Summary of Day 4

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1 Objectives

- Explore more geometric properties of $\mathbb{R}^n$ by looking at dot products to capture the notions of lengths and angles.
- Calculate dot products and norms of vectors.
- Write parametric and normal equations for lines and planes in $\mathbb{R}^2$ and $\mathbb{R}^3$.
- Understand the connection between lines/planes and linear combinations of vectors.
- Define span and the geometric intuition.

2 Summary

- Recall: In $\mathbb{R}^2$ a vector can be viewed as a directed line segmented. We can ask two questions about that line segment:
  - What is the length?
  - What is the angle it makes (with another vector, for instance)?
- We define a type of multiplication between vectors called the **dot product** (or **scalar product**) which is an operation:
$$\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$$
  That is, it is an operation between vectors of $\mathbb{R}^n$ that returns a scalar in $\mathbb{R}$ (hence the name scalar product). It is define as follows:

  If $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ were:
$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \qquad \mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}$$
  Then we define:
$$\mathbf{v} \cdot \mathbf{w} := v_1 w_1 + v_2 w_2 + \cdots + v_n w_n = \sum_{i=1}^{n} v_i w_i$$
- This type of product will be generalized to other vector spaces; in an abstract vector space, this type of operation is called a **inner product**. Inner products are traditionally written as $\langle \mathbf{u}, \mathbf{v} \rangle$ instead of $\mathbf{u} \cdot \mathbf{v}$. We'll use the latter notation because it is more specific: it is the dot product, which happens to be an inner product.
- There are some properties of the dot product we'd like to write down and prove.

  Theorem Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Then :

  1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$.
  2. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \cdot \mathbf{v}) + (\mathbf{u} \cdot \mathbf{w})$.

3. $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v})$.
4. $\mathbf{u} \cdot \mathbf{u} \geq 0$ and $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

Proof. We will prove just property 4, because it's a little important for something we are about to define.

There are two things we must show, so we will be begin by proving that $\mathbf{u} \cdot \mathbf{u} \geq 0$. We first can write down what $\mathbf{u}$ is as it is a vector in $\mathbb{R}^n$ therefore we can write it in the following form:

$$\mathbf{u} = [u_1, \ldots, u_n]$$

Therefore, by the definition of the dot product:

$$\mathbf{u} \cdot \mathbf{u} = u_1 u_1 + \cdots u_n u_n = u_1^2 + \cdots + u_n^2 = \sum_{i=1}^{n} u_i^2$$

It is true that for every real number c we have that $c^2 \geq 0$; therefore, the above is the sum of n non-negative numbers, therefore itself is non-negative. Thus $\mathbf{u} \cdot \mathbf{u} \geq 0$ which is what we wanted.

Now we need to show the next condition: $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$. For this, as it is an 'if and only if' we must show two direction: that the left implies thr right, and the right implies the left.

We begin by showing that the left implies the right. So we assume that $\mathbf{u} \cdot \mathbf{u} = 0$ and hope to show that $\mathbf{u} = 0$. Let $\mathbf{u}$ be as above, and then, as above, $\mathbf{u} \cdot \mathbf{u} = \sum_{i=1}^{n} u_i^2$. Suppose, for sake of contradiction that this quantity was non-zero. Then it must be that at least one of the things in the sum is non-zero; so $u_i^2 \neq 0$. This holds only when $u_i \neq 0$, which means that $\mathbf{u} \neq \mathbf{0}$ as the ith component is nonzero.

Next we show the right implies the left. This direction is easier; we need only show that $\mathbf{0} \cdot \mathbf{0} = 0$, which it does as $\sum_{i=0}^{n} (0)(0) = 0$ □

- We now define a **norm** on a the vector space on $\mathbb{R}^n$; we write the norm of vector $\mathbf{v}$ as $\|\mathbf{v}\|$ and define it as:

  $$\|\mathbf{v}\| := \sqrt{\mathbf{v} \cdot \mathbf{v}}$$

  Note that this makes sense; $\mathbf{v} \cdot \mathbf{v}$ is always a real number, and by property 4 above it is always non-negative, so it has a square root.

- The norm of a vector is suppose to give a measurement of length. We already know from geometry was the length of one of these line segments is in $\mathbb{R}^2$ and $\mathbb{R}^3$; we can check that this notion of length coincides with out expectations:

  Example $\|[v_1, v_2]\| = \sqrt{v_1^2 + v_2^2}$, which is what we'd expect from the Pythagorean Theorem.

- The norm has several properties that we'd like to pick out an identify.

  Theorem Let $\mathbf{v} \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Then:

  1. $\|\mathbf{v}\| = 0$ if and only if $\mathbf{v} = 0$.
  2. $\|c\mathbf{v}\| = c\,\|\mathbf{v}\|$.

  Proof. You should try to write out a formal proof, but these follow pretty straightforwardly from the inner product properties 3 and 4 above, and the definition of the norm. □

- There are two fundamental properties involving norms and inner products: the **Triangle inequality** and the **Cauchy-Schwarz inequality**. We will prove the former using the latter, and revisit Cauchy-Schwarz later in the course.

