Summary of Day 23

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1 Objectives

- Explore Abstract Vector Spaces
- Explore subspaces of vector spaces
- Generalize notions of bases, dimension, linear independence, etc. to arbitrary vector spaces.

2 Summary

- Studying vector spaces gives us a change to make very broad theorems above a large class of structures. We will see that a lot of theorems we have already done carry over to all vector spaces. For now, here are some:

  Theorem Let V be any vector space, $\mathbf{u}$ a vector and c a scalar. Then:

  1. $0\mathbf{u} = \mathbf{0}$
  2. $c\mathbf{0} = \mathbf{0}$
  3. $(-1)\mathbf{u} = -\mathbf{u}$
  4. If $c\mathbf{u} = \mathbf{0}$ then $c = 0$ or $\mathbf{u} = \mathbf{0}$.

  Proof.

  □

- We can also generalize the notion of a subspace: W is a subspace of V if W is a subset of V and W is itself a vector space with the same operations as V.

  To check something is a subspace, it really amounts to checking closure since V was alreay known to be a subspace:

  Theorem W is a subspace of V if W is closed under addition (i.e. $\mathbf{u} + \mathbf{v} \in W$ if $\mathbf{u}, \mathbf{v} \in W$) and scalar multiplication (i.e. $c\mathbf{u} \in W$ if c is a scalar and $\mathbf{u} \in W$).

  Example $m \times n$ symmetric (real) matrices are a subspace of the space of $m \times n$ (real) matrices.

Example Integrable functions is a subspace of the space of real valued function on $\mathbb{R}$.

Example The set of solutions to the differential equation

$$f'' + f = 0$$

is a subspace of the differentiable function.

Example The set of all 2×2 matrices with determinant 0 is not a subspace of the vector space of 2×2 matrices.

- The notion of linear combination is exactly the same as in $\mathbb{R}^n$. So the notion of span carries over.

  Example For P_2, the space of polynomials of degree ≤ 2, we have the set $\{1, x, x^2\}$ spans the space, since every polynomial is $ax^2 + bx + c$ which is $a(x^2) + b(x) + c(1)$. What is a set that spans P, the space of polynomials?

  Example As with subspace of $\mathbb{R}^n$ we have:

  Theorem If $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are vectors from V then

  1. $\text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_k)$ is a subspace of V.
  2. $\text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_k)$ is the smallest subspace of V which contains $\mathbf{v}_1, \ldots, \mathbf{v}_k$.

- Along with the notion of linear combinations and span, the notions of linear dependence and linear independence are exactly the same. The following theorem's proof is actually exactly the same in this more general case:

  Theorem $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are vectors from V which are linearly dependent if and only if at least one of the vectors can be expressed as a linear combination of the others.

  Example In P_2, the set $\{1 + x + x^2, 1 - x + 3x^2, 1 + 3x - x^2\}$ is linearly dependent since:

  $$2(1 + x + x^2) - (1 - x + 3x^2) - (1 + 3x - x^2) = 0$$

  Example In the space of real valued functions $\{\sin^2(x), \cos^2(x), 1\}$ are linearly dependent since

  $$\sin^2(x) + \cos^2(x) - 1 = 0$$

  Example in P^n, the set $\{1, x, \ldots, x^n\}$ is linear independent.

Example In the space of function, $\{\sin(x), \cos(x)\}$ is linearly dependent.

Example In the space of polynomials, P, the set $\{1, x, x^2, \ldots\}$ is linearly independent.

- Now with the idea of span and linear dependence, we can carry over the idea of a basis. A basis for a vector space V is one which spans V and is linearly independent, just as in the case for $\mathbb{R}^n$.

  Example $\{1, x, x^2, \ldots\}$ is a basis for P.

  Example What is a basis for the space of 2×2 polynomials?

- We have the following theorem as we did for $\mathbb{R}^n$:

  Theorem If B is a basis for V then there exists a unique way to write every $\mathbf{v} \in V$ as a linear combination of vectors from B.

  The proof is the same as in $\mathbb{R}^n$.

- Using this notion, we can generalize the notion of a coordinate vector.

  Example Taking the basis $B = \{1, x, x^2\}$ a coordinate vector for $2 + 4x - x^2$ in P_2 is

$$\begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix}_B$$

- For most vector space over the real numbers, we can use all of the facts we've learned about real numbers to reason about the space by looking at coordinate vectors in $\mathbb{R}^n$.

  Theorem If $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis for V and $\mathbf{u}, \mathbf{v}$ are vectors in V and c a scalar then:

  1. $[\mathbf{v} + \mathbf{u}]_B = [\mathbf{v}]_B + [\mathbf{u}]_B$
  2. $[c\mathbf{v}]_B = c[\mathbf{v}]_B$

We omit the proof. But this tells use we can reason about the vector space V by using coordinate vectors, which live in $\mathbb{R}^n$. In fact:

Theorem If $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis for V and $\mathbf{u}_1, \ldots, \mathbf{u}_k$ are vectors in V then $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is linearly independent in V if and only if the coordinate vectors with respect to this basis are linearly independent in $\mathbb{R}^n$.

- This should tell us something: we haven't wasted out time this semester. Studying R^n was actually studying a broad class of vector spaces.

- Moving on, we'd like to generalize the notion of dimensions as well. This is where we reach a snag. We can't do it right away just by making a definition, because remember that dimension only make sense if we know that all bases have the same size. If all bases didn't have the same size, we'd be in trouble.

  We're trouble with this for the following reason though:

  Theorem $\left\{1, x, x^2, \ldots\right\}$ is a basis for the space of polynomials.

  Proof. We already established it is linearly independent, and it clearly spans the space. □

  So, what does it mean for two infinite sets to have the same size? If you've taken concepts, you know that we can invoke cardinality here. The space of polynomial has a countable basis, and therefore it's dimension should be countably infinite (if we can make sense out of dimension). The point however is that this is a difficult theorem for spaces with infinite bases.

  The theorem easily carries over if there's a finite basis:

  Theorem if V is a vector space with basis $\mathbf{v}_1, \ldots, \mathbf{v}_n$ then set of $< n$ vectors cannot span the space, and any set of $> n$ vectors is linearly independent. Therefore, all bases have the same size.

  Proof. The same proof as in $\mathbb{R}^n$. □

- A vector space which has a finite bases is **finite dimensional**. Otherwise, it is **infinite dimensional**

  Example All the coordinate spaces are finite dimensional. P_n is as well (what is its dimension?). The space of polynomials is infinite dimensional (have we proved it?). The space of real valued function, differentiable function, etc is infinite dimensional.

  Remark The theorem about dimension does carry over with respect to cardinality, but it is beyond the scope of this course. All bases for the space of polynomials, therefore, are countable. All bases for the space of real valued functions have a much higher cardinality (which is uncountable).

  Theorem If V is a finite dimensional vector space of dimension n then

  1. Any linearly indepdent set in V contains at most n vectors.
  2. Any spanning set for V contains at least n vectors.
  3. Any linearly indepdent set which exactly n vectors is a basis.
  4. Any spanning set for V consisting of exactly n vectors is a basis.
  5. Any linear independent set can be extended to be a basis.
  6. Any spanning set can be reduced to be a basis.

  Theorem If W is a subspace of a finite dimensional vector space V then:

  1. W is finite dimensional, and $\dim(W) \leq \dim(V)$
  2. $\dim W = \dim V$ if and only if $V = W$.