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Quelques problèmes variationnels issus de la physique de la matière condensée

THÈSE

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par

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à Lucrecia

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Introduction

1 Avant-propos

Cette thèse est une exposition des différents travaux de recherche effectués par l'auteur au cours de son doctorat. Chaque chapitre pourra donc être considéré indépendamment des autres. Nous avons toutefois décidé de regrouper les références bibliographiques par soucis de présentation. Les chapitres 1, 2 et 5 sont respectivement extraits de [70], [71] et [48]. Les résultats du chapitre 3 sont annoncés dans [60].

2 Présentation de la thèse

2.1 Energie avec poids des applications à valeurs dans S^2 et singularités prescrites

Dans le premier chapitre, nous étudions un problème variationnel inspiré d'un célèbre article de H. Brezis, J.M. Coron et E.H. Lieb [30].

Pour N points distincts $a_1, \dots, a_N$ dans un domaine borné régulier $\Omega \subset \mathbb{R}^3$ (ou $\Omega = \mathbb{R}^3$) et N entiers non nuls $d_1, \dots, d_N$ tels que $\sum d_i = 0$, nous considérons la classe

$$\mathcal{E} = \Big\{ u \in C^1\big(\overline{\Omega} \setminus \cup_i\{a_i\}, S^2\big),\ u = \text{constante sur } \partial\Omega, \\ \int_\Omega |\nabla u(x)|^2 dx < +\infty,\ \deg(u, a_i) = d_i \quad \text{pour } i = 1, \dots, N \Big\}$$

(sans condition au bord si $\Omega = \mathbb{R}^3$). La condition $\sum d_i = 0$ nous assure ici que $\mathcal{E} \neq \emptyset$ (ce qui n'est pas le cas dans l'hypothèse inverse, voir [30]). On se donne une fonction mesurable $w : \Omega \to \mathbb{R}$ satisfaisant

$$0 < \lambda \leq w \leq \Lambda \quad \text{presque partout dans } \Omega \tag{1}$$

pour deux constantes λ et Λ. Notre objectif est de déterminer une formule (explicite si possible) nous permettant de calculer

$$E_w\left((a_i, d_i)_{i=1}^N\right) = \operatorname*{Inf}_{u \in \mathcal{E}} \int_\Omega |\nabla u(x)|^2 w(x) dx. \tag{2}$$

Dans [30], H. Brezis, J.M. Coron et E.H. Lieb ont étudié le cas $w \equiv 1$ et ont montré que

$$E_1\left((a_i, d_i)_{i=1}^N\right) = 8\pi L_1$$

où L_1 désigne la *longueur d'une connexion minimale* associée à la configuration $(a_i, d_i)_{i=1}^N$ et à la distance géodésique euclidienne d_Ω sur $\overline{\Omega}$. Le problème était motivé par des questions se rattachant à la théorie des cristaux liquides (cf. [43, 50]). Peu après, F. Bethuel, H. Brezis et J.M. Coron ont mis en évidence l'importance de la notion de connexion minimale en ce qui concerne l'approximation pour la topologie forte de H^1 des applications de $H^1(\Omega, S^2)$ par des applications régulières (cf. [16, 18]). Plus récemment, cette notion s'est révélée très utile pour l'étude des applications à valeurs dans S^1 en liaison avec la minimisation de la fonctionnelle de Ginzburg-Landau tridimensionnelle (voir J. Bourgain, H. Brezis et P. Mironescu [23] et H. Brezis, P. Mironescu et A.C. Ponce [32]). En étudiant le problème (2), nous chercherons à définir une notion de connexion minimale adaptée aux problèmes posés dans des milieux inhomogènes discontinus lorsque l'inhomogénéité peut être modélisée par une fonction de densité w.

Rappelons brièvement la définition générale de longueur d'une connexion minimale. Dans un espace métrique M muni d'une distance D et pour une configuration donnée $(a_i, d_i)_{i=1}^N \in M^N \times (\mathbb{Z}^*)^N$ telle que $\sum d_i = 0$, nous assignons le signe de d_i à chaque point a_i que nous écrivons $|d_i|$ fois. Nous obtenons alors une liste de points positifs $(p_1, \ldots, p_K)$ et une liste de points négatifs $(n_1, \ldots, n_K)$ (ces deux listes ont le même nombre d'éléments puisque $\sum d_i = 0$). La longueur L_D d'une connexion minimale associée à $(a_i, d_i)_{i=1}^N$ est définie par la formule :

$$L_D = \operatorname*{Min}_{\sigma \in \mathcal{S}_K} \sum_{j=1}^{K} D(p_j, n_{\sigma(j)})$$

où $\mathcal{S}_K$ est l'ensemble des permutations de K indices.

Dans la situation dite du *dipôle*, c'est à dire pour une configuration prescrite de la forme $\big((a, +1), (b, -1)\big)$, la valeur de L_1 est simplement donnée par $d_\Omega(a, b)$. Lorsque la fonction w est régulière, nous verrons que $E_w\big((a, +1), (b, -1)\big) = 8\pi\delta_w(a, b)$ où δ_w désigne la distance (riemannienne) sur $\overline{\Omega}$ définie par

$$\delta_w(a, b) = \operatorname{Inf} \int_0^1 w\big(\gamma(t)\big)|\dot{\gamma}(t)|\, dt, \tag{3}$$

l'infimum étant pris sur toutes les courbes lipschitziennes $\gamma : [0, 1] \to \overline{\Omega}$ satisfaisant $\gamma(0) = a$ et $\gamma(1) = b$. Pour une fonction mesurable w, nous observons que la formule (3) n'a plus de sens puisque w n'est pas bien définie sur les courbes qui sont des objets de mesure nulle. Nous démontrons que pour toute fonction mesurable w, la quantité $(1/8\pi)E_w\big((a, +1), (b, -1)\big)$ définit une distance sur $\overline{\Omega}$ notée $d_w(a, b)$ qui est équivalente à la distance géodésique euclidienne. De plus, nous établissons le caractère géodésique de cette distance : la distance géodésique associée à d_w coïncide avec d_w.

Dans le cas d'une configuration générale, nous montrons que

$$E_w\left((a_i,d_i)_{i=1}^N\right) = 8\pi L_w$$

où L_w désigne la longueur d'une connexion minimale associée à $(a_i,d_i)_{i=1}^N$ et à la distance d_w. Nous présentons ensuite quelques propriétés de stabilité et d'approximation par rapport à w de (2) ainsi que des résultats partiels concernant une version anisotrope de (2) (le problème général restant ouvert).

2.2 Energie relaxée des applications à valeurs dans S^2 et poids mesurables

Comme application des résultats que nous venons de présenter, nous étudions un problème de relaxation rattaché au phénomène de non densité des fonctions régulières dans $H^1(\Omega,S^2)$ muni de sa topologie forte (voir [22]).

Etant donnés un domaine borné régulier $\Omega\subset\mathbb{R}^3$, une fonction mesurable $w:\Omega\to\mathbb{R}$ satisfaisant la condition (1), et une application régulière $g:\partial\Omega\to S^2$ de degré topologique nul, nous chercherons à expliciter la fonctionnelle

$$E_w(u)=\mathrm{Inf}\big\{\liminf_{n\to+\infty}\int_\Omega|\nabla u_n(x)|^2w(x)dx,\ u_n\in H^1_g(\Omega,S^2)\cap C^1(\overline{\Omega}),\ u_n\rightharpoonup u \text{ dans } H^1 \text{ faible}\big\}$$

définie pour $u\in H^1_g(\Omega,S^2)$.

Dans [18], F. Bethuel, H. Brezis et J.M. Coron ont montré que pour $w\equiv 1$,

$$E_1(u) = \int_\Omega |\nabla u(x)|^2dx + 8\pi L_1(u),$$

où $L_1(u)$ désigne la *longueur d'une connexion minimale* relative à la distance géodésique Euclidienne d_Ω sur $\overline{\Omega}$ connectant les singularités topologiques de u. Plus précisément, $L_1(u)$ est définie par la formule

$$L_1(u) = \frac{1}{4\pi}\,\mathrm{Sup}\left\{\langle T(u),\zeta\rangle,\ \zeta:\overline{\Omega}\to\mathbb{R}\ \text{1-Lipschitz par rapport à } d_\Omega\right\}, \qquad (4)$$

$T(u)$ désignant la distribution

$$\langle T(u),\zeta\rangle = \int_\Omega D(u)\cdot\nabla\zeta - \int_{\partial\Omega}(D(u)\cdot\nu)\,\zeta$$

où $D(u) = \big(u\cdot\partial_2u\wedge\partial_3u, u\cdot\partial_3u\wedge\partial_1u, u\cdot\partial_1u\wedge\partial_2u\big)$. Lorsque l'application u a un nombre fini de singularités $a_1,\dots,a_N$ dans Ω, $T(u)$ s'écrit sous la forme (voir [30])

$$T(u) = 4\pi\sum_{i=1}^N d_i\,\delta_{a_i}$$

où $d_i = \deg(u, a_i)$. Dans cette situation, $L_1(u)$ coïncide avec la longueur d'une connexion minimale associée à la configuration $(a_i, d_i)_{i=1}^N$ et à la distance d_Ω.

Nous montrons que pour tout $u \in H^1_g(\Omega, S^2)$,

$$E_w(u) = \int_\Omega |\nabla u(x)|^2 w(x) dx + 8\pi L_w(u),$$

où

$$L_w(u) = \frac{1}{4\pi} \operatorname{Sup} \left\{ \langle T(u), \zeta \rangle,\ \zeta : \overline{\Omega} \to \mathbb{R} \text{ 1-Lipschitz par rapport à } d_w \right\}.$$

Nous étudions comme pour le problème (2), certaines propriétés de stabilité et d'approximation par rapport à w de la fonctionnelle E_w. Le cas d'une fonctionnelle sans donnée prescrite sur le bord est également traité.

