Summary of Day 21

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1 Objectives

- Look at projections between two vectors, and generalize to projection of a vector on a space.

2 Summary

- In another class you might have explored the idea of a projection of one vector onto another. Let us explore that idea

- We can see the projection of $\mathbf{v}$ onto $\mathbf{u}$ is given by:

$$\text{proj}_{\mathbf{u}} \mathbf{v} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|^2} \right) \mathbf{u}$$

- We can extent this idea to the projection of a vector onto a space.

  Let $\mathbf{v}$ be a vector of $\mathbb{R}^n$ and W a subspace and $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is an orthogonal basis for W then we say the **orthogonal projection of v onto** W is defined to be:

$$\text{proj}_W(\mathbf{v}) = \text{proj}_{\mathbf{u_1}}(\mathbf{v}) + \cdots + \text{proj}_{\mathbf{u_k}}(\mathbf{v})$$

  A worry, of course, is that this might depend on the basis. We will see that it does not.

  Further we define the **component of v orthogonal to** W as:

$$\text{perp}_W(\mathbf{v}) = \mathbf{v} - \text{proj}_W(\mathbf{v})$$

  Example Let W be a plane in $\mathbb{R}^3$ given by the following orthogonal basis:

$$\mathbf{u_1} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \qquad \mathbf{u_2} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

Let

$$\mathbf{v} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$$

Find the orthogonal projection of $\mathbf{v}$ onto W and the component of $\mathbf{v}$ orthogonal to W.

- Notice that $\text{proj}_W(\mathbf{v}) + \text{perp}_W(\mathbf{v}) = \mathbf{v}$. That is, there is a decomposition of $\mathbf{v}$ in terms of a vector on the subspace W and some other vector.

  Theorem (Orthogonal Decomposition Theorem) If W is a subspace of $\mathbb{R}^n$ and $\mathbf{v}$ is a vector of $\mathbb{R}^n$ then there are **unique** vectors $\mathbf{w}$ and $\mathbf{w}^\perp \in W^\perp$ such that

  $$\mathbf{v} = \mathbf{w} + \mathbf{w}^\perp$$

  Proof.

  □

  Remark There is a problem with this proof that we'll have to fix later. Do you see what it is?

  This gives us the following as well:

  Theorem Let W be a subspace of $\mathbb{R}^n$. Then:

  $$\dim(W) + \dim(W^\perp) = n$$

  Proof.

  □

- The above actually gives a quite short proof of the rank nullity theorem since $(\text{row}(A))^{\perp} = (A)$.

- We will now try to fix the problem presented in the last section: we don't know how to find orthogonal bases for spaces. We actually don't even know if it is in principle always to find them.

- Idea: We want to be able to take a basis for some subspace W of $\mathbb{R}^n$ and transform it to an orthogonal set of vectors. We will do this using an algorithm called **The Gram-Schmidt process**.

  The Gram-Schmidt process is iterative. We will construct our vectors one at a time. The input to our algorithm is a basis $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$. The output of our algorithm will be a list of k (why k?) many vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$. Here's how it goes:

  1. First choose $\mathbf{v}_1 := x_1$. Note: $\text{span}(\mathbf{v}_1) = \text{span}(\mathbf{x}_1)$. Set $W_1 := \{\mathbf{x}_1\}$.
  2. For choose $\mathbf{v}_2$ we choose it as follows:

     $$\mathbf{v}_2 = \mathbf{x}_2 - \text{proj}_{\mathbf{v}_1}(\mathbf{x}_2) = \text{perp}_{\mathbf{v}_1}(\mathbf{x}_2)$$

     Note: $\text{span}(\mathbf{v}_1, \mathbf{v}_2) = \text{span}(\mathbf{x}_1, \mathbf{x}_2)$ Set $W_2 := \{\mathbf{x}_1, \mathbf{x}_2\}$
  3. Iterating... Choose $\mathbf{v}_i$ as follows:

     $$\begin{aligned} \mathbf{v}_i &= \mathbf{x}_i - \sum_{j=1}^{i-1} \text{proj}_{\mathbf{v}_j}(\mathbf{x}_i) \\ &= \mathbf{x}_i - \text{proj}_{W_{i-1}}(x_i) \\ &= \text{perp}_{W_{i-1}}(x_i) \end{aligned}$$

  Theorem Grahm-Schmidt is correct; meaning, the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a orthogonal basis.

  Proof. (this will be a more informal argument)

  □

Example Use Grahm-Schmidt to construct an orthonormal basis for the span of:

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} \qquad \mathbf{x}_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \end{pmatrix} \qquad \mathbf{x}_3 = \begin{pmatrix} 2 \\ 2 \\ 1 \\ 2 \end{pmatrix}$$