Methods of Integration

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In this we will go over some of the techniques of integration, and when to apply them.

1 Simple Rules

So, remember that integration is the inverse operation to differentation. Thuse we get a few rules for free:

Sum/Difference $\int (f(x) \pm g(x))\ dx = \int f(x)dx \pm \int g(x)\ dx$

Scalar Multiplication $\int cf(x)\ dx = c \cdot \int f(x)\ dx$ for $c \in \mathbb{R}$

Product Rule $\int x^n\ dx = \dfrac{x^{n+1}}{n+1} + C$ for $n \neq -1$

The above allows us to integrate any polynomials and roots. The only think we don't yet know how to integrate is $\int \frac{1}{x}\ dx$. Luckily, we know $\frac{d}{dx} \ln(x) = \frac{1}{x}$. From this, and other knowledge we know about derivatives, we know:

Trig

$\int \sin(x)\ dx = -\cos(x) + C$

$\int \cos(x)\ dx = \sin(x) + C$

$\int \sec^2(x)\ dx = \tan(x) + C$

$\int \sec(x)\tan(x)\ dx = \sec(x) + C$

Exponentials

$\int e^x\ dx = e^x + C$

$\int \frac{1}{x}\ dx = \ln|x| + C.$

!!EXAMPLES!!

2 u-substitution

Notice, if $f(x)$ and $g(x)$ are functions, then the chain rule says

$$\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x)$$

So, we know:

$$\int f'(g(x)) \cdot g'(x)\ dx = f(g(x))$$

Writing this out in a better way, we get let $u = g(x)$. Then $du = g'(x)\ dx$, meaning we can trade a $g'(x)\ dx$ for a du and substitute u for $g(x)$ in the integral. The goal is to eliminate all occurrences of x in the integral, and then your entire integral is in terms of u, and is simplier.

Example 1. Let us solve the integral

$$\int \sin(2x)\ dx$$

We do this by doing the substitution $u = 2x$. Then $du = 2\ dx$. Thus we can trade a $2\ dx$ for a du. So we write the integral in the following way:

$$\int \sin(2x)\ dx = \frac{1}{2}\int \sin(2x)(2\ dx)$$

Then:

$$\frac{1}{2}\int \sin(2x)(2\ dx) = \frac{1}{2}\int \sin(u)\ du$$

Doing the integration:

$$\frac{1}{2}\int \sin(u)\ du = \frac{1}{2}(-\cos(u)) + C$$

As the problem was given in terms of x, we want the answer in terms of x. So we substitute $2x$ for u.

$$\frac{1}{2}(-\cos(u)) + C = -\frac{cos(2x)}{2} + C$$

We do the following integrals with less exposition:

Example 2.

$$\int x\cos(x^2)\ dx$$

Set $u = x^2$. Then $du = 2x\ dx$.

$$\begin{aligned}\int x\cos(x^2)\ dx &= \frac{1}{2}\int \cos(x^2)2x\ dx\\ &= \frac{1}{2}\int \cos(u)\ du\\ &= \frac{1}{2}(\sin(u)) + C\\ &= \frac{\sin(x^2)}{2} + C\end{aligned}$$

Example 3.

$$\int \frac{\cos(\ln(x))}{x}\ dx$$

Set $u = \ln(x)$. Then $du = \frac{1}{x}\ dx$.

$$\begin{aligned}\int \frac{\cos(\ln(x))}{x}\ dx &= \int \cos(\ln(x))\frac{1}{x}\ dx\\ &= \int \cos(u)\ du\\ &= \sin(u) + C\\ &= \sin(\ln(x)) + C\end{aligned}$$

Example 4.

$$\int 3\cos(x)e^{\sin(x)}\ dx$$

Let $u = \sin(x)$. Then $du = \cos(x)\ dx$.

$$\begin{aligned}\int 3\cos(x)e^{\sin(x)}\ dx &= 3\cdot\int \cos(x)e^{\sin(x)}\ dx\\ &= 3\cdot\int e^u\ du\\ &= 3\cdot e^u + C\\ &= 3e^{\sin(x)} + C\end{aligned}$$

Example 5.

$$\int x(x+5)^{10}\ dx$$

Here, we can solve the integral by expanding. But, expanding the 10th power is rather annoying. So instead:

Let $u = x + 5$. Then we get $x = u - 5$, and $du = dx$. All we have done is a linear transformation. Note, in general we can not solve for x when we do a substitution. When the substitution is linear we can.