2.3 Tourbillons dans un condensat de Bose-Einstein bidimensionnel en rotation (en collaboration avec R. Ignat)

Le phénomène de condensation de Bose-Einstein a donné lieu à une recherche intense depuis sa première réalisation dans des gaz alcalins en 1995. Un condensat de Bose-Einstein (BEC) est un gaz quantique pouvant être décrit par une seule fonction d'onde complexe. La présence de tourbillons est une particularité majeure de ces systèmes, ils sont définis comme les zéros de la fonction d'onde autour desquels il y a une circulation de phase. Expérimentalement, ces tourbillons peuvent être obtenus par la rotation du piège regroupant les atomes (voir [1, 68, 69]). Les premiers tourbillons sont observés à partir d'une certaine vitesse de rotation, puis leur nombre croît progressivement quand la vitesse augmente. Les tourbillons se répartissent alors régulièrement autour du centre du condensat.

Un modèle bidimensionnel de BEC en rotation a été utilisé par Y. Castin et R. Dum [40]. Ce modèle correspond à un piège confinant fortement les atomes dans la direction de l'axe de rotation. Dans le cas axisymétrique, la fonction d'onde u_ε minimise l'énergie de Gross-Pitaevskii

$$F_\varepsilon(u) = \int_{\mathbb{R}^2} \left\{ \frac{1}{2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}\left[(|u|^2 - a(x))^2 - (a^-(x))^2\right] - \Omega x^\perp \cdot (iu, \nabla u) \right\} dx$$

sous la contrainte de masse

$$\int_{\mathbb{R}^2} |u|^2 = 1$$

où $\varepsilon > 0$ est un petit paramètre d'échelle, $\Omega = \Omega(\varepsilon) \geq 0$ désigne la vitesse de rotation et $a(x) = a_0 - |x|^2$ avec a_0 determinée par $\int_{\mathbb{R}^2} a^+(x) = 1$ (i.e. $a_0 = \sqrt{2/\pi}$), représente le potentiel de piégeage.

Notre but est d'étudier le nombre et la position des tourbillons en fonction de la vitesse angulaire $\Omega(\varepsilon)$ quand $\varepsilon \to 0$. Nous nous plaçons dans la situation où Ω est au plus de l'ordre de $|\ln \varepsilon|$, ce qui correspond au régime critique pour l'existence de tourbillons.

Lorsque $\varepsilon \to 0$, la minimisation de F_ε force $|u_\varepsilon|$ à se rapprocher de $\sqrt{a^+}$. La densité de masse est donc asymptotiquement localisée dans

$$\mathcal{D} := \big\{x \in \mathbb{R}^2,\, a(x) > 0\big\} = B(0, \sqrt{a_0}\,).$$

Nous montrons également que $|u_\varepsilon|$ décroît exponentiellement vers 0 en dehors de $\mathcal{D}$. Nous limitons la recherche des tourbillons au disque $\mathcal{D}$. Un développement asymptotique de $F_\varepsilon(u_\varepsilon)$ nous permet d'estimer la vitesse critique Ω_d pour laquelle le $d$ième tourbillon devient énergétiquement favorable et aussi de calculer l'*énergie renormalisée* (i.e. l'énergie d'interaction) gouvernant la position des tourbillons.

2.4 Sur une énergie de Ginzburg-Landau avec un poids dépendant de ε

Le quatrième chapitre est consacré à l'étude des minimiseurs u_ε de la fonctionnelle de type Ginzburg-Landau avec poids

$$E_\varepsilon(u) = \frac{1}{2}\int_G |\nabla u(x)|^2 dx + \frac{1}{4\varepsilon^2}\int_G a_\varepsilon(x)(1-|u(x)|^2)^2 dx$$

définie pour $u \in H^1_g(G, S^1)$ où $G \subset \mathbb{R}^2$ est un domaine borné régulier simplement connexe, $g : \partial G \to S^1$ est une donnée régulière de degré topologique $d > 0$ et $\varepsilon > 0$ est un petit paramètre. La fonction de poids $a_\varepsilon(x)$ que nous considérons est de la forme

$$a_\varepsilon(x) = \varepsilon^{-\alpha} \;\text{ si }\; x \in G^+ \;\text{ et }\; a_\varepsilon(x) = 1 \;\text{ si }\; x \in G^-,$$

où α est une constante strictement positive, G^+ et G^- sont deux ouverts disjoints de G tels que $\overline{G^+} \cup \overline{G^-} = \overline{G}$ et $\Sigma = \overline{G^+} \cap \overline{G^-}$ définisse une courbe régulière.

Lorsque $a_\varepsilon(x) \equiv 1$, F. Bethuel, H. Brezis et F. Hélein [20] ont montré que pour toute suite $\varepsilon_n \to 0$, il existe une sous-suite (ε_{n_k}) et d points $a_1, \ldots, a_d \in G$ tels que $u_{\varepsilon_{n_k}}$ converge dans certaines topologies vers l'application harmonique u_0 donnée par

$$u_0(z) = \frac{z-a_1}{|z-a_1|} \cdots \frac{z-a_d}{|z-a_d|}\, e^{i\varphi(z)} \quad \text{dans } \; G \setminus \{a_1, \ldots, a_d\}$$

où

$$\begin{cases} \Delta\varphi = 0 & \text{dans } \; G, \\ u_0 = g & \text{sur } \; \partial G. \end{cases}$$

Il est également montré dans [20] que les singularités limites $a_1, \ldots, a_d$ peuvent être localisées dans G comme une configuration minimisante d'une certaine *énergie renormalisée* $W(\cdot)$ associée à la fonction g.

Dans notre situation, nous obtenons un résultat de convergence similaire et nous montrons que toutes les singularités limites se situent dans $G^- \cup \Sigma$, celles-ci pouvant être localisées au moyen de l'énergie renormalisée $W(\cdot)$ restreinte à l'ensemble $G^- \cup \Sigma$.

2.5 Stabilisation en temps fini pour un système d'oscillateurs amortis (en collaboration avec J.I. Díaz)

Dans le dernier chapitre, nous présentons des résultats obtenus en collaboration avec J.I. Díaz [48]. Dans cette étude, nous avons cherché à déterminer certaines conditions entrainant l'arrêt en temps fini de processus où interagissent les phénomènes de frottement de Coulomb (ou frottement solide) et d'oscillation. De telles situations se présentent dans de nombreuses formulations allant de la plus élémentaire, correspondant au mouvement d'un oscillateur harmonique soumis à un amortissement solide et visqueux

$$m\ddot{x}(t) + 2kx(t) + \mu_\beta \beta(\dot{x}(t)) + \mu_g g(\dot{x}(t)) \ni 0,$$

à celle plus complexe d'une corde vibrante amortie occupant un intervalle borné Ω

$$u_{tt} - u_{xx} + \mu_\beta \beta(u_t) + \mu_g g(u_t) \ni 0.$$

Dans chaque cas, β désigne le graphe maximal monotone de $\mathbb{R}^2$ associé à la fonction signe

$$\beta(r) = \begin{cases} \{1\} & \text{si } r > 0, \\ [-1, 1] & \text{si } r = 0, \\ \{-1\} & \text{si } r < 0, \end{cases}$$

g désigne une fonction lipschitzienne satisfaisant certaines conditions auxiliaires et les paramètres m, k, μ_β et μ_g sont supposés strictement positifs.

Nous nous intéressons principalement au cas intermédiaire à N degrés de liberté ($1 \leq N < +\infty$) se présentant lors de la discrétisation spaciale par différences finies de la corde vibrante et lors de l'étude de N oscillateurs couplés amortis. Un système modèle, admettant de nombreuses variantes, peut être formulé de la façon suivante

$$(P_N) \begin{cases} m\ddot{x}_i(t) + k(-x_{i-1}(t) + 2x_i(t) - x_{i+1}(t)) + \mu_\beta \beta(\dot{x}_i(t)) + \mu_g g(\dot{x}_i(t)) \ni 0, \\ x_i(0) = u_{0,i}, \\ \dot{x}_i(0) = v_{0,i}. \end{cases}$$

L'objectif principal de notre analyse est de montrer que la présence de la fonction g peut générer deux types d'orbite qualitativement distincts : en fonction des données initiales, l'état du système atteint un état d'équilibre soit en temps fini soit de façon asymptotique (lorsque $t \to +\infty$). Cette dichotomie constraste avec le phénomène d'*extinction en temps fini* pour les équations paraboliques non linéaires de premier ordre en temps.

Chapitre 1

Energy with weight for S^2-valued maps with prescribed singularities

1.1 Introduction and main results

Let Ω be a smooth bounded and connected open set of $\mathbb{R}^3$ or $\Omega = \mathbb{R}^3$ and let $w : \Omega \to \mathbb{R}$ be a measurable function such that

$$0 < \lambda \leq w \leq \Lambda \quad \text{a.e. in } \Omega \tag{1.1}$$

for some constant λ and Λ. We consider N distinct points $a_1, \ldots, a_N$ in Ω and we define the following class of S^2-valued maps

$$\mathcal{E} = \Big\{ u \in C^1\big(\overline{\Omega} \setminus \cup_i\{a_i\}, S^2\big),\ u = \text{const on } \partial\Omega, \\ \int_\Omega |\nabla u(x)|^2 dx < +\infty,\ \deg(u, a_i) = d_i \quad \text{for } i = 1, \ldots, N \Big\}$$

(without boundary condition if $\Omega = \mathbb{R}^3$) where the d_i's are given in $\mathbb{Z} \setminus \{0\}$ and such that $\sum d_i = 0$ (which is a necessary and sufficient condition for $\mathcal{E}$ to be non-empty, see [30]). Our goal is to establish a formula for

$$E_w\left((a_i, d_i)_{i=1}^N\right) = \operatorname*{Inf}_{u \in \mathcal{E}} \int_\Omega |\nabla u(x)|^2 w(x) dx. \tag{1.2}$$

In [30], H. Brezis, J.M. Coron and E.H. Lieb have proved that for $w \equiv 1$ this quantity is equal to $8\pi L$ where L is the *length of a minimal connection* associated to the configuration $(a_i, d_i)_{i=1}^N$ and the Euclidean geodesic distance d_Ω on $\overline{\Omega}$ (see also [8, 27, 28, 53]). The first motivation for studying such a problem comes from the theory of liquid crystals (see [43, 50]). Later F. Bethuel, H. Brezis and J.M. Coron have shown that the notion of minimal connection is very useful when dealing with questions of approximation of S^2-maps by smooth S^2-maps in the strong H^1-topology (see [16, 18]). We also refer to the results of J. Bourgain, H. Brezis, P. Mironescu [23] and H. Brezis, P. Mironescu, A.C.