  Theorem (The Cauchy Schwarz inequality)

  $$\mathbf{u} \cdot \mathbf{v} \leq \|\mathbf{u}\|\,\|\mathbf{v}\|$$

  Theorem (The Triangle inequality)

  $$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\|\,\|\mathbf{v}\|$$

Proof. (of Triangle inequality).

$$\begin{aligned}
\|\mathbf{u}+\mathbf{v}\|^2 &= (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) && \text{by dfn of norm}\\
&= (\mathbf{u}+\mathbf{v})\cdot\mathbf{u} + (\mathbf{u}+\mathbf{v})\cdot\mathbf{v} && \text{dot product property}\\
&= \mathbf{u}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{u} + \mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{v} && \text{same property}\\
&= \|\mathbf{u}\|^2 + 2(\mathbf{v}\cdot\mathbf{u}) + \|\mathbf{v}\|^2 && \text{communitivity of dot product and dfn of norm}\\
&= \|\mathbf{u}\|^2 + 2|\mathbf{v}\cdot\mathbf{u}| + \|\mathbf{v}\|^2 && |x| \geq x \text{ for all } x \in \mathbb{R}\\
&\leq \|\mathbf{u}\|^2 + 2\,\|\mathbf{v}\|\,\|\mathbf{u}\| + \|\mathbf{v}\|^2 && \text{Cauchy-Schwarz}\\
&= (\|\mathbf{u}\| + \|\mathbf{v}\|)^2 && \text{factor}
\end{aligned}$$

Therefore, $\|\mathbf{u}+\mathbf{v}\|^2 \leq (\|\mathbf{u}\| + \|\mathbf{v}\|)^2$. As all quantities are positive, we can conclude that:

$$\|\mathbf{u}+\mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$$

- We can also measure the length between two vectors using the norm:

  The distance between the tips of the vectors $\mathbf{v}$ and $\mathbf{u}$ is $\|\mathbf{v}-\mathbf{u}\|$.

- We can also measure angles with the dot product

  You can use the law of cosines to get the following formula for θ

$$\cos(\theta) = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}$$

- The most important part of the above calculation is we can now describe what it means for 2 angles to be orthogonal to each other. Two vectors $\mathbf{v}$ and $\mathbf{u}$ are **orthogonal** if $\mathbf{u}\cdot\mathbf{v} = 0$.

  This is the definition of orthogonal; you can see it coincides with what you'd expect. Namely $\mathbf{u}\cdot\mathbf{v} = 0$ if and only if the angle between them is 90 degrees.

- We can also use vectors to describe lines and planes in $\mathbb{R}^n$ (but in particular, we'll stick to $\mathbb{R}^2$ and $\mathbb{R}^3$ because those are the only ones that we mere mortals can easily visualize).

- Recall (from earlier math classes) that a line in $\mathbb{R}^2$ is given by the equation $ax + by = c$ (or sometimes $y = mx + b$).

  It is the set of all points that go through a particular point (which we can describe by the vector $\mathbf{p}$ pointing at the point) with a particular slope (which we can describe by a vector $\mathbf{d}$ parallel to the slope of the line).

  Let $\mathbf{x}$ signify a point on the line. What relationship should hold between $\mathbf{x}$, $\mathbf{p}$ and $\mathbf{d}$? Well, it should be the case that if you move the line to the origin (by subtracting $\mathbf{p}$) you should be able to stretch $\mathbf{d}$ by some quantity to hit the point. That is:

$$\mathbf{x} - \mathbf{p} = t\mathbf{d}$$

  t in this instance is called the **parameter**; we can imagine t varying and as it does it 'draws' the line in $\mathbb{R}^2$. Solving for $\mathbf{x}$ you get the following equation (which should look like $y = mx + b$):

$$\mathbf{x} = t\mathbf{d} + \mathbf{p}$$

- We could also describe a line by finding a vector $\mathbf{v}$ which is orthogonal to the line. Let's call $\mathbf{n}$ a vector which is orthogonal to the line (this is called a **normal vector** to the line). Then we want that if you dot $\mathbf{n}$ with a vector pointing at point on the line offset by the point $\mathbf{p}$ you should get 0; that is:

$$\mathbf{n}\cdot(\mathbf{x}-\mathbf{p}) = 0$$

  This should look like $ax + bx = c$; particular if you move the constants to the right hand side:

$$\mathbf{n}\cdot\mathbf{x} = \mathbf{n}\cdot\mathbf{p}$$

- The last equation actually would describe a plane in $\mathbb{R}^3$; there is a vector $\mathbf{n}$ which is orthogonal to all points on a place. Therefore, if you knew this vector and a point on the plane, the above would describe all such points.

  Given two vector $\mathbf{u}$ and $\mathbf{v}$ on the plane (non-parallel), you could find a vector $\mathbf{n}$ which is orthogonal to both (using perhaps the **cross product**, which we will not talk about this his course; you could also use the dot product and solve some equations) and you'd get the following parametric equation:

  $$\mathbf{x} - \mathbf{p} = s\mathbf{u} + t\mathbf{v}$$

  Here, s and t are both parameters. If you fix one of the parameters then you can see that you are drawing a line. As both vary though, you are drawing a plane.

- This last section is really to help you build geometric intuition for $\mathbb{R}^n$. It is a useful skill to be able to visualize particular sets of points as geometric objects, like lines and planes.

  Example Consider a system where this is the augmented matrix:

  $$\left(\begin{array}{cc|c} 1 & 3 & 1 \\ 0 & 0 & 0 \end{array}\right)$$

  The only restriction for the solution is that $x + 3y = 1$. This is a linear in $\mathbb{R}^2$.

- We define the **span** of a set of vectors as the set of all linear combinations of these vectors.