$$\begin{aligned}
\int x(x+5)^{10}\ dx &= \int (u-5)(u)^{10}\ du \\
&= \int (u^{11} - 5u^{10})\ du \\
&= \frac{u^{12}}{12} - \frac{5u^{11}}{11} + C \\
&= \frac{(x+5)^{12}}{12} - \frac{5(x+5)^{11}}{11} + C
\end{aligned}$$

3 Integration by Parts

Recall the product rule:

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$$

Moving things around, we see

$$f'(x)g(x) = \frac{d}{dx}[f(x)g(x)] - f(x)g'(x)$$

Integrating both sides, we see

$$\int f'(x)g(x)\ dx = f(x)g(x) - \int f(x)g'(x)dx$$

Renaming $v = f(x)$ and $u = g(x)$ we have $dv = f'(x)\ dx$ and $du = g'(x)\ dx$ and our formula becomes

$$\int u\ dv = uv - \int v\ du$$

Here, we seperate our integral into two parts: one part we differentiate, and the other we integrate. Then we apply the formula, and get a new integral with these new parts (the derivative of the one part and the integral of the other).

As a strategy, we tend to choose our u (the part we differentiate) so that the new integral is easier to integrate. We also need to take care that the dv (the part we integrate) can actually be integrated by us.

Example 6.

$$\int x \cdot e^x\ dx$$

Here, we see that when we take the derivate of x it vanishes completely making our next integral simplier.

$$\int x \cdot e^x\ dx \qquad\qquad \begin{matrix} u = x & dv = e^x\ dx \\ du = dx & v = e^x \end{matrix}$$

$$= xe^x - \int e^x\ dx$$

$$= xe^x - e^x + C$$

As a heuristic (rule of thumb) we choose logarithms and inverse trigonometric functions to be our u before any others since their integrals are hard to calculate and complicated. After those, we like polynomials (or really anything algebraic) as those derivatives often get simplier. We rarely want to choose exponentials to be our u since integrating an exponential is virtually the same as deriving it.

Unless we deviate from this heuristic, the u shall be chosen without exposition:

Example 7.

$$\int \ln(x)\ dx$$

We do this by parts:

$$\int \ln(x)\ dx \qquad\qquad \begin{matrix} u = \ln(x) & dv = dx \\ du = \frac{1}{x}\ dx & v = x \end{matrix}$$

$$\begin{aligned} &= x\ln(x) - \int \frac{1}{x}x\ dx \\ &= x\ln(x) - \int\ dx \\ &= x\ln(x) - x + C \end{aligned}$$

Example 8.

$$\int_0^1 (x^2+1)e^{-x}\ dx$$

We do this by parts:

$$\int_0^1 (x^2+1)e^{-x}\ dx \qquad\qquad \begin{matrix} u = x^2+1 & dv = e^{-x}\ dx \\ du = 2x\ dx & v = -e^{-x} \end{matrix}$$

$$\begin{aligned} &= (x^2+1)(-e^{-x}) - \int (-e^{-x})(2x)\ dx \\ &= -(x^2+1)e^{-x} + 2\int xe^{-x}\ dx \qquad\qquad \begin{matrix} u = x & dv = e^{-x}\ dx \\ du = dx & v = -e^{-x} \end{matrix} \\ &= -(x^2+1)e^{-x} + 2(-xe^{-x} - \int (-e^{-x})\ dx) \\ &= -(x^2+1)e^{-x} - 2xe^{-x} + 2\int e^{-x}\ dx \\ &= -(x^2+1)e^{-x} - 2xe^{-x} - 2e^{-x} + C \end{aligned}$$

Sometimes, we can do a nice subsitution before finishing the problem using parts:

Example 9.

$$\int x^3 \cdot 3^{x^2}\ dx$$

First, we re-write it to have e as the exponential base. Note that in general $a^b = e^{b\ln(a)}$ So

$$\int x^3 \cdot 3^{x^2}\ dx = \int x^3 \cdot e^{x^2\ln(3)}\ dx$$

Let $u = x^2$. Then $du = 2x\ dx$. So $\frac{1}{2}\ du = x\ dx$.

$$\begin{aligned} \int x^3 \cdot e^{x^2\ln(3)}\ dx &= \int (x^2)\cdot e^{(x^2)\ln(3)}(x\ dx) \\ &= \int u \cdot e^{u\ln(3)} du \end{aligned}$$

Now, we do parts.

4 Trig Functions

Let's recall what we know about trig.