Ponce [32] for some similar problems involving S^1-valued maps. In the *dipole case*, namely when we have two prescribed points P and N of degree $+1$ and -1 respectively, the value of L is equal to $d_\Omega(P,N)$. When w is continuous, we prove that $E_w(P,N) = 8\pi\delta_w(P,N)$ where δ_w denotes the Riemannian distance on $\overline{\Omega}$ defined by

$$\delta_w(P,N) = \text{Inf} \int_0^1 w\left(\gamma(t)\right)|\dot\gamma(t)|dt, \tag{1.3}$$

where the infimum is taken over all curves $\gamma \in \text{Lip}_{P,N}\left([0,1],\overline{\Omega}\right)$. Here $\text{Lip}_{P,N}\left([0,1],\overline{\Omega}\right)$ denotes the set of all Lipschitz maps γ from $[0,1]$ with values into $\overline{\Omega}$ such that $\gamma(0) = P$ and $\gamma(1) = N$. For a general measurable function w, we prove that $E_w(P,N)$ induces a geodesic distance on $\overline{\Omega}$ (in the sense defined in Section 1.2.1). We call the attention of the reader to the fact that, in the measurable case, there is no way to define a distance by a formula like (1.3) since w is not well defined on curves which are sets of null Lebesgue measure. To overcome this difficulty, we construct a kind of "length structure" in which the general idea is to thicken the curves. We proceed as follows. For two points x and y in Ω, we consider the class $\mathcal{P}(x,y)$ of all finite collections of segments $\mathcal{F} = ([\alpha_k,\beta_k])_{k=1}^{n(\mathcal{F})}$ such that $\beta_k = \alpha_{k+1}$, $\alpha_1 = x$, $\beta_{n(\mathcal{F})} = y$ and $[\alpha_k,\beta_k] \subset \Omega$. We define "the length" of an element $\mathcal{F} \in \mathcal{P}(x,y)$ by

$$\ell_w(\mathcal{F}) = \sum_{k=1}^{n(\mathcal{F})} \liminf_{\varepsilon\to 0^+} \frac{1}{\pi\varepsilon^2}\int_{\Xi([\alpha_k,\beta_k],\varepsilon)\cap\Omega} w(\xi)d\xi.$$

where $\Xi([\alpha_k,\beta_k],\varepsilon) = \{\xi \in \mathbb{R}^3,\ \text{dist}(\xi,[\alpha_k,\beta_k]) \le \varepsilon\}$ and then we consider the function $d_w : \Omega\times\Omega \to \mathbb{R}_+$ defined by

$$d_w(x,y) = \underset{\mathcal{F}\in\mathcal{P}(x,y)}{\text{Inf}} \ell_w(\mathcal{F}).$$

In Section 1.2, we extend d_w to $\overline{\Omega}\times\overline{\Omega}$ and we prove the metric and geodesic character of d_w. We also show that d_w agrees with δ_w whenever w is continuous. In Section 1.3, we give the proof of the following result.

Theorem 1.1. *We have*

$$E_w\left((a_i,d_i)_{i=1}^N\right) = 8\pi L_w$$

where L_w is the length of a minimal connection associated to the configuration $(a_i,d_i)_{i=1}^N$ and the distance d_w on $\overline{\Omega}$.

The geodesic character of the distance d_w implies that d_w coincides with the distance induced by the length functional associated to the Finsler metric φ_w obtained by differentiation of d_w (cf. Section 1.2.2). More precisely, for every P and N in $\overline{\Omega}$, we prove that

$$d_w(P,N) = \text{Min}\left\{\int_0^1 \varphi_w\left(\gamma(t),\dot\gamma(t)\right)dt,\ \gamma\in\text{Lip}_{P,N}\left([0,1],\overline{\Omega}\right)\right\}. \tag{1.4}$$

Formula (1.4) shows that, for a non-smooth w, the quantity $E_w\left((a_i,d_i)_{i=1}^N\right)$ is still given in terms of shortest paths between the a_i's but the metric we compute the lengths with might be non-isotropic (a metric φ is said to be isotropic if $\varphi(x,\nu)=p(x)|\nu|$ for some positive function p).

We recall that the length L_w of a minimal connection is computed as follows (see [30]). We relabel the points a_i, taking into account their multiplicity $|d_i|$, as two lists of positive and negative points say $(p_1,\ldots,p_K)$ and $(n_1,\ldots,n_K)$ (note that this two lists have the same number of elements since $\sum d_i=0$). Then we have

$$L_w=\underset{\sigma\in\mathcal{S}_K}{\mathrm{Min}}\sum_{j=1}^{K}d_w(p_j,n_{\sigma(j)}) \tag{1.5}$$

where $\mathcal{S}_K$ denotes the set of all permutations of K indices. Another way to compute L_w is to use the following formula (see [30]),

$$L_w=\mathrm{Max}\sum_{j=1}^{K}\zeta(p_j)-\zeta(n_j), \tag{1.6}$$

where the supremum is taken over all functions $\zeta:\overline{\Omega}\to\mathbb{R}$ which are 1-Lipschitz with respect to d_w, i.e., $|\zeta(x)-\zeta(y)|\leq d_w(x,y)$ for any $x,y\in\overline{\Omega}$. In Section 1.2.3, we give a characterization of 1-Lipschitz functions for the distance d_w. Combining this characterization with formula (1.6), we obtain the lower bound of the energy following the approach in [30]. The upper bound is obtained using explicit test functions based on a *dipole construction.*

Section 1.4.1 concerns a stability property of problem (1.2). We investigate the following question. Given an arbitrary sequence $(w_n)_{n\in\mathbb{N}}$ of real measurable functions, under which condition on $(w_n)_{n\in\mathbb{N}}$, can we conclude that $\left\{E_{w_n}\left((a_i,d_i)_{i=1}^N\right)\right\}_{n\in\mathbb{N}}$ converges to $E_w\left((a_i,d_i)_{i=1}^N\right)$? From Theorem 1.1, we infer that the convergence of the sequence $\left\{E_{w_n}\left((a_i,d_i)_{i=1}^N\right)\right\}_{n\in\mathbb{N}}$ is strictly related to the convergence of the variational problems

$$\mathrm{Min}\left\{\int_0^1\varphi_{w_n}\left(\gamma(t),\dot\gamma(t)\right)dt,\ \gamma\in\mathrm{Lip}_{P,N}\left([0,1],\overline{\Omega}\right)\right\}$$

where $P,N\in\Omega$ and φ_{w_n} denotes the Finsler metric derived from w_n. The same question involving the class $\mathrm{Lip}_{P,N}\left([0,1],\Omega\right)$ instead of the class $\mathrm{Lip}_{P,N}\left([0,1],\overline{\Omega}\right)$ has been studied in [34] by G. Buttazzo, L. De Pascale and I. Fragalà in the Γ-convergence framework. Adapting their result to our setting, we give a necessary and sufficient condition on $(w_n)_{n\in\mathbb{N}}$ under which $\left\{E_{w_n}\left((a_i,d_i)_{i=1}^N\right)\right\}_{n\in\mathbb{N}}$ converges to $E_w\left((a_i,d_i)_{i=1}^N\right)$. In Section 4.2, we concentrate on the approximation procedure by smooth weights. If one requires that w_n is continuous and converges to w uniformly in $\overline{\Omega}$ then we get easily the convergence using formula (1.3) but such an assumption implies that w is continuous and this is quite restrictive in our setting. On the other hand if one assumes that $w_n\to w$ almost everywhere in Ω, we show that the convergence of the problems does not hold in general (c.f. Remark 1.4). However, we prove that $E_w\left((a_i,d_i)_{i=1}^N\right)$ is the limit of a sequence $\left\{E_{w_n}\left((a_i,d_i)_{i=1}^N\right)\right\}_{n\in\mathbb{N}}$ where w_n obtained from w by a regularization procedure.

In the last section, we present a partial result on a similar problem involving a matrix field $M = (m_{kl})_{k,l=1}^3$ instead of a weight :

$$E_M\left((a_i, d_i)_{i=1}^N\right) = \operatorname*{Inf}_{u\in\mathcal{E}} \int_\Omega \sum_{k,l=1}^3 m_{kl}(x)\frac{\partial u}{\partial x_k}\cdot\frac{\partial u}{\partial x_l}\,dx.$$

Throughout this chapter, a sequence of smooth mollifiers means any sequence $(\rho_n)_{n\in\mathbb{N}}$ satisfying

$$\rho_n \in C^\infty(\mathbb{R}^3,\mathbb{R}), \quad \operatorname{Supp}\rho_n \subset B_{1/n}(0), \quad \int_{\mathbb{R}^3}\rho_n = 1, \quad \rho_n \geq 0 \text{ on } \mathbb{R}^3.$$

1.2 Preliminary results : Metric properties of d_w

1.2.1 Metric and geodesic character of d_w

First of all we recall that for any metric space (M, d), we may associate the length functional $\mathbb{L}_d$ defined by

$$\mathbb{L}_d(\gamma) = \operatorname{Sup}\left\{\sum_{k=1}^{m-1} d\left(\gamma(t_k),\gamma(t_{k+1})\right),\ 0 = t_0 < t_1 < \ldots < t_m = 1,\ m \in \mathbb{N}\right\}$$

where $\gamma : [0, 1] \to M$ is any continuous curve. Note that $\mathbb{L}_d$ is lower semicontinuous on $C^0([0, 1], M)$ endowed with the topology of the uniform convergence on $[0, 1]$.

