

Exam 3 Review

June 24, 2014

1 Technique Questions

- Let W be a subspace of \mathbb{R}^n .
 - Prove that W^\perp is a subspace of \mathbb{R}^n .
 - If $\mathbf{v}_1, \dots, \mathbf{v}_m$ is a basis for W and $\mathbf{u}_1, \dots, \mathbf{u}_k$ is a basis for W^\perp then $\mathbf{v}_1, \dots, \mathbf{v}_m, \mathbf{u}_1, \dots, \mathbf{u}_k$ is a basis for \mathbb{R}^n .

(Note: These are both theorems proved in the book and argued in class).

- Let W be a subspace of \mathbb{R}^n given by the basis $\{x_1, x_2\}$. Prove that $\{x_1, x_2 - \text{proj}_{x_1}(x_2)\}$ is an orthogonal basis of W . (Note: This is a special case of the theorem that the Gram-Schmidt process works).
- Let Q be an orthogonal matrix. Prove that if λ is an eigenvalue for Q then $|\lambda| = 1$.
- A $n \times n$ matrix is called a permutation matrix if (a) every entry is either 0 or 1, (b) in every row there is exactly one 1, and (c) in every column there is exactly one 1. Prove that every permutation matrix is orthogonal.
- Let Q be an upper triangular matrix which is orthogonal. Prove that Q is diagonal.
- Let W be a subspace of \mathbb{R}^n and $\mathbf{x} \in \mathbb{R}^n$.
 - Prove that \mathbf{x} is in W if and only if $\text{proj}_W(\mathbf{x}) = \mathbf{x}$.
 - Prove that $\text{proj}_W(\text{proj}_W(\mathbf{x})) = \text{proj}_W(\mathbf{x})$.
- Let V be the space of polynomials with real number coefficients with the usual operations. Define an operator $T : V \rightarrow V$ by:

$$T \left(\sum_{i=0}^n a_i x^i \right) = \sum_{i=0}^n b_i x^i$$

where

$$b_i = \begin{cases} 0 & \text{if } i \text{ is odd} \\ a_i & \text{if } i \text{ is even} \end{cases}$$

For example, $T(2x^2 + 3x + 4) = 2x^2 + 4$.

- Prove that T is a linear operator.
 - Describe the range and the kernel.
- Let P_3 be the vector space of polynomials of degree ≤ 3 . Prove that

$$\{p \in P_3 \mid p(1) = 0\}$$

is a subspace of P_3 . Find its dimension.

- If V is a vector space and T is a linear operator on V we say that λ is an **eigenvalue of T** if there is some $\mathbf{x} \in V$ nonzero such that $T\mathbf{x} = \lambda\mathbf{x}$.

Suppose that V is the vector space of differentiable function $f : \mathbb{R} \rightarrow \mathbb{R}$. Let $D : V \rightarrow V$ be the linear operator which takes the derivative of f . Prove that every real number is an eigenvalue for this transformation.

2 Definitions

Gram-Schmidt process, basis, bijection, closure under scalar multiplication, closure under vector addition, complex numbers, coordinate space, coordinate vector, dimension, field, finite dimensional, infinite dimensional, invertible function, kernel, linear dependence, linear independence, linear transformation, one-to-one, onto, orthogonal complement, orthogonal matrix, orthogonal projection, range, scalar multiplication, space of matrices, space of polynomials, space of real valued functions, span, subspace, trivial subspace, vector addition, vector projection, vector space, vector space axioms, zero subspace

3 Major Theorems

A component of the test will be able to state/use easy corollaries to theorems in a true/false manner. For example, a question on the test might be:

True/False: Every linear transformation on a finite vector space is 1-1 if and only if it is invertible.

This test will have a larger theoretical component than other tests. This is simply because the material on abstract vector spaces lends itself to theoretical thinking. So be prepared for lots of questions which test your understanding of the definitions and theorems involving vector spaces.

Similarly, you can expect some creative questions which ask you to provide short proof of things which are not listed in the technique questions. Practicing the Technique Questions will help you prepare for these questions.

4 Major Computational Problems

We've done a few types of computational problems in the unit. Here's some things you should be able to do (this is not exhaustive):

- Find the orthogonal complement of a subspace of \mathbb{R}^n .
- Find the orthogonal projection of a vector on a subspace (or vector) of \mathbb{R}^n .
- Verify that a matrix is orthogonal.
- Turn an orthogonal basis into an orthonormal basis by normalizing.
- Get an orthogonal basis for a subspace by using Gram-Schmidt.