Approximating Integrals

June 3, 2013

Goals:

- Discuss what exactly a differential equation is, and how to verify a function is a solution to a differential equation.
- Discuss what exactly an initial value problem is, and how to find a solution to an initial value problem from the most general form a differential equation.
- Visualize differential equations with slope fields.
- Solve separable differential equations.

1 Motivation

Recall the fundamental theorem of calculus:

$$\frac{d}{dx}\int_a^x f(y)\,dy = f(x)$$

Stated differently, if $g(x)$ is the function which gives the area under f from a to x then

$$g'(x) = f(x)$$

So, g with the specifications above is a function which satisfies the following:

$$y' = f(x)$$

That is, if one substitutes g for y then the above is satisfied.

Definition 1. An **(ordinary) differential equation** is an equation involving an independent variable (usually x or t) and any number of derivatives of a dependent variable (usually y, i.e. y, y', y'', $y^{(3)}$, $\ldots$).

A function g which satisfies a differential equation is a **solution**.

Example 1. Verify that, for any c_1, c_2, the function $g(x) = c_1e^{-3t} + c_2e^{2t}$ is a solution to:

$$y'' + y' - 6y = 0$$

Example 2. Guess solutions to the following differential equations:

1. $y' = y$
2. $y' = 2\sqrt{y}$
3. $y'' = -y$

Notice, in the first two, the highest number of derivatives appearing is 1, and in the third we have a second derivative. The highest derivative used in a differential equation is called the **order** of the differential equation.

Thus, the fundamental theorem of calculus states that a function finding the area (in the Riemann sense) under $f(x)$ is a solution to $y' = f(x)$. The converse is also true; in the field of differential equations, there are **uniqueness** theorems which state that solutions are unique, with some degree of freedom.

For example, a solution to $y' = f(x)$ is not unique. For instance, if $f(x) = x$ we have already seen that

$$g(x) = \frac{1}{2}x^2 + C$$

is a solution. In fact, on the first day of class we argued that all solutions have this form. Therefore, there are infinitely many solutions to $y' = x$, and the above is the **most general solution**. But, if we give a little more information, for instance maybe $y(0) = 1$ then we also have the constraint that:

$$1 = C$$

Therefore, the only solution to $y' = x$ satisfying $y(0) = 1$ is $\frac{1}{2}x^2 + 1$.

Definition 2. An **initial value problem** is a differential equation given with an equation $y(a) = b$ which also may be satisfied.

A fact is that every sufficiently nice, first order differential equation has a unique solution when an initial value is specified. In a differential equation course, (21-261) much time will be spent giving more theorems such as this one, and more clarity on what 'sufficiently nice' means; this type of theorem is called an 'Existence and Uniqueness Theorem' as it both says there is a solution, and that solution is unique.

Example 3. Solve the following initial value problem:

1. $y' = y$ and $y(0) = 0$.
2. $y' = 2\sqrt{y}$ and $y(0) = 1$.

Notice, I didn't ask for a solution to $y'' = -y$ by giving an initial value problem. This is because second order differential equations, in general, need more than one value to specify an unique solution.

Example 4. As it turns out, the most general solution to $y'' = y$ is:

$$g(x) = c_1 \sin(x) + c_2 \cos(x)$$

Verify this is a solution, and then solve the initial value problem where $y(0) = 1$ and $y'(0) = 1$.

2 Visualizing Solutions

In class, we will talk about visualizing solutions by slope fields and isoclines.

Example 5.

- $y' = y$
- $y' = x$
- $y' = ty$
- $y' = x^2 + y^2 - 1$.

Example 6. Sketch the slope field, and then a solution going through the point:

1. $y' = y - 2x$ through $(1, 0)$.
2. $y' = y + xy$ through $(0, 1)$.

3 Separable Differential Equations

A first order differential equation is **separable** if one can separate the y variables from the x variables (multiplicitively). That is, all first order differential equations have the following form:

$$y' = g(x)f(y)$$

As it turns out, all differential equations like the above are easy to solve. We begin by rewriting the left hand side in the Leibniz derivative notation:

$$\frac{dy}{dx} = g(x)f(y)$$

Now, assume that $f(y) \neq 0$, we can divide both sides by $f(y)$ and 'multiply' both sides by dx, getting:

$$\frac{dy}{f(y)} = g(x)dx$$

Integrating both sides, we then have an implicit solution. Solving for y will give an explicit solution.

Example 7.

$$y' = \frac{x^2}{y^2}$$

Example 8.

1. $y' = xy^2$
2. $\dfrac{dP}{dt} = \sqrt{Pt}$ where $P(1) = 2$
3. $y' - \dfrac{x + x^3}{y + \sin(y)} = 0$
4. $y' = \dfrac{yt^2}{(4 - x^2)^{3/2}}$
5. $y' = \dfrac{yx\sec(x)\tan(x)}{(y+1)^2}$
6. $\dfrac{du}{dt} = \dfrac{2t + \sec^2(t)}{2u}$ where $u(0) = -5$