Definition 1.1. A distance d is said to be *geodesic* on M if for any $x, y \in M$,

$$d(x, y) = \operatorname{Inf}\ \mathbb{L}_d(\gamma)$$

where the infimum is taken over all continuous curves $\gamma : [0, 1] \to M$ such that $\gamma(0) = x$ and $\gamma(1) = y$.

Proposition 1.1. *d_w defines a geodesic distance on $\overline{\Omega}$ which is equivalent to the Euclidean geodesic distance d_Ω and d_w agrees with δ_w whenever w is continuous.*

Proof. Step 1. Let $x, y \in \Omega$ and let $\mathcal{F} = ([\alpha_1, \beta_1], \ldots, [\alpha_n, \beta_n])$ be an element of $\mathcal{P}(x, y)$. From assumption (1.1), we get that

$$\ell_w(\mathcal{F}) \geq \sum_{k=1}^n \lim_{\varepsilon\to 0^+} \frac{\lambda}{\pi\varepsilon^2}\int_{\Xi([\alpha_k,\beta_k],\varepsilon)\cap\Omega} d\xi = \lambda\sum_{k=1}^n |\alpha_k - \beta_k| \geq \lambda\, d_\Omega(x, y). \tag{1.7}$$

By the definition of d_w and (1.1), for any $\mathcal{F} = ([\alpha_1, \beta_1], \ldots, [\alpha_n, \beta_n])$ in $\mathcal{P}(x, y)$, we have

$$d_w(x, y) \leq \Lambda\sum_{k=1}^n \lim_{\varepsilon\to 0^+} \frac{1}{\pi\varepsilon^2}\int_{\Xi([\alpha_k,\beta_k],\varepsilon)\cap\Omega} d\xi = \Lambda\sum_{k=1}^n |\alpha_k - \beta_k|.$$

Taking the infimum over all $\mathcal{F} \in \mathcal{P}(x,y)$, we infer that

$$d_w(x,y) \leq \Lambda\, d_\Omega(x,y). \tag{1.8}$$

From (1.7) and (1.8), we deduce that $d_w(x,y) = 0$ if and only if $x = y$. Now let us now prove that d_w is symmetric. Let $x, y \in \Omega$ and $\delta > 0$ arbitrary small. We can find $\mathcal{F}_\delta = ([\alpha_1, \beta_2], \ldots, [\alpha_n, \beta_n])$ in $\mathcal{P}(x,y)$ satisfying

$$\ell_w(\mathcal{F}_\delta) \leq d_w(x,y) + \delta.$$

Then for $\mathcal{F}'_\delta = ([\beta_n, \alpha_n], \ldots, [\beta_1, \alpha_1]) \in \mathcal{P}(y,x)$, we have

$$d_w(y,x) \leq \ell_w(\mathcal{F}'_\delta) = \ell_w(\mathcal{F}_\delta) \leq d_w(x,y) + \delta.$$

Since δ is arbitrary, we obtain $d_w(y,x) \leq d_w(x,y)$ and we conclude that $d_w(y,x) = d_w(x,y)$ inverting the roles of x and y. The triangle inequality is immediate since the juxtaposition of $\mathcal{F}_1 \in \mathcal{P}(x,z)$ with $\mathcal{F}_2 \in \mathcal{P}(z,y)$ is an element of $\mathcal{P}(x,y)$. Hence d_w defines a distance on Ω verifying

$$\lambda d_\Omega(x,y) \leq d_w(x,y) \leq \Lambda d_\Omega(x,y) \quad \text{for any } x, y \in \Omega. \tag{1.9}$$

Therefore distance d_w extends uniquely to $\overline{\Omega} \times \overline{\Omega}$ into a distance function that we still denote by d_w. By continuity, d_w satisfies (1.9) on $\overline{\Omega}$.

If w is continuous, it is easy to see that for a segment $[\alpha, \beta] \subset \Omega$ we have

$$\lim_{\varepsilon \to 0^+} \frac{1}{\pi \varepsilon^2} \int_{\Xi([\alpha,\beta],\varepsilon) \cap \Omega} w(\xi) d\xi = \int_{[\alpha,\beta]} w(s) ds,$$

and we obtain for $\mathcal{F} = ([\alpha_1, \beta_1], \ldots, [\alpha_n, \beta_n]) \in \mathcal{P}(x,y)$ and $x, y \in \Omega$,

$$\ell_w(\mathcal{F}) = \int_{\cup_{k=1}^n [\alpha_k, \beta_k]} w(s) ds. \tag{1.10}$$

Since w is continuous, the infimum in (1.3) can be taken over all piecewise affine curves $\gamma : [0,1] \to \Omega$ such that $\gamma(0) = x$ and $\gamma(1) = y$ and we infer from (1.10) that

$$d_w(x,y) = \delta_w(x,y).$$

Then $d_w \equiv \delta_w$ on $\Omega \times \Omega$ which implies that the equality holds on $\overline{\Omega} \times \overline{\Omega}$ by continuity.

Step 2. We prove the geodesic character of d_w on $\overline{\Omega}$. Since d_w is equivalent to d_Ω, $\overline{\Omega}$ endowed with d_w remains complete. By Theorem 1.8 in [55], it suffices to prove that for any $x, y \in \overline{\Omega}$ and any $\delta > 0$, we can find a point $z \in \overline{\Omega}$ verifying

$$\max(d_w(x,z), d_w(z,y)) \leq \frac{1}{2}\, d_w(x,y) + \delta.$$

We fix $x, y \in \overline{\Omega}$ and then $\tilde{x}, \tilde{y} \in \Omega$ such that $d_w(x,\tilde{x}) + d_w(y,\tilde{y}) \leq \delta/2$. We choose some $\mathcal{F} = ([\alpha_1,\beta_1],\dots,[\alpha_n,\beta_n])$ in $\mathcal{P}(\tilde{x},\tilde{y})$ satisfying $\ell_w(\mathcal{F}) \leq d_w(\tilde{x},\tilde{y}) + \delta/2$. For every $1 \leq m \leq n$, we set $\mathcal{F}_m = ([\alpha_1,\beta_1],\dots,[\alpha_m,\beta_m])$. We consider $n_\star \in \mathbb{N}$ defined by

$$n_\star = \begin{cases} \mathrm{Max}\left\{ m,\, 2 \leq m \leq n,\, \ell_w\left(\mathcal{F}_{m-1}\right) < \frac{1}{2}\,\ell_w(\mathcal{F}) \right\} & \text{if } \ell_w\left(\mathcal{F}_1\right) < \frac{1}{2}\,\ell_w(\mathcal{F}), \\ 1 & \text{otherwise,} \end{cases}$$

and $s \in (0,1)$ defined by

$$s = \begin{cases} \dfrac{\ell_w(\mathcal{F}) - 2\,\ell_w(\mathcal{F}_{n_\star-1})}{2\,\ell_w([\alpha_{n_\star},\beta_{n_\star}])} & \text{if } n_\star > 1, \\ \dfrac{\ell_w(\mathcal{F})}{2\ell_w([\alpha_{n_\star},\beta_{n_\star}])} & \text{if } n_\star = 1. \end{cases}$$

Let $\varepsilon_k \to 0^+$ as $k \to +\infty$ such that

$$\ell_w([\alpha_{n_\star},\beta_{n_\star}]) = \lim_{k\to+\infty} \frac{1}{\pi\varepsilon_k^2} \int_{\Xi([\alpha_{n_\star},\beta_{n_\star}],\varepsilon_k)\cap\Omega} w(\xi)d\xi.$$

For each $k \in \mathbb{N}$, we choose $z_k \in [\alpha_{n_\star},\beta_{n_\star}]$ verifying

$$\frac{1}{\pi\varepsilon_k^2} \int_{\Xi([\alpha_{n_\star},z_k],\varepsilon_k)\cap\Omega} w(\xi)d\xi = \frac{s}{\pi\varepsilon_k^2} \int_{\Xi([\alpha_{n_\star},\beta_{n_\star}],\varepsilon_k)\cap\Omega} w(\xi)d\xi + \mathcal{O}(\varepsilon_k),$$

and

$$\frac{1}{\pi\varepsilon_k^2} \int_{\Xi([z_k,\beta_{n_\star}],\varepsilon_k)\cap\Omega} w(\xi)d\xi = \frac{1-s}{2\pi\varepsilon_k^2} \int_{\Xi([\alpha_{n_\star},\beta_{n_\star}],\varepsilon_k)\cap\Omega} w(\xi)d\xi + \mathcal{O}(\varepsilon_k).$$

Extracting a subsequence if necessary, we may assume that $z_k \underset{k\to+\infty}{\longrightarrow} z$ with $z \in [\alpha_{n_\star},\beta_{n_\star}]$. Then we have

$$\frac{1}{\pi\varepsilon_k^2} \int_{\Xi([\alpha_{n_\star},z],\varepsilon_k)\cap\Omega} w(\xi)d\xi = \frac{s}{\pi\varepsilon_k^2} \int_{\Xi([\alpha_{n_\star},\beta_{n_\star}],\varepsilon_k)\cap\Omega} w(\xi)d\xi + \mathcal{O}(\varepsilon_k) + \mathcal{O}(|z-z_k|),$$

and

$$\frac{1}{\pi\varepsilon_k^2} \int_{\Xi([z,\beta_{n_\star}],\varepsilon_k)\cap\Omega} w(\xi)d\xi = \frac{1-s}{2\pi\varepsilon_k^2} \int_{\Xi([\alpha_{n_\star},\beta_{n_\star}],\varepsilon_k)\cap\Omega} w(\xi)d\xi + \mathcal{O}(\varepsilon_k) + \mathcal{O}(|z-z_k|).$$

