Summary of Day 18

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1 Objectives

- Do an example of diagonalization.
- Rediscover inner products and talk about orthogonality

2 Summary

- We begin where we left off yesterday: exploring diagonalization.

  Theorem If A is a $n \times n$ matrix with n eigenvalues with multiplicity, then the following are equivalent:

  1. A is diagonalizable
  2. $\mathbb{R}^n$ has a basis of eigenvectors of A.
  3. Each eigenvalue of A has algebraic multiplicity equal to its geometric multiplicity.

  Example Determine if these matrices are diagonalizable. If they are, they diagonalize them.

  (a)
  $$\begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}$$

  (b)
  $$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix}$$

- We will now talk about orthogonality (this is 5.1). We begin with revisiting the notion of the dot product.

- Recall: For vectors $\mathbf{v} = [v_1, \ldots, v_n]$ and $\mathbf{u} = [u_1, \ldots, u_n]$ of $\mathbb{R}^n$ we define the **dot product** of $\mathbf{u}$ and $\mathbf{v}$ by:

$$\mathbf{v} \cdot \mathbf{u} = \sum_{i=1^n} v_i u_i$$

  We say they are **orthogonal** if $\mathbf{v} \cdot \mathbf{u} = 0$. We now extend this definition to a set.

- A set of vectors $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ of $\mathbb{R}^n$ is an **orthogonal set** if the vectors in the set are pairwise orthogonal. That is:

$$\mathbf{v_i} \cdot \mathbf{v_j} = 0 \text{ if } i \neq j$$

  Example The following three vectors form an orthogonal set:

$$\mathbf{v_1} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \qquad \mathbf{v_2} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \qquad \mathbf{v_3} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

- Geometrically, the next theorem is fairly intuitive:

  Theorem If $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is an orthogonal set of vectors then S is linearly independent.

  Proof.

  □

- Recall that a basis is a linearly independent set that spans the space. The most used bases is the standard basis which is:

$$\mathbf{e_1} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \mathbf{e_n} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

  These vectors form an orthogonal set and a basis. Such bases are very useful, which motivates the next definition:

- A basis B is an **orthogonal basis** of a subspace W of $\mathbb{R}^n$ if it is also orthogonal.

  Example

- The standard basis is pretty useful because we can easily write vectors as a linear combination of it. For example. $[2,3] = 2[1,0]+3[0,1]$. All bases enjoy the property of being able to write every member uniquely, but most of the time it requires solving a system to find the coefficents. For the standard basis, this is not the case.

  This is a property of all orthogonal bases.

  Theorem Let W be subspace of $\mathbb{R}^n$ with orthogonal basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$. Then for each $\mathbf{w} \in W$ there is a unique $c_1, \ldots, c_k$ such that

  $$c_1\mathbf{v_1} + \cdots + c_k\mathbf{v_k} = \mathbf{w}$$

  Moreover:

  $$c_i = \frac{\mathbf{w} \cdot \mathbf{v_i}}{\mathbf{v_i} \cdot \mathbf{v_i}}$$

  Proof.

  Remark The formula above may look familiar if you took any classes that talked about vector geometry. It is the projection of $\mathbf{w}$ onto $\mathbf{v_i}$. We will talk about this soon, no worries.

- Something else from the above formula looks familiar. Recall that we can define a **norm** of $\mathbb{R}^n$ in the following way:

  $$\|\mathbf{x}\| = \mathbf{x} \cdot \mathbf{x}$$

  We say that a vector is a **unit vector** if it has norm 1.

  Remark The standard basis consists of orthogonal unit vectors. This motivates the next definition:

- A basis is called an **orthonormal basis** if it is an orthogonal basis consisting of unit vectors.

  Remark Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ be such a basis. Complete the following formula:

  $$\mathbf{v_i} \cdot \mathbf{v_j} = \Bigg\{$$

  In the event we have an orthonormal basis, the above theorem gets simpler:

  Theorem Let W be subspace of $\mathbb{R}^n$ with orthonormal basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$. Then for each $\mathbf{w} \in W$ there is a unique $c_1, \ldots, c_k$ such that

  $$c_1\mathbf{v_1} + \cdots + c_k\mathbf{v_k} = \mathbf{w}$$

  Moreover:

  $$c_i = \mathbf{w} \cdot \mathbf{v_i}$$