Summary of Day 22

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1 Objectives

- Explore Abstract Vector Spaces

2 Summary

- This semester we have explored the vector space $\mathbb{R}^n$ and subspaces of $\mathbb{R}^n$. We now move on to more abstract vector spaces whose geometric nature is either more subtle or perhaps absent all together.
- It's important to note that *almost* everything we did with $\mathbb{R}^n$ will carry over to talking about abstract vector spaces. We'll talk about which things won't. The worst things that won't will be the idea of a matrix representing a linear transformation in an infinite dimensional vector space.
- First, let's define what a vector space is. First, we have to define what a field is.

  A **field** is an algebraic structure over a set F equipped with an addition operation $+$ and a multiplication operation $\cdot$ such that:

  - The operations are complete; you can add and multiply any two elements in the field to get another element in the field.
  - The addition and multiplication operations are commutative and associative.
  - The multiplication operation distributes over the addition operation.
  - There is an additive identity, which we call 0. Similarly, there is a multiplicative identity, which we call 1. 1 and 0 must be different.
  - For every element a there is an additive inverse, which we call $-a$. $(a+(-a) = (-a)+a = 0)$
  - For every element a *except for the additive identity* this is a multiplictive inverse, which we call a^{-1}. $(a \cdot a^{-1} = a^{-1} \cdot a = 1)$

  Example The following are fields:

  - $\mathbb{Q}$
  - $\mathbb{R}$
  - $\mathbb{C}$
  - $\mathbb{Z}_p$ (integers modulo a prime).

- Vector spaces are always vector spaces over some field. We can now define what a vector space is:

  A **vector space** is a set V equipped with an binary operation of addition $+$, a field F which is called the **scalar field**, and a binary operation $\cdot$ between elements of F and V called scalar multiplication. Elements from V are called vectors. The operations must satisfy the following:

  – The operations are complete; meaning adding two vectors or multiplying a vector by a scalar results in a vector from V.

  – The addition operation is commutative and associative.

  – There is an additive identity, which we call $\mathbf{0}$.

  – For ever vector $\mathbf{a}$ there is an additive inverse $-\mathbf{a}$. $(\mathbf{a} + (-\mathbf{a}) = (-\mathbf{a}) + \mathbf{a} = \mathbf{0})$.

  – The operations of the scalar field respect that of the vector space, and vice-versa. That is to say:

    * $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
    * $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
    * $c(d\mathbf{u}) = (cd)\mathbf{u}$
    * $1\mathbf{u} = \mathbf{u}$

- We will now explore some examples of vector spaces.

  Example

  1. $\mathbb{R}^n$: n-tuples of real numbers with operations of coordinate-wise addition and scalar multiplication with scalar field $\mathbb{R}$. This is actually an instance of a more general phenomenon we will soon explore.
  2. $\mathbb{C}^n$: n-tuples of complex numbers defined in the same way, with scalar field $\mathbb{C}$.
  3. $\mathbb{Z}^n$: n-tuples of integers modulo n defined in the same way, with scalar field $\mathbb{Z}_n$.
  4. The above are all examples of **coordinate spaces**. They are: take a field, and consider n-tuples defined by coordinate-wise.
  5. Polynomials of of degree $\leq n$ with coefficients from some field F with the usual addition and scalar multiplication.

  6. The set of polynomials with coefficents from some field F with the usual addition and scalar multiplication.

7. The set of functions $f : \mathbb{R} \to \mathbb{R}$ with scalar field $\mathbb{R}$ (you can change $\mathbb{R}$ to any field F, but this is a particularly useful example) with the usual addition and scalar multiplication.

8. The set of continuous function $f : \mathbb{R} \to \mathbb{R}$ with scalar field $\mathbb{R}$ with the usual addition and scalar multiplication.
9. The set of differentiable function $f : \mathbb{R} \to \mathbb{R}$ with scalar field $\mathbb{R}$ with the usual addition and scalar multiplication.

10. Here's an odd one: real valued $m \times n$ matrices over $\mathbb{R}$ with the usual addition and scalar multiplication.

- It's also useful to see some non-examples.

  1. The following is not a vector space: $\mathbb{R}^2$ with usual addition, but scalar multiplication as:

$$c\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} cx \\ 0 \end{pmatrix}$$

  2. $m \times n$ invertible real values matrices over $\mathbb{R}$ with usual operations are not a subspace

- Studying vector spaces gives us a change to make very broad theorems above a large class of structures. We will see that a lot of theorems we have already done carry over to all vector spaces. For now, here are some:

  <u>**Theorem**</u> Let V be any vector space, $\mathbf{u}$ a vector and c a scalar. Then:

  1. $0\mathbf{u} = \mathbf{0}$

2. $c\mathbf{0} = \mathbf{0}$
3. $(-1)\mathbf{u} = -\mathbf{u}$
4. If $c\mathbf{u} = \mathbf{0}$ then $c = 0$ or $\mathbf{u} = \mathbf{0}$.

Proof.

□

- We can also generalize the notion of a subspace: W is a subspace of V if W is a subset of V and W is itself a vector space with the same operations as V.

  To check something is a subspace, it really amounts to checking closure since V was alreay known to be a subspace:

  Theorem W is a subspace of V if W is closed under addition (i.e. $\mathbf{u} + \mathbf{v} \in W$ if $\mathbf{u}, \mathbf{v} \in W$) and scalar multiplication (i.e. $c\mathbf{u} \in W$ if c is a scalar and $\mathbf{u} \in W$).

  Example $m \times n$ symmetric (real) matrices are a subspace of the space of $m \times n$ (real) matrices.

  Example Integrable functions is a subspace of the space of real valued function on $\mathbb{R}$.

  Example The set of solutions to the differential equation

$$f'' + f = 0$$

  is a subspace of the differentiable function.