Taking the $\liminf$ in k, we derive

$$\ell_w([\alpha_{n_\star},z]) \leq s\ell_w([\alpha_{n_\star},\beta_{n_\star}]) \quad \text{and} \quad \ell_w([z,\beta_{n_\star}]) \leq (1-s)\ell_w([\alpha_{n_\star},\beta_{n_\star}]).$$

Hence we deduce that the elements $\mathcal{F}_{\tilde{x}} = ([\alpha_1,\beta_1],\dots,[\alpha_{n_\star},z]) \in \mathcal{P}(\tilde{x},z)$ and $\mathcal{F}_{\tilde{y}} = ([z,\beta_{n_\star}],\dots,[\alpha_n,\beta_n]) \in \mathcal{P}(z,\tilde{y})$ verify

$$d_w(\tilde{x},z) \leq \ell_w(\mathcal{F}_{\tilde{x}}) \leq \frac{1}{2}\,\ell_w(\mathcal{F}) \leq \frac{1}{2}\,d_w(\tilde{x},\tilde{y}) + \delta/4,$$
$$d_w(\tilde{y},z) \leq \ell_w(\mathcal{F}_{\tilde{y}}) \leq \frac{1}{2}\,\ell_w(\mathcal{F}) \leq \frac{1}{2}\,d_w(\tilde{x},\tilde{y}) + \delta/4,$$

and we conclude that

$$\max(d_w(x,z), d_w(y,z)) \le \max(d_w(\tilde{x},z), d_w(\tilde{y},z)) + \frac{\delta}{2} \le \frac{1}{2}\, d_w(\tilde{x},\tilde{y}) + \frac{3\delta}{4}$$
$$\le \frac{1}{2}\, d_w(x,y) + \delta,$$

i.e., the point z meets the requirement. ■

Remark 1.1. The geodesic character of d_w implies that two arbitrary points of $\left(\overline{\Omega}, d_w\right)$ can be linked by a minimizing geodesic. We mean by a minimizing geodesic any curve $\gamma : I \to \overline{\Omega}$ such that

$$d_w(\gamma(t), \gamma(t')) = |t - t'| \quad \text{for any } t, t' \in I,$$

where I is some interval of $\mathbb{R}$. In particular we obtain the existence for any $x, y \in \overline{\Omega}$ of a curve $\gamma_{xy} \in \mathrm{Lip}_{x,y}\left([0,1], \overline{\Omega}\right)$ satisfying

$$d_w\left(\gamma_{xy}(t), \gamma_{xy}(t')\right) = \mathbb{L}_{d_w}(\gamma_{xy})|t - t'| \quad \text{for any } t, t' \in [0,1]$$

(and then $d_w(x,y) = \mathbb{L}_{d_w}(\gamma_{xy})$). Indeed, $\left(\overline{\Omega}, d_w\right)$ defines a complete and locally compact metric space and since d_w is of geodesic type, the existence of a minimizing geodesic is ensured by the Hopf-Rinow Theorem (see [55], Chapter 1). Moreover we deduce from (1.9) that any minimizing geodesic for the distance d_w is a λ^{-1}-Lipschitz curve for the Euclidean geodesic distance.

1.2.2 Integral representation of the length functional

In this section, we show that d_w is actually induced by a Finsler metric in the sense defined below.

Definition 1.2. A Borel measurable function $\varphi : \overline{\Omega} \times \mathbb{R}^3 \to [0, +\infty)$ is said to be a *Finsler metric* if $\varphi(x, \cdot)$ is positively 1-homogeneous for every $x \in \overline{\Omega}$ and convex for almost every $x \in \overline{\Omega}$.

Proposition 1.2. *There exists a Finsler metric $\varphi_w : \overline{\Omega} \times \mathbb{R}^3 \to [0, +\infty)$ such that for any Lipschitz curve $\gamma : [0,1] \to \overline{\Omega}$,*

$$\mathbb{L}_{d_w}(\gamma) = \int_0^1 \varphi_w\left(\gamma(t), \dot{\gamma}(t)\right) dt. \tag{1.11}$$

Moreover, for any $x, y \in \overline{\Omega}$ we have

$$d_w(x,y) = \mathrm{Min}\left\{\int_0^1 \varphi_w\left(\gamma(t), \dot{\gamma}(t)\right) dt,\ \gamma \in \mathrm{Lip}_{x,y}\left([0,1], \overline{\Omega}\right)\right\}. \tag{1.12}$$

Proof. Step 1. First, we assume that $\Omega = \mathbb{R}^3$. To distance d_w we associate the function $\varphi_w : \mathbb{R}^3 \times \mathbb{R}^3 \to [0, +\infty)$ defined by

$$\varphi_w(x, \nu) = \limsup_{t \to 0^+} \frac{d_w(x, x + t\nu)}{t}.$$

In [83], it is proved that φ_w defines a Finsler metric and the proof of (1.11) is given in [42], Theorem 2.5. Then (1.12) directly follows from Remark 1.1.

Step 2. Assume that Ω is a smooth bounded and connected open set of $\mathbb{R}^3$. For $\delta > 0$, we consider $\Omega_\delta = \{x \in \mathbb{R}^3, \operatorname{dist}(x, \Omega) < \delta\}$ where "dist" denotes the usual Euclidean distance on $\mathbb{R}^3$. We choose δ sufficiently small for the projection Πx of $x \in \Omega_\delta$ on $\overline{\Omega}$ to be well defined and smooth. Setting $x_\perp = x - \Pi x$ for $x \in \Omega_\delta$, we define the function $d_{w,\delta} : \Omega_\delta \times \Omega_\delta \to [0, +\infty)$ by

$$d_{w,\delta}(x, y) = d_w(\Pi x, \Pi y) + |x_\perp - y_\perp|.$$

We easily check that $d_{w,\delta}$ defines a distance on Ω_δ. Then we consider for $x, y \in \Omega_\delta$,

$$\overline{d}_{w,\delta}(x, y) = \operatorname{Inf} \mathbb{L}_{d_{w,\delta}}(\gamma),$$

where the infimum is taken over all $\gamma \in C^0([0,1], \Omega_\delta)$ satisfying $\gamma(0) = x$ and $\gamma(1) = y$. We also easily verify that $\overline{d}_{w,\delta}$ defines a distance on Ω_δ and it follows from Proposition 1.6 in [55] that

$$\mathbb{L}_{\overline{d}_{w,\delta}} = \mathbb{L}_{d_{w,\delta}} \quad \text{on } C^0([0,1], \Omega_\delta). \tag{1.13}$$

Therefore $\overline{d}_{w,\delta}(x, y)$ is a geodesic distance on Ω_δ. Moreover we infer from (1.9) that $\overline{d}_{w,\delta}$ is equivalent to the Euclidean geodesic distance on Ω_δ. We consider $\varphi_{w,\delta} : \Omega_\delta \times \mathbb{R}^3 \to [0, +\infty)$ defined by

$$\varphi_{w,\delta}(x, \nu) = \limsup_{t \to 0^+} \frac{\overline{d}_{w,\delta}(x, x + t\nu)}{t}.$$

By the results in [83], $\varphi_{w,\delta}$ is Borel measurable, positively 1-homogeneous in ν for every $x \in \Omega_\delta$ and convex in ν for almost every $x \in \Omega_\delta$. By Theorem 2.5 in [42], we have for any Lipschitz curve $\gamma : [0,1] \to \Omega_\delta$,

$$\mathbb{L}_{\overline{d}_{w,\delta}}(\gamma) = \int_0^1 \varphi_{w,\delta}(\gamma(t), \dot{\gamma}(t))\, dt. \tag{1.14}$$

Since $d_{w,\delta} = d_w$ on $\overline{\Omega}$, we deduce that

$$\mathbb{L}_{d_{w,\delta}} = \mathbb{L}_{d_w} \quad \text{on } C^0\big([0,1], \overline{\Omega}\big). \tag{1.15}$$

If we denote by φ_w the restriction of $\varphi_{w,\delta}$ to $\overline{\Omega} \times \mathbb{R}^3$, we obtain (1.11) combining (1.13), (1.14) and (1.15). Then (1.12) follows from Remark 1.1. ■

Remark 1.2. If we assume that w is continuous in Ω, we have

$$\varphi_w(x,\nu) = w(x)|\nu| \quad \text{for any } (x,\nu) \in \Omega \times \mathbb{R}^3.$$

Indeed, fix $(x,\nu) \in \Omega \times \mathbb{R}^3 \setminus \{0\}$, $t > 0$ such that $B(x, 2t\lambda^{-1}|\nu|) \subset \Omega$ and consider a sequence $\gamma_n \in \mathrm{Lip}([0,1], \overline{\Omega})$ verifying

$$\int_0^1 w(\gamma_n(s))\, |\dot{\gamma}_n(s)| ds \to d_w(x, x+t\nu) \quad \text{as } n \to +\infty.$$

Since $d_w \geq \lambda d_\Omega$, we infer that $\gamma_n([0,1]) \subset B(x, 2t\lambda^{-1}|\nu|)$ and therefore

$$\int_0^1 w(\gamma_n(s))\, |\dot{\gamma}_n(s)| ds \geq w(x) \int_0^1 |\dot{\gamma}_n(s)| ds - o(t) \geq w(x)t|\nu| - o(t).$$

Letting $n \to +\infty$, we obtain

$$\frac{d_w(x, x+t\nu)}{t} \geq w(x)|\nu| - o(1).$$

But we trivially have

$$\frac{d_w(x, x+t\nu)}{t} \leq \frac{1}{t} \int_0^t w(x+s\nu)|\nu| ds = w(x)|\nu| + o(1).$$

We derive the result from these two last inequalities letting $t \to 0$.

1.2.3 Characterization of 1-Lipschitz functions

Proposition 1.3. *Assume that (1.1) holds. Then for any $\zeta : \overline{\Omega} \to \mathbb{R}$, the following properties are equivalent :*

i) $|\zeta(x) - \zeta(y)| \leq d_w(x,y)$ *for any $x, y \in \overline{\Omega}$.*

ii) ζ is Lipschitz continuous and $|\nabla\zeta(x)| \leq w(x)$ for a.e. $x \in \Omega$.