Type	Formula	How it is obtained
Simple	$\sin^2(x) + \cos^2(x) = 1$	Unit circle + Pythagorean Theorem
	$\tan^2(x) + 1 = \sec^2(x)$	Divide by $\cos^2(x)$ in previous
	$1 + \cot^2(x) = \csc^2(x)$	Divide by $\sin^2(x)$ in previous
Double Angle	$\sin(2x) = 2\sin(x)\cos(x)$	Memorize/Geometry
	$\cos(2x) = \cos^2(x) - \sin^2(x)$	Memorize/Geometry
	$\cos(2x) = 2\cos^2(x) - 1$	Rewrite previous using $\sin^2(x) = 1 - \cos^2(x)$
	$\cos(2x) = 1 - 2\sin^2(x)$	Rewrite previous using $\cos^2(x) = 1 - \sin^2(x)$
Half Angle	$\sin^2(x) = \frac{1-\cos(2x)}{2}$	Use cosine double angle in terms of sine.
	$\cos^2(x) = \frac{1+\cos(2x)}{2}$	Use cosine double angle in terms of cosine
Derivatives	$\frac{d}{dx}\sin(x) = \cos(x)$	Memorize/Limit definition of derivative
	$\frac{d}{dx}\cos(x) = -\sin(x)$	Memorize/Chain rule: $\cos(x) = \sin(\frac{\pi}{2} + x)$
	$\frac{d}{dx}\sec(x) = \sec(x)\tan(x)$	Quotient Rule: $\sec(x) = \frac{1}{\cos(x)}$
	$\frac{d}{dx}\tan(x) = \sec^2(x)$	Quotient Rule: $\tan(x) = \frac{\sin(x)}{\cos(x)}$
Integrals	$\int \sin(x)\, dx = -\cos(x) + C$	Fundamental Theorem of Calculus
	$\int \cos(x)\, dx = \sin(x) + C$	Fundamental Theorem
	$\int \sec(x)\, dx = \ln\lvert\sec(x) + \tan(x)\rvert + C$	u-sub and cleverness
	$\int \tan(x)\, dx = \ln\lvert\sec(x)\rvert + C$	u-sub ($u = \cos(x)$)

Sometimes you have powers of sines and consines and you want to integrate them. Here is how:
You are doing the integral:

$$\int \sin^m(x)\cos^n(x)\, dx$$

Then:

If n is odd then save a cosine, and change the rest of the cosine's into sines using $\cos^2(x) = 1 - \sin^2(x)$. then you can do a u-substitution where $u = \sin(x)$.

$$\int \underbrace{\cos(x)}_{\text{save}}\ \underbrace{\cos^{n-1}(x)}_{\text{change to sine}}\ \sin^m(x)\, dx$$

If m is odd then save a sine, and change the rest of the sine's into cosines $\sin^2(x) = 1 - \cos^2(x)$. then you can do a u-substitution where $u = \cos(x)$.

$$\int \underbrace{\sin(x)}_{\text{save}}\ \underbrace{\sin^{m-1}(x)}_{\text{change to cosine}}\ \cos^n(x)\, dx$$

If both even then use half angle formulas to reduce problems

Example 10. See examples 1, 2 and 3 on page 310 and 311 of Stewart.

Sometimes you have to integrate powers of secant and tangents too. Here is how:
You are doing the integral:

$$\int \sec^n(x)\tan^m(x)\, dx$$

If n is even then save a $\sec^2(x)$, and change the rest of the secands into tangents by $\sec^2(x) = \tan^2(x) + 1$ then do a u-sub where $u = \tan(x)$

$$\int \underbrace{\sec^2(x)}_{\text{save}}\ \underbrace{\sec^{n-2}(x)}_{\text{change to tangents}}\ \tan^m(x)\, dx$$

If m is odd Then save a $\tan(x)$ and a $\sec(x)$, and change all the tangents into secants by $\tan^2(x) = \sec^2(x) - 1$ then do a u-sub where $u = \sec(x)$.

$$\int \underbrace{\sec(x)\tan(x)}_{\text{save}} \quad \underbrace{\tan^{m-1}(x)}_{\text{change to secants}} \quad \sec^{n-1}(x)\, dx$$

If n is odd and m is even try something else; usually these integrals generally ad hoc, and do come up from time to time. Integration by parts can be helpful (like for $\int \sec^3(x)\, dx$).

Example 11. See examples 5, 6, 7, 8 in Stewart on p.312-314.

5 Partial Fractions