be the space of continuous functions on the interval a,b. (Why is this a vector space?). Consider the function:

$$S(f) = \int_a^b f(x)dx$$

- Here's some non-examples:

  1. $T : M_{22} \to \mathbb{R}$ by $T(A) = \det(A)$
  2. $T : \mathbb{R} \to \mathbb{R}$ by $T(x) = 2^x$
  3. $T : \mathbb{R} \to \mathbb{R}$ by $T(x) = x + 1$

- There are some transformations that are useful:

  – If V and W are vector spaces then **zero transformation** from V to W, defined by:

  $$T(\mathbf{v}) = \mathbf{0}$$

  is a linear transformation.

  – If V is a vector space the **identity transformation** is a linear operator on V defined by

  $$T(\mathbf{v}) = \mathbf{v}$$

  – If W is a subspace of V then the **inclusion transformation** is a transformation from W to V defined by

  $$T(\mathbf{v}) = \mathbf{v}$$

- Linear transformations (as in the case of $\mathbb{R}^n$) are completely determined by a basis.

  Theorem Suppose V and W are vector spaces. Let B be a basis for V, and let $T' : B \to W$ be a function from B to W. Then there exists a unique linear transformation $T : V \to W$.

  Proof.

$\square$

Theorem Suppose V and W are vector spaces. Let $T : V \to W$ be a linear transformation, and S a spanning set for V. Then $T(S) = \{ T(\mathbf{v}) \mid \mathbf{v} \in S \}$ spans the range.

Proof.

$\square$