Proof. i) $\Rightarrow$ ii). Let $\zeta : \overline{\Omega} \to \mathbb{R}$ satisfying *i)*. From Proposition 1.1, we infer that ζ is Lipschitz continuous. Fix $x_0 \in \Omega$ and $R > 0$ such that $B_{3R}(x_0) \subset \Omega$. Let $(\rho_n)_{n\in\mathbb{N}}$ be a sequence of smooth mollifiers and consider, for $n > 1/R$, the smooth function $\zeta_n = \rho_n * \zeta : B_R(x_0) \to \mathbb{R}$. We write

$$\zeta_n(x) = \int_{B_{1/n}} \rho_n(-z)\zeta(x+z)dz$$

and then for any $x, y \in B_R(x_0)$,

$$\begin{aligned}
|\zeta_n(x) - \zeta_n(y)| &\leq \int_{B_{1/n}} \rho_n(-z)\, |\zeta(x+z) - \zeta(y+z)|\, dz \\
&\leq \int_{B_{1/n}} \rho_n(-z)\, d_w(x+z, y+z) dz \\
&\leq \int_{B_{1/n}} \rho_n(-z)\, \ell_w([x+z, y+z])\, dz.
\end{aligned}$$

Taking an arbitrary sequence $\varepsilon_k \to 0^+$ as $k \to +\infty$ and using Fatou's lemma, we get that

$$\begin{aligned}|\zeta_n(x) - \zeta_n(y)| &\leq \int_{B_{1/n}} \rho_n(-z) \left(\liminf_{k\to+\infty} \frac{1}{\pi\varepsilon_k^2} \int_{\Xi([x+z,y+z],\varepsilon_k)\cap\Omega} w(\xi)d\xi \right) dz \\ &\leq \liminf_{k\to+\infty} \frac{1}{\pi\varepsilon_k^2} \int_{B_{1/n}} \int_{\Xi([x+z,y+z],\varepsilon_k)\cap\Omega} \rho_n(-z)w(\xi)d\xi dz.\end{aligned}$$

For $k \in \mathbb{N}$ sufficiently large, we have $\Xi([x+z, y+z], \varepsilon_k) \subset B_{3R}(x_0)$ and accordingly

$$\begin{aligned}\int_{B_{1/n}} \int_{\Xi([x+z,y+z],\varepsilon_k)} \rho_n(-z)w(\xi)d\xi dz &= \int_{\Xi([x,y],\varepsilon_k)} \int_{B_{1/n}} \rho_n(-z)w(\xi+z)dzd\xi \\ &= \int_{\Xi([x,y],\varepsilon_k)} \rho_n * w(\xi)d\xi.\end{aligned}$$

Since $\rho_n * w$ is smooth, we obtain as in the proof of Proposition 1.1,

$$\frac{1}{\pi\varepsilon_k^2} \int_{\Xi([x,y],\varepsilon_k)} \rho_n * w(\xi)d\xi \to \int_{[x,y]} \rho_n * w(s)ds \quad \text{as } k \to +\infty.$$

Thus for each $x, y \in B_R(x_0)$ we have

$$|\zeta_n(x) - \zeta_n(y)| \leq \int_{[x,y]} \rho_n * w(s)ds.$$

Then for $x \in B_R(x_0)$, $h \in S^2$ fixed and $\delta > 0$ small, we derive

$$\frac{|\zeta_n(x+\delta h) - \zeta_n(x)|}{\delta} \leq \frac{1}{\delta} \int_{[x,x+\delta h]} \rho_n * w(s)ds \underset{\delta\to 0^+}{\to} \rho_n * w(x)$$

and we conclude, letting $\delta \to 0$, that $|\nabla\zeta_n(x) \cdot h| \leq \rho_n * w(x)$ for each $x \in B_R(x_0)$ and $h \in S^2$ which implies that $|\nabla\zeta_n| \leq \rho_n * w$ on $B_R(x_0)$. Since $\nabla\zeta_n \to \nabla\zeta$ and $\rho_n * w \to w$ a.e. on $B_R(x_0)$ as $n \to +\infty$, we deduce that $|\nabla\zeta| \leq w$ a.e. on $B_R(x_0)$. Since x_0 is arbitrary in Ω, we get the result.

ii) $\Rightarrow$ *i)* The reverse implication follows from the lemma below.

Lemma 1.1. *Let $\zeta : \overline{\Omega} \to \mathbb{R}$ be a Lipschitz continuous function. For any $a, b \in \Omega$ with $[a,b] \subset \Omega$ and all $\varepsilon > 0$ sufficiently small, we have*

$$|\zeta(a) - \zeta(b)| \leq \frac{1}{\pi\varepsilon^2} \int_{\Xi([a,b],\varepsilon)\cap\Omega} |\nabla\zeta(z)|dz + 2\varepsilon \|\nabla\zeta\|_\infty .$$

Proof of ii) $\Rightarrow$ *i) completed.* Indeed, let ζ be a Lipschitz continuous function satisfying *ii).* We deduce from Lemma 1.1 and (1.1) that for any $\mathcal{F} = ([\alpha_1, \beta_1], \ldots, [\alpha_n, \beta_n]) \in \mathcal{P}(x,y)$ and any parameters $\varepsilon_1, \ldots, \varepsilon_n > 0$ sufficiently small, we have

$$|\zeta(x)-\zeta(y)| \leq \sum_{k=1}^n |\zeta(\beta_k)-\zeta(\alpha_k)| \leq \sum_{k=1}^n \left(\frac{1}{\pi\varepsilon_k^2} \int_{\Xi([\alpha_k,\beta_k],\varepsilon_k)\cap\Omega} w(z)dz + 2\Lambda\varepsilon_k \right).$$

Taking successively the lim inf in $\varepsilon_k \to 0^+$ for each parameter ε_k, we get that

$$|\zeta(x) - \zeta(y)| \leq \ell_w(\mathcal{F}).$$

We obtain the result for $x, y \in \Omega$ taking the infimum over all $\mathcal{F} \in \mathcal{P}(x, y)$. We conclude that *i)* holds in all $\overline{\Omega}$ by continuity. ■

Proof of Lemma 1.1. First note that we just have to prove the inequality for smooth functions ζ, the general case follows by a density argument. Let ζ be a smooth real valued function. Without loss of generality, we may assume that $a = (0, 0, 0)$ and $b = (0, 0, R)$. Then for any $\varepsilon > 0$ such that the 3D-cylinder $B_\varepsilon^{(2)}(0) \times [0, R]$ is included in Ω, and any $(x_1, x_2) \in B_\varepsilon^{(2)}(0)$, we have

$$\begin{aligned}|\zeta(b) - \zeta(a)| \leq & |\zeta(0, 0, R) - \zeta(x_1, x_2, R)| + |\zeta(x_1, x_2, R) - \zeta(x_1, x_2, 0)| \\ & + |\zeta(x_1, x_2, 0) - \zeta(0, 0, 0)| \\ \leq & \int_0^R |\nabla\zeta(x_1, x_2, x_3)| \, dx_3 + 2\varepsilon \|\nabla\zeta\|_\infty .\end{aligned}$$

Integrating the last inequality in $(x_1, x_2) \in B_\varepsilon^{(2)}(0)$ yields

$$\pi\varepsilon^2 |\zeta(b) - \zeta(a)| \leq \int_{B_\varepsilon^{(2)}(0)\times[0,R]} |\nabla\zeta(x_1, x_2, x_3)| \, dx_1 dx_2 dx_3 + 2\pi\varepsilon^3 \|\nabla\zeta\|_\infty .$$

Dividing by $\pi\varepsilon^2$, we get the result since $B_\varepsilon^{(2)}(0) \times [0, R] \subset \Xi([a, b], \varepsilon) \cap \Omega$. ■

Remark 1.3. In [38], F. Camilli and A. Siconolfi study the Hamilton-Jacobi equation

$$H(x, \nabla u) = 0 \quad \text{a.e. in } \Omega$$

where the Hamiltonian $H(x, \nu)$ is measurable in x, continuous and quasiconvexe in ν. They construct the *optical length function* $L^\Omega : \overline{\Omega} \times \overline{\Omega}$ giving a class of "fundamental solutions". They show that for every $y_0 \in \overline{\Omega}$, $L^\Omega(y_0, \cdot)$ is the maximal element of the set

$$\mathcal{C}(y_0) = \left\{ v \in W^{1,\infty}(\Omega, \mathbb{R}),\ H(x, \nabla v) \leq 0 \text{ a.e in } \Omega,\ v(y_0) = 0 \right\}.$$

In the case $H(x, \nu) = |\nu| - w(x)$, Proposition 1.3 shows that d_w and the optical length function L^Ω coincide i.e., $d_w(x, y) = L^\Omega(x, y)$ for any $x, y \in \overline{\Omega}$.

1.3 Energy estimates - Proof of Theorem 1.1

Theorem 1.1 follows from the combination of Lemma 1.2 and Lemma 1.5 below. In Section 1.3.2, we give an explicit *dipole construction*.

