Exam 2 - Technique Questions

June 18, 2012

The following are Exam 2 technique questions. Of this list, you will be asked to prove a specific few of them on the exam. This will constitute the majority of the points on Exam 2. Therefore, it is highly advised that you know these questions very well. It is not advised that you simply memorize the proof, as there are too many problems here to make that feasible.

You are welcome to study these problems with each other, and use any resources as per the Homework Resource Policy on the syllabus. You may not ask me any questions about them specifically except if you think there is an error or poor phrasing. However, as always, if you feel like you are having difficulties with a concept, you should come to me for help.

1. Suppose that R is a symmetric and transitive relation on S. Prove that the following are equivalent:
   (a) R is reflexive.
   (b) For every $a \in S$ there is a $b \in S$ such that aRb.
2. Define the relation $\sim$ on $\mathbb{Z}$ by
   $$a \sim b \iff \gcd(a, b) \neq 1$$
   Is $\sim$ an equivalence relation? Justify your answer. If it is, describe the equivalence classes.
3. Define the relation $\sim$ on $\mathbb{Z}$ by
   $$a \sim b \iff a^2 = b^2$$
   Is $\sim$ an equivalence relation? Justify your answer. If it is, describe the equivalence classes.
4. Fix a nonempty set Ω. Define a relation $\cong$ on $\wp(\Omega)$ by
   $$A \cong B \iff A \subseteq B$$
   Is $\cong$ an equivalence relation? Justify your answer. If it is, describe the equivalence classes.
5. Prove if $a \equiv b \mod n$ then $a^2 \equiv b^2 \mod n$
6. State and Prove Fermat's Little Theorem (Question 1 on Homework 5).
7. Let $f : A \to B$. Prove that for all $X, Y \subseteq A$
   $$f[X \cup Y] = f[X] \cup f[Y]$$
8. Let $f : A \to B$. Prove that for every $X, Y \subseteq A$
   $$f[X \setminus Y] \supseteq f[X] \setminus f[Y]$$
   Show that equality need not hold. Show that equality holds if f is injective.
9. Let $f : \mathbb{Z}_6 \to \mathbb{Z}_2$ by
   $$f([x]_6) = [x]_2$$
   Prove that f is well defined. Is it surjective? Injective?

10. Let $f : \mathbb{Z}_2 \to Z_6$ by
$$f([x]_2) = [x]_6$$
Show that f is not a well-defined function.

11. Let $f : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Z}^+$ by
$$f(a, b) = a + b$$
Is f surjective? Injective?

12. Fix A, B nonempty sets. Define $f : \wp(A \cup B) \to \wp(A)$ by
$$f(X) = X \cap A$$
Show that f is well-defined. Is it surjective? Injective?

13. Let $f : A \to B$. Prove that the following are equivalent:
    - f is not surjective.
    - $\exists C \subseteq B \,\mathbf{.}\, (B \setminus C \neq \emptyset) \wedge (f^{-1}[C] = A)$

14. Suppose $f : A \to A$. Prove that f is injective if and only if $f \circ f$ is injective.

15. Suppose that $f : A \to B$ and $g : B \to C$. Prove that if $g \circ f$ is injective then f must be injective. Show that the converse is not true, in general.

16. Suppose that $f : A \to B$ and $g : B \to C$. Prove that if $g \circ f$ is surjective then g must be surjective. Show that the converse is not true, in general.

17. Fix a function $f : A \to B$. Define a relation $\sim$ on A by
$$a \sim b \iff f(a) = f(b)$$
Prove that $\sim$ is an equivalence relation. What are the equivalence classes if f is injective?

18. Fix a function $f : A \to B$ Define a relation $\sim$ on B by
$$a \sim b \iff f^{-1}[\{a\}] = f^{-1}[\{b\}]$$
Prove that $\sim$ is an equivalence relation. What are the equivalence classes? What are the equivalence classes if f is surjective?

19. Prove that $\mathbb{N} \times \mathbb{N}$ is countable.

20. Let S be the set of infinite binary sequence, ie. sequences of 0's and 1's indexed by natural numbers. Prove that $|S \times S| = |S|$.

21. Construct an explicit bijection between (0,1) and [1,2].

22. Prove that the set $\{\, x \in \mathbb{R} \mid \sin(x) = \cos(x) \,\}$ is countable. Does this imply that $\{\, x \in \mathbb{R} \mid \sin(x) \neq \cos(x) \,\}$ is uncountable? Why or why not?