Homework 3 Solutions

3.1.19 A factory produced three products and ships them to two warehouses for storage. The number of units of each product shipped to each warehouse is given by the matrix:

$$A = \begin{pmatrix} 200 & 75 \\ 150 & 100 \\ 100 & 125 \end{pmatrix}$$

Where a_{ij} is the number of units of product i sent to warehouse j.

The cost of shipping one unit of each product is \$1.50, \$1.00, and \$2.00 (respective to the order of the rows of the above matrix). The cost to ship by train are \$1.75, \$1.50, and \$1.00. Organize this information in matrices and use matrix multiplication to analyze the cost.

Solution. The cost to ship can be visualized as this matrix:

$$B = \begin{pmatrix} \$1.50 & \$1.00 & \$2.00 \\ \$1.75 & \$1.50 & \$1.00 \end{pmatrix}$$

Where the fist row is cost by truck, and second row is the cost by train, and the rows signify the different product. We want to analyze how much it would cost to ship the products to each of the two warehouses by the methods. So we do:

$$BA = \begin{pmatrix} \$650.00 & \$462.50 \\ \$675.00 & \$406.25 \end{pmatrix}$$

The columns of the matrix represents the warehouses, and the rows represents the shipping mechanism. Therefore, it is better to ship to the first warehouse using trucks and to the second warehouse by train.

3.2.29 Prove that the product of two upper triangular matrices is upper triangular.

Solution. Let A and B be upper triangular square matrices of the same dimensions. Recall this means that if a_{ij} and b_{ij} are zero when $i > j$.

We want to show that AB is upper triangular. Consider the i, j coordinate of AB where $i > j$; it suffices to show that this is 0. This comes from the ith row of A, call this $\mathbf{a_i}$ dotted with the jth row of B, call this $\mathbf{b_j}$. As A is upper triangular, for all $1 \leq k < i$, the vector $\mathbf{a_i}$ is zero at those components. Similarly, for all $j < k \leq n$ we have that $\mathbf{b}_j$ is zero at those components.

$$\mathbf{a_i} = [0, 0, \ldots \underbrace{0}_{i-1}, \ldots] \qquad \mathbf{b_j} = [\ldots, \underbrace{0}_{j+1}, 0, \ldots, 0]$$

Thus, consider the kth term of the sum of the dot product. If $k < i$ then the term is 0 since the kth component of $\mathbf{a_i}$ is 0. If $k > i$, then $k > j$ so the term is 0 since the kth component of $\mathbf{b_j}$ is 0. So the dot product is 0. □

3.2.36 a. Give an example of two symmetric matrices which whose product is non-symmetric. b. Then prove that the product of two symmetric matrices is symmetric if and only if $AB = BA$

(a) It turns out, the product of two symmetric 2 by 2 matrices is communitive. So you'll have to go to 3 by 3. There, lots of things work.

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 3 & 3 \\ 4 & 4 & 4 \\ 3 & 3 & 3 \end{pmatrix}$$

(b) *Proof.* Suppose that $AB = BA$ where A and B are symmetric. Well, $(AB)^T = B^T A^T = BA$ since A And B are symmetric, and $BA = AB$ by assumption. Therefore $(AB)^T = AB$, so AB is symmetric.

Conversely, suppose that AB is symmetric and A and B are symmetric. Then

$$AB = (AB)^T = B^T A^T = BA$$

3.3.42 (a) Prove that if A And invertible and $AB = O$ then $B = O$.

Proof. As A is invertible, there is a A^{-1} such that $A^{-1}A = I$. So, as $AB = O$, we can apply the inverse of A to the left on both sides and get $A^{-1}AB = A^{-1}O$. As anything times the zero matrix is the zero matrix, and we have that the RHS is O. Moreover, as matrix multiplication is associative, we have that $A^{-1}AB = (A^{-1}AB) = IB = B$. Therefore, $B = O$. □

(b) Find a counterexample to show that the result in (a) may fail if A is not invertible.

Solution. Probably the easiest solution is to take $A = O$. You can be less trivial though if you desire.

3.3.44 A matrix is idempotent if $A^2 = A$.

(a) Find three idempotent 2×2 matrices.

Solution.

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

(b) Prove that A is invertible and idempotent if and only if $A = I$.

Proof. Clearly I is an invertible idempotent matrix. Thus we need only show that if A is an invertible idempotent matrix then $A = I$. Well, $A^2 = A$. Apply A^{-1} on the left to both sides, and get $A^{-1}AA = A^{-1}A$. Then using associativity, and the fact that $A^{-1}A = I$ we get $IA = I$. Therefore, $A = I$. □

3.3.52 Find the inverse of the given matrix.

$$\begin{pmatrix} 2 & 0 & -1 \\ 1 & 5 & 1 \\ 2 & 3 & 0 \end{pmatrix}$$

Solution. We set up an matrix augmented with the identity matrix:

$$\left(\begin{array}{ccc|ccc} 2 & 0 & -1 & 1 & 0 & 0 \\ 1 & 5 & 1 & 0 & 1 & 0 \\ 2 & 3 & 0 & 0 & 0 & 1 \end{array}\right)$$

We then do row reductions to get the matrix in rref.

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & -3 & 5 \\ 0 & 1 & 0 & 2 & 2 & -3 \\ 0 & 0 & 1 & -7 & -6 & 10 \end{array}\right)$$

This tells us that this is the inverse:

$$\begin{pmatrix} -3 & -3 & 5 \\ 2 & 2 & -3 \\ -7 & -6 & 10 \end{pmatrix}$$