Summary of Day 17

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1 Objectives

- Begin the discussion of similarity and diagonalization.
- Explore a few more properties of eigenvalues/eigenvectors to aid in exploration of diagonalization.
- Get necessary and sufficent conditions for a matrix to be diagonalized.

2 Summary

- For now though, we will explore another equivalence relation on matrices. We have explored one already: having the same reduced row echelon form. This preserves lots of nice properties, like invertibility, rank, nullity, and even the row space and the null space (but *not* the column space, although it will preserve the dimension of this space, which is exactly the rank).

  The bad thing of this equivalence relation is it does *not* preserve most spectral properties. That is, elementary row operations in general do change the spectrum or eigenvectors (although it does preserve some properties of the spectrum; for example elementary row operations will never introduce/eliminate a 0 eigenvalue). It also does not preserve the determinant (although it does preserve the nonzero-ness of it).

- We say a matrix B is **similar to** (or **conjugate to**) matrix A if there is some invertible matrix P such that:

  $$B = P^{-1}AP$$

  We notate this as $A \sim B$.

  Example

  Theorem For any A, B square matrices:

  - $A \sim A$.
  - If $A \sim B$ then $B \sim A$.
  - If $A \sim B$ and $B \sim C$ then $A \sim C$.

- Like the equivalence relation of row equivalence, similarity preserves lots of properties of a matrix. Let's write them down:

  Theorem If A and B are $n \times n$ matrices where $A \sim B$ then:

- $\det(A) = \det(B)$
- A is invertible if and only if B is.
- A and B have the same rank.
- A and B have the same characteristic polynomial.
- A and B have the same eigenvalues.

Proof.

- This theorem gives us some easy ways to determine that matrices are not similar.

  Example The following matrices are not similar:

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \qquad \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

- We have seen that having upper triangular and diagonal matrices helps a lot with computations. To that end we make the following definition: a matrix A is **diagonalizable** if there is some diagonal matrix B such that $A \sim B$.

- This might seem artificial, but it is of computational significance. For example, suppose A was diagonalizable, and we wanted to calculate A^{100} (which is actually not the most uncommon thing to do. We might even want to look at $\lim A^n$ as $n \to \infty$!). Well, if A is diagonalizable by B then there is P such that $A = P^{-1}BP$. Then:

$$A^{100} = \underbrace{(P^{-1}BP)(P^{-1}BP)\cdots(P^{-1}BP)}_{100}$$

  But, this is just:

$$A^{100} = P^{-1}B^{100}P$$

  But raising a diagonal matrix to the 100 power is just raising the diagonal entries by that power!

- If A is diagonalizable, then the matrix B must have the eigenvalues of A in its diagonal entries. Can you see why?

Theorem If a matrix A is diagonalizable then it's determinant is the product of its eigenvalues.

- We will explore when matrices are diagonalizable. This is related to properties regarding the spectrum of a matrix. Here's a helpful property:

  Theorem Let $\mathbf{x}_1, \ldots, \mathbf{x}_m$ be eigenvectors corresponding to eigenvalues $\lambda_1, \ldots, \lambda_m$. Then $\mathbf{x}_1, \ldots, \mathbf{x}_m$ is linearly independent.

  Proof. Omitted for class. It's in the book at the end of 4.3. □

  So, the big takeaway here is that eigenvectors that come from different eigenvalues are linearly independent. So the eigenspaces corresponding to each of these are these disjoint (except for $\mathbf{0}$) subspaces of $\mathbb{R}^n$.

  Theorem If A has n distinct eigenvalues then there is a basis for $\mathbb{R}^n$ from the eigenvectors of A.

  Proof.

  □

  Remark This is not an if and only if. The converse is false because there still could be a basis. We can strengthen the two previous theorems:

  Theorem Let A be an $n \times n$ matrix. Let $\lambda_1, \ldots, \lambda_m$ be eigenvalues, and like B_m be bases, where basis B_i is a basis for the eigenspace of λ_i. Then $\bigcup B_1, \ldots, B_i$ is linearly independent.

  Proof. Omitted for class. It is 4.24 in the book. □

  Theorem Let A be a $n \times n$ matrix with k many distinct eigenvalues. There is a basis for eigenvectors of $\mathbb{R}^n$ if and only if the sum of the geometric multiplicity of each eigenvalue is n.

  Proof. The proof is really rather the same. Each eigenvalue comes as associated with some l-dimensional subspace (the eigenspace). This has a basis of l eigenvectors. If the sum of the geometric multiplicity is n then the sum of the dimensions of all of these vector spaces is n. Therefore we can choose a basis of eigenvectors by choosing a basis for each eigenspace and then putting them together. By the above theorem, they are linearly independent. As there are n of them, they must span the entire space of $\mathbb{R}^n$. □

- The purpose for all this exploration is currently unclear, but here is a theorem that will relate what we have just discovered to diagonalizability:

  Theorem A matrix A is diagonalizable if and only if it has n linearly independent eigenvectors. Moreover, if $P^{-1}AP = D$ then

  - The diagonal of A is just the eigenvalues of A
  - P is the collection of eigenvectors in the same order.

$\square$

Theorem If A is a $n \times n$ matrix with n eigenvalues with multiplicity, then the following are equivalent:

1. A is diagonalizable
2. $\mathbb{R}^n$ has a basis of eigenvectors of A.
3. Each eigenvalue of A has algebraic multiplicity equal to its geometric multiplicity.