1.3.1 Lower bound of the energy

Lemma 1.2. *For any $u \in \mathcal{E}$, we have*

$$\int_\Omega |\nabla u|^2 w(x)dx \geq 8\pi L_w.$$

Proof. The proof is essentially the same as in [30] once we have the results of Section 1.2. We introduce for each $u \in \mathcal{E}$ the vector field D defined by

$$D = \left(u \cdot \frac{\partial u}{\partial x_2} \wedge \frac{\partial u}{\partial x_3} \, , \; u \cdot \frac{\partial u}{\partial x_3} \wedge \frac{\partial u}{\partial x_1} \, , \; u \cdot \frac{\partial u}{\partial x_1} \wedge \frac{\partial u}{\partial x_2} \right). \tag{1.16}$$

As in [30], we have $2|D| \leq |\nabla u|^2$ and $D \in L^1(\Omega)$ defines a distribution which satisfies

$$\operatorname{div} D = 4\pi \sum_{i=1}^{N} d_i \delta_{a_i} \quad \text{in } \mathcal{D}'(\Omega). \tag{1.17}$$

Relabelling the points (a_i) as positive and negative points taking into account their multiplicity $|d_i|$, we get a list (p_j) of positive points and a list (n_j) of negative points. Since $\sum d_i = 0$, we have as many positive points as negative points. Then we write (1.17) as

$$\operatorname{div} D = 4\pi \sum_{j=1}^{K} \delta_{p_j} - \delta_{n_j}. \tag{1.18}$$

From Proposition 1.3 and the properties of D, we deduce that for any function $\zeta : \overline{\Omega} \to \mathbb{R}$ which is 1-Lipschitz with respect to d_w,

$$\int_\Omega |\nabla u|^2 w(x)dx \geq 2 \int_\Omega |D| w(x)dx \geq -2 \int_\Omega D \cdot \nabla \zeta. \tag{1.19}$$

Using (1.18), we get that

$$\int_\Omega |\nabla u|^2 w(x)dx \geq 8\pi \left(\sum_{j=1}^{K} \zeta(p_j) - \zeta(n_j) \right) - 8\pi \int_{\partial\Omega} (D \cdot \eta)\zeta \, d\sigma$$

without the boundary term if $\Omega = \mathbb{R}^3$. On $\partial\Omega$, we have $D \cdot \eta = \operatorname{Jac}_2(u_{/\partial\Omega})$ where η denotes the outward normal and $\operatorname{Jac}_2(u_{/\partial\Omega})$ denotes the 2×2 Jacobian determinant of u restricted to $\partial\Omega$. Since each $u \in \mathcal{E}$ is constant on $\partial\Omega$, we have $D \cdot \eta \equiv 0$ on $\partial\Omega$ and therefore we derive

$$\int_\Omega |\nabla u|^2 w(x)dx \geq 8\pi \operatorname{Max} \sum_{j=1}^{K} \zeta(p_j) - \zeta(n_j)$$

where the maximum is taken over all functions ζ which 1-Lipschitz with respect to d_w. By (1.6) we conclude that

$$\int_\Omega |\nabla u|^2 w(x)dx \geq 8\pi L_w$$

for any map $u \in \mathcal{E}$ which completes the proof of the lower bound. ∎

1.3.2 The dipole construction

Lemma 1.3. *Let P, N be two distinct points in Ω. For any $\delta > 0$, there exists a map $u_\delta \in C^1\left(\overline{\Omega} \setminus \{P, N\}, S^2\right)$ such that* $\deg(u_\delta, P) = +1$, $\deg(u_\delta, N) = -1$ *and*

$$\int_\Omega |\nabla u_\delta|^2 w(x) dx \leq 8\pi d_w(P, N) + \delta.$$

Moreover u_δ is constant outside a small neighborhood of a polygonal curve running between P and N.

Proof. For $\varepsilon > 0$, we consider the map $\omega_\varepsilon : \mathbb{R}^2 \to S^2$ defined by

$$\omega_\varepsilon(x, y) = \begin{cases} \dfrac{2\varepsilon^2}{\varepsilon^4 + r^2}\left(x, y, -\varepsilon^2\right) & \text{if } r \leq \varepsilon \\ \left(A(r)\cos\theta, A(r)\sin\theta, C(r)\right) & \text{if } \varepsilon \leq r \leq 2\varepsilon \\ (0, 0, 1) & \text{if } 2\varepsilon \leq r \end{cases} \tag{1.20}$$

where $(x, y) = (r\cos\theta, r\sin\theta)$ and

$$A(r) = \frac{-2\varepsilon^2}{\varepsilon^4 + \varepsilon^2} r + \frac{4\varepsilon^3}{\varepsilon^4 + \varepsilon^2}\,, \quad C(r) = \sqrt{1 - (A(r))^2}\,.$$

According to the results in [29], ω_ε is Lipschitz continuous and $\deg \omega_\varepsilon = +1$ when one identifies $\mathbb{R}^2 \cup \{\infty\}$ with S^2. As in [30], the map ω_ε will be the main ingredient in our construction. First we define the following objects. For two distinct points $\alpha, \beta \in \Omega$ with $[\alpha, \beta] \subset \Omega$, we denote by $p_{\alpha,\beta}(x)$ the projection of $x \in \mathbb{R}^3$ on the straight line passing by α and β and

$$r_{\alpha,\beta}(x) = \operatorname{dist}\left(x, [\alpha, \beta]\right), \quad h_{\alpha,\beta}(x) = \operatorname{dist}\left(p_{\alpha,\beta}(x), \{\alpha, \beta\}\right),$$

where "dist" denotes the Euclidean distance in $\mathbb{R}^3$. For some small $\sigma > 0$, we consider the following sets :

$$C^\sigma_\varepsilon(\alpha, \beta) = \left\{x \in \mathbb{R}^3,\, p_{\alpha,\beta}(x) \in]\alpha, \beta[,\, \sigma r_{\alpha,\beta}(x) \leq h_{\alpha,\beta}(x),\, 0 \leq h_{\alpha,\beta}(x) \leq \sigma\varepsilon\right\},$$

$$T^\sigma_\varepsilon(\alpha, \beta) = \left\{x \in \mathbb{R}^3,\, p_{\alpha,\beta}(x) \in [\alpha, \beta],\, r_{\alpha,\beta}(x) \leq \varepsilon,\, h_{\alpha,\beta}(x) \geq \sigma\varepsilon\right\},$$

$$V_\varepsilon(\alpha, \beta) = \left\{x \in \mathbb{R}^3,\, p_{\alpha,\beta}(x) \in [\alpha, \beta],\, r_{\alpha,\beta}(x) \leq \varepsilon\right\}.$$

We choose ε small enough such that $C^\sigma_{2\varepsilon}(\alpha, \beta) \cup T^\sigma_{2\varepsilon}(\alpha, \beta) \cup V_{2\varepsilon}(\alpha, \beta) \subset \Omega$. We fix $\delta > 0$ and we consider $\mathcal{F} = ([\alpha_1, \beta_1], \ldots, [\alpha_n, \beta_n]) \in \mathcal{P}(P, N)$ such that the curve $\gamma = \cup_k [\alpha_k, \beta_k]$ has no self-intersection points. Then for each $k \in \{1, \ldots, n\}$, we fix two unit vectors i_k and j_k

in the orthogonal plane to $\beta_k - \alpha_k$ such that $(i_k, j_k, \frac{\beta_k-\alpha_k}{|\beta_k-\alpha_k|})$ defines a direct orthonormal basis of $\mathbb{R}^3$ and we consider $u_\varepsilon^{(k)} : \Omega \to S^2$ defined by

$$u_\varepsilon^{(k)}(x) = \begin{cases} \omega_\varepsilon\left(X_k(x), Y_k(x)\right) & \text{if } x \in C_{2\varepsilon}^\sigma(\alpha_k, \beta_k), \\ \omega_\varepsilon\left((x - p_{\alpha_k,\beta_k}(x)) \cdot i_k\,, (x - p_{\alpha_k,\beta_k}(x)) \cdot j_k\right) & \text{if } x \in T_{2\varepsilon}^\sigma(\alpha_k, \beta_k), \\ (0,0,1) & \text{otherwise} \end{cases}$$

with

$$X_k(x) = \frac{2\sigma\varepsilon}{h_{\alpha_k,\beta_k}(x)}(x - p_{\alpha_k,\beta_k}(x)) \cdot i_k\,, \quad Y_k(x) = \frac{2\sigma\varepsilon}{h_{\alpha_k,\beta_k}(x)}(x - p_{\alpha_k,\beta_k}(x)) \cdot j_k\,.$$

We check that $u_\varepsilon^{(k)} \in W^{1,\infty}_{\text{loc}}\left(\overline{\Omega} \setminus \{\alpha_k, \beta_k\}, S^2\right)$, $\deg(u_\varepsilon^{(k)}, \alpha_k) = +1$, $\deg(u_\varepsilon^{(k)}, \beta_k) = -1$. Using coordinates in the basis $(i_k, j_k, \frac{\beta_k-\alpha_k}{|\beta_k-\alpha_k|})$, some classical computations (see [27]) lead to

$$|\nabla u_\varepsilon^{(k)}(x)|^2 \leq (1+C\varepsilon^2)\frac{4\sigma^2\varepsilon^2}{h^2_{\alpha_k,\beta_k}(x)}\left|\nabla\omega_\varepsilon\left(X_k(x), Y_k(x)\right)\right|^2 \quad \text{in } C_{2\varepsilon}^\sigma(\alpha_k, \beta_k). \tag{1.21}$$

By the results in [29], we have

$$\int_{B_{2\varepsilon}(0)\setminus B_\varepsilon(0)} |\nabla\omega_\varepsilon|^2 = \mathcal{O}(\varepsilon)\,,\ \int_{B_\varepsilon(0)} |\nabla\omega_\varepsilon|^2 = 8\pi + \mathcal{O}(\varepsilon) \tag{1.22}$$

and therefore

$$\int_{(T_{2\varepsilon}^\sigma \setminus T_\varepsilon^\sigma)(\alpha_k,\beta_k)} |\nabla\omega_\varepsilon((x - p_{\alpha_k,\beta_k}(x))\cdot i_k\,, (x - p_{\alpha_k,\beta_k}(x))\cdot j_k)|^2 dx = \mathcal{O}(\varepsilon), \tag{1.23}$$

$$\int_{C_{2\varepsilon}^\sigma(\alpha_k,\beta_k)} \frac{4\sigma^2\varepsilon^2}{h^2_{\alpha_k,\beta_k}(x)}\left|\nabla\omega_\varepsilon\left(X_k(x), Y_k(x)\right)\right|^2 dx = \mathcal{O}(\varepsilon). \tag{1.24}$$

