Exam 3 - Technique Questions

June 29, 2012

The following are Exam 3 technique questions. Of this list, you will be asked to prove a specific few of them on the exam. This will constitute the majority of the points on Exam 3. Therefore, it is highly advised that you know these questions very well. It is not advised that you simply memorize the proof, as there are too many problems here to make that feasible.

You are welcome to study these problems with each other, and use any resources as per the Homework Resource Policy on the syllabus. You may not ask me any questions about them specifically except if you think there is an error or poor phrasing. However, as always, if you feel like you are having difficulties with a concept, you should come to me for help.

1. Consider 5 digits numbers. (Note: numbers less than ten thousand are read with preceeding 0's. For example, 431 is read 00431, which as 4 distinct digits: 0, 4, 3, and 1).
   (a) Count how many said digits have exactly 3 distinct digits.
   (b) Count how many said digits have exactly 4 distinct digits.
   (c) Count how many said sigits have exactly 4 distinct digits and are even.
2. Supposed you were setting up a tennis tourtament.
   (a) Count the number of ways to set up n tennis matches with $2n$ total people (A match is different only if different people are player).
   (b) Count the number of ways to assign n matches to courts if there are n distinguishable courts.
   (c) Count the number of ways to assign n matches to courts if there are clay, grass, and hard courts and courts of the same type are indistiguishable.
3. Suppose there are 12 people, two of them are Alice and Bob.
   (a) Count the number of ways to line them up so that Alice and Bob are standing next to each other.
   (b) Count the number of ways to line them up so that Alice and Bob are not standing next to each other.
   (c) Count the number of ways to line them up so there is exactly i people between Alice and Bob, where $0 \leq i \leq 10$
4. Consider functions from $[m]$ to $[n]$.
   (a) Count the number of said bijections.
   (b) Count the number of said injections.
   (c) Count the number of said surjections.
5. You have twenty books and you will place them on a bookshelf with 5 shelves. Each shelf is capable of holding twenty books.
   (a) Count the number of ways if all you care about is how many books are on each shelf.
   (b) Count the number of ways if all you care about is which book is on which shelf.
   (c) Count the number of ways if you care about which books are on which shelf, and the order they are in on that shelf.

6. $\sum_{i=0}^{k} \binom{n}{i}\binom{m}{k-i} = \binom{m+n}{k}$

7. $\binom{n+1}{m} = \binom{n}{m-1} + \binom{n-1}{m} + \binom{n-1}{m-1}$

8. Let $A = \{1, 4, 7, \ldots, 1+3i, \ldots 100\}$. Prove that for any 19 elements chosen from these set, there is two distinct integers chosen whose sum is 104.

9. Let $n \geq 3$ be odd. Prove there is a number in $\left\{2^1 - 1, 2^2 - 1, \ldots, 2^{n-1} - 1\right\}$ which is divisble by n. *Hint: if $n \mid ab$, n odd, and a is a power of two, then n divides b. You may use this without proof.*

10. A bag contains 100 quarters, dimes, nickels, and pennies. Every minute, a coin is drawn from the bag. Determine and prove how long it will take to ensure that you have a dozen of one kind of object. *Note: You need to show the result is best possible as well*

11. Prove that if for any choice of $2^{n-1} + 1$ subsets of $[n]$ then there are two sets that are disjoint. Prove this result is best possible by finding 2^{n-1} subsets of $[n]$ which are all, pairwise, have an element in common.

12. Suppose there are 100 gold bars, and 3 pirates. Determine the nunber of ways to divide the gold among the pirates so that no one recieves more than half of the tresure.

13. Count the number of positive integers less than 1000 are not divisible by 2, 3, or 5.

14. Suppose there are n different classes, which will be taught in the spring and the fall. Count the number of ways to assign n professors to teach the courses, ensuring that a professor doesn't teach the same course in both semesters.

15. Prove the following are equivalent:

    (a) T is a tree (ie. T is minmally connected).

    (b) T for any two points on the tree a and b, there is a unique path connecting a and b

    (c) T is connected and acyclic.

    (d) T is connected with $n - 1$ vertices.

16. Prove the chromatic number of any tree with more than one vertex is 2.

17. Prove a graph is Eulerian if and only if every vertex has even degree.