Day 25

June 27, 2012

1 Complete Graphs

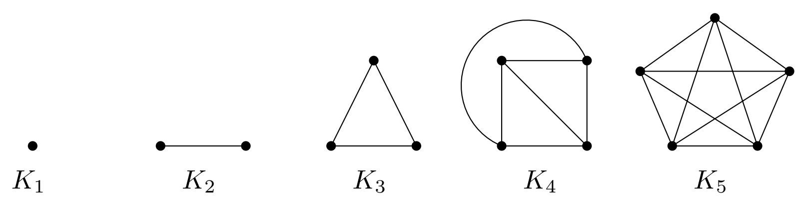
Theorem 1. *The most number of edges a (simple) graph can have is $\binom{n}{2}$ where n is the number of vertices.*

Proof. An edge is exactly determined for by its vertices. So every two vertices determines an edge, and there are $\binom{n}{2}$ ways to pick two vertices. $\square$

Corollary 1. *There are $2^{\binom{n}{2}}$ graphs on n vertices.*

Definition 1. There **complete graph on n vertices** is a graph with n where every edge is present. (So it has $\binom{n}{2}$ edges). We notate this K_n

Example 1. Here are the first 5 complete graphs.

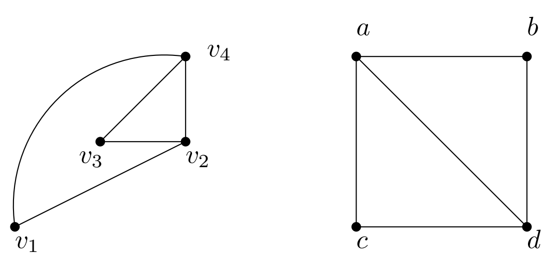


Definition 2. We say two graphs are **isomorphic** if there is a bijective function between the two sets of vertices which preserves edge relation.

More formally: G with vertex set V and edge relation E is isomorphic to G' with vertex set V' and edge relation E' if and only if there is a bijection $f : V \to V'$ such that for every $v, w \in V$ if vEw then $f(v)Ef(w)$.

Remark 1. The idea is an isomorphism spans every area of math. Two structures are isomorphic if they are "identical" with respect to the structure.

Example 2. The following two graphs are isomorphic; can you see the isomorphism?



Definition 3. A subgraph of a graph which is isomorphic to K_m is called a m-clique. That is, a set of m vertices for which every pair of vertices is connected.

Example 3. A 3-clique is also called a **triangle** since that is what it looks like graphically.

2 Coloring

Definition 4. A **(vertex)** n**-coloring** of a graph G with vertex V and edge relation E is function $f : V \to [n]$ that has the property that for every v, v' if vEv' then $f(v) \neq f(v')$. That is, adjacent vertices are colored different colors. In this context, elements on $[n]$ are called **colors**.

The minimum number n for which there is an n-coloring of G is called the **(vertex) chromatic number** for G. We notate this $\chi(G)$

Why is this an important idea?

Example 4. Suppost that you had a list of job $J_1, J_2, J_3, \ldots, J_n$ and you wanted to assign them to different time slots, but perhaps all jobs cannot be done at the same time. Maybe J_1 and J_2, for instance, both need to be done my Paul, and Paul cannot multi-task them. In this case, we say the jobs have a conflict.

We represent this by a graph where the vertices are the jobs, and two vertices are connected if the jobs they represent are in conflict. This is called a **conflict graph**. A coloring of this graph represents an assignment of the jobs to time slots in a valid way (ie. so that no conflict occurs). Therefore, the chromatic number of a conflict represents the minimial length of time necessary to complete your task.

Now with some idea about what this is important, let's prove some basic properties of it.

Theorem 2. *If G has at least one edge then $\chi(G) > 1$.*

Theorem 3. *If G has n vertices then $\chi(G) \leq n$.*

Proof. This is obvious; a valid color of G would be to assign every vertex a different color. □

Is it possible to prove this bound? We need a way of coloring a graph with a valid coloring. We look at the following algorithm:

Definition 5 (Greedy Coloring Algorithm)**.** We do recursion on the number of vertices to color the graph. If there is one vertex, just color it 1.

Otherwise, we proceed by induction. Pick a vertex v, and remove it. We are left with a graph on $n-1$ vertices, which our alogirthm colors. Then color v the minimum color that is valid; that is, look at the neights of v, and color v the smallest color which is not used.

Theorem 4. *Let G be a graph, and let d be the maximum degree of a vertex of G. Then $\chi(G) \leq d+1$.*

Proof. We claim that the Greedy Coloring Algorithm uses only colors in $[d+1]$. It is clear in the base case (when there is 1 vertex).

Otherwise, we picked a vertex v and removed it, and the Greedy algorithm gave us a valid coloring the the graph G using only colors in $[d+1]$. When we put back in v, as it's degree is at most d, there are at most d vertices that v is adjacent to. And thus, there is only d many colors that cannot be used to color v. Thus we can find a color in $[d+1]$ to color v. □

Is it possible to improve upon this result in general for all graphs? Or, are there some graphs that require exactly this many colors?

Theorem 5.

$$\chi(K_n) = n$$

Proof. We need to show two parts: We need to show $\chi(K_n) \leq n$ and $\chi(K_n) \geq n$. The first part is easy, as we can just assign every vertex a different color (and it is implied by the last two theorems).

To show the other part, suppose that we colored the graph with $n-1$ colors. By pigeon-hole, two vertices must have the same color. As the graph is complete, they are adjacent. Thus the color is not valid. □

This actually proves that any graph with a m clique has chromatic number at least m

Theorem 6. *If T is a tree with at least two vertices then $\chi(T) = 2$*

Proof. This is an exam technique question; since it is a result about trees, how do you think you should prove it? □

Theorem 7. *If G contains a cycle with n vertices, where $n > 1$, then $\chi(G) \geq 2$ if n is even, and $\chi(G) \geq 3$ if n is odd.*

Proof. This we leave as an exercise; the idea however is to show that a cycle requires the given number of colors. Name a vertex on the cycle, and say that without loss of generality this has color 1 (you can make it so by renaming the colors of the graph). Then you immediately get the result for n is even. Continue for n is odd by contradiction, and you will see a contradiction when you have only one vertex remaining to color. $\square$

2.1 Edge Coloring

An edge coloring is similar to a vertex coloring except where we color the edges instead. We can impose restrictions similar to the above (no edges connected to the same vertex are assigned the same color) and define a concept similar to the chromatic number called the edge chromatic number.

Instead though, we will go a different way and prove the following:

Theorem 8. *In any group of 5 people, assuming friendship is symmetric, there is either a group of 3 people where all are friends with each other, or 3 people who all are not.*