We infer from (1.21-1.24) that

$$\int_\Omega |\nabla u_\varepsilon^{(k)}|^2 w(x) dx \leq$$
$$\leq \int_{T_\varepsilon^\sigma(\alpha_k,\beta_k)} |\nabla\omega_\varepsilon\left((x - p_{\alpha_k,\beta_k}(x)) \cdot i_k\,, (x - p_{\alpha_k,\beta_k}(x)) \cdot j_k\right)|^2 w(x) dx + \mathcal{O}(\varepsilon).$$

Since we have

$$|\nabla\omega_\varepsilon(x,y)|^2 = \frac{8\varepsilon^4}{(\varepsilon^4 + x^2 + y^2)^2} \quad \text{for } (x,y) \in B_\varepsilon(0),$$

we conclude that

$$\int_\Omega |\nabla u_\varepsilon^{(k)}|^2 w(x) dx \leq 8\int_{V_\varepsilon(\alpha_k,\beta_k)} \frac{\varepsilon^4 w(x)}{\left(\varepsilon^4 + r^2_{\alpha_k,\beta_k}(x)\right)^2}\, dx + \mathcal{O}(\varepsilon). \tag{1.25}$$

Then we set

$$\tilde{\ell}_w(\mathcal{F}) = \sum_{k=1}^{n} \liminf_{\varepsilon \to 0^+} \frac{1}{\pi} \int_{V_\varepsilon(\alpha_k,\beta_k)} \frac{\varepsilon^4 w(x)}{\left(\varepsilon^4 + r^2_{\alpha_k,\beta_k}(x)\right)^2} dx. \tag{1.26}$$

By (1.25) and (1.26), we can choose $\varepsilon_1, \ldots, \varepsilon_n > 0$ arbitrarily small to have

$$\sum_{k=1}^{n} \int_\Omega |\nabla u^{(k)}_{\varepsilon_k}|^2 w(x) dx \leq 8\pi \tilde{\ell}_w(\mathcal{F}) + \frac{\delta}{4}. \tag{1.27}$$

We choose σ and then each ε_k for $\left\{C^\sigma_{2\varepsilon_k}(\alpha_k,\beta_k) \cup T^\sigma_{2\varepsilon_k}(\alpha_k,\beta_k)\right\}_{k=1}^n$ to define a family of disjoint sets (which is possible since the curve γ has no self intersection points) and such that (1.27) holds. Then we consider the map $\tilde{u}_\delta : \Omega \to S^2$ defined by

$$\tilde{u}_\delta(x) = \begin{cases} u^{(k)}_{\varepsilon_k} & \text{if } x \in C^\sigma_{2\varepsilon_k}(\alpha_k,\beta_k) \cup T^\sigma_{2\varepsilon_k}(\alpha_k,\beta_k), \\ (0,0,1) & \text{if } x \notin \cup_k C^\sigma_{2\varepsilon_k}(\alpha_k,\beta_k) \cup T^\sigma_{2\varepsilon_k}(\alpha_k,\beta_k). \end{cases}$$

By construction, we have $\tilde{u}_\delta \in W^{1,\infty}_{\text{loc}}\left(\overline{\Omega} \setminus \{P, \alpha_2, \ldots, \alpha_n, N\}, S^2\right)$ and $\deg(\tilde{u}_\delta, P) = 1$, $\deg(\tilde{u}_\delta, N) = -1$, $\deg(\tilde{u}_\delta, \alpha_k) = 0$ for $k = 2, \ldots, n$. From (1.27), we derive that

$$\int_\Omega |\nabla \tilde{u}_\delta|^2 w(x) dx \leq 8\pi \tilde{\ell}_w(\mathcal{F}) + \frac{\delta}{4}.$$

Since $\deg(\tilde{u}_\delta, \alpha_k) = 0$ for $k = 2, \ldots, n$, we can smoothen $\tilde{u}_\delta$ around γ, using the result in [16], in order to obtain a new map $u_\delta \in \mathcal{C}^1_{\text{loc}}\left(\overline{\Omega} \setminus \{P, N\}, S^2\right)$ verifying $\deg(u_\delta, P) = 1$, $\deg(u_\delta, N) = -1$ and

$$\int_\Omega |\nabla u_\delta|^2 w(x) dx \leq 8\pi \tilde{\ell}_w(\mathcal{F}) + \frac{\delta}{2}. \tag{1.28}$$

Now we recall that the collection $\mathcal{F} = ([\alpha_1, \beta_1], \ldots, [\alpha_n, \beta_n]) \in \mathcal{P}(P, N)$ such that the curve $\gamma = \cup_k [\alpha_k, \beta_k]$ has no self-intersection points, can be chosen for the construction of u_δ. From Lemma 1.4 below, we can find $\mathcal{F}$ such that

$$\tilde{\ell}_w(\mathcal{F}) \leq d_w(P, N) + \frac{\delta}{16\pi}$$

and according to (1.28), the map u_δ satisfies the required properties. ■

Lemma 1.4. *For $x, y \in \Omega$, let $\mathcal{P}'(x,y)$ be the class of elements $\mathcal{F} = ([\alpha_1, \beta_1], \ldots, [\alpha_n, \beta_n])$ in $\mathcal{P}(x,y)$ such that the curve $\gamma = \cup_k [\alpha_k, \beta_k]$ has no self intersection points. Then*

$$\tilde{d}_w(x,y) = \operatorname*{Inf}_{\mathcal{F} \in \mathcal{P}'(x,y)} \tilde{\ell}_w(\mathcal{F}) \leq d_w(x,y),$$

where $\tilde{\ell}_w(\mathcal{F})$ is defined in (1.26).

Proof. Step 1. First we prove that $\tilde{d}_w$ defines a distance. As for distance d_w, we infer that $\tilde{d}_w(x,y) = 0$ if and only if $x = y$ and $\tilde{d}_w$ is symmetric. Then we just have to check the triangle inequality. We remark that the juxtaposition of $\mathcal{F}_1 \in \mathcal{P}'(x,z)$ with $\mathcal{F}_2 \in \mathcal{P}'(z,y)$ is not an element of $\mathcal{P}'(x,y)$ in general and we can't proceed as for d_w. Let x, y, z be three distinct points in Ω. We consider two arbitrary elements $\mathcal{F}_1 = ([\alpha_1^1, \beta_1^1], \dots, [\alpha_{n_1}^1, \beta_{n_1}^1]) \in \mathcal{P}'(x,z)$, $\mathcal{F}_2 = ([\alpha_1^2, \beta_1^2], \dots, [\alpha_{n_2}^2, \beta_{n_2}^2]) \in \mathcal{P}'(z,y)$, and the curves $\gamma_1 = \cup_k [\alpha_k^1, \beta_k^1]$ and $\gamma_2 = \cup_k [\alpha_k^2, \beta_k^2]$. We have to prove that we can construct $\mathcal{F}_3 \in \mathcal{P}'(x,y)$ such that $\tilde{\ell}_w(\mathcal{F}_3) \le \tilde{\ell}_w(\mathcal{F}_1) + \tilde{\ell}_w(\mathcal{F}_2)$.

First Case : If the curve $\gamma_1 \cup \gamma_2$ has no self intersection points then we take $\mathcal{F}_3 = ([\alpha_1^1, \beta_1^1], \dots, [\alpha_{n_1}^1, \beta_{n_1}^1], [\alpha_1^2, \beta_1^2], \dots, [\alpha_{n_2}^2, \beta_{n_2}^2]) \in \mathcal{P}'(x,y)$ and we have

$$\tilde{\ell}_w(\mathcal{F}_3) = \tilde{\ell}_w(\mathcal{F}_1) + \tilde{\ell}_w(\mathcal{F}_2).$$

Second Case : If $\gamma_1 \cup \gamma_2$ has self intersection points then we rewrite the curves γ_1 and γ_2 as $\gamma_1 = \cup_{k=1}^{\tilde{n}_1} [\tilde{\alpha}_k^1, \tilde{\beta}_k^1]$ and $\gamma_2 = \cup_{k=1}^{\tilde{n}_2} [\tilde{\alpha}_k^2, \tilde{\beta}_k^2]$ such that

a) $(\alpha_k^i)_{k=1}^{n_i} \subset (\tilde{\alpha}_k^i)_{k=1}^{\tilde{n}_i}$ for $i = 1, 2$,

b) if S is a connected component of $\gamma_1 \cap \gamma_2$ then one of the following cases holds :

b1) $S \subset \left(\cup_{k=1}^{\tilde{n}_1} \{\tilde{\alpha}_k^1, \tilde{\beta}_k^1\} \right) \cap \left(\cup_{k=1}^{\tilde{n}_1} \{\tilde{\alpha}_k^2, \tilde{\beta}_k^2\} \right)$,

b2) $S \in \left\{ [\tilde{\alpha}_1^1, \tilde{\beta}_1^1], \dots, [\tilde{\alpha}_{\tilde{n}_1}^1, \tilde{\beta}_{\tilde{n}_1}^1] \right\} \cap \left\{ [\tilde{\alpha}_1^2, \tilde{\beta}_1^2], \dots, [\tilde{\alpha}_{\tilde{n}_2}^2, \tilde{\beta}_{\tilde{n}_2}^2] \right\}$,

c) $\tilde{\mathcal{F}}_1 = ([\tilde{\alpha}_1^1, \tilde{\beta}_1^1], \dots, [\tilde{\alpha}_{\tilde{n}_1}^1, \tilde{\beta}_{\tilde{n}_1}^1]) \in \mathcal{P}'(x,z)$,

d) $\tilde{\mathcal{F}}_2 = ([\tilde{\alpha}_1^2, \tilde{\beta}_1^2], \dots, [\tilde{\alpha}_{\tilde{n}_2}^2, \tilde{\beta}_{\tilde{n}_2}^2]) \in \mathcal{P}'(z,y)$.

By construction, we can write for every $k = 1, \dots, n_i$ and $i = 1, 2$,

$$[\alpha_k^i, \beta_k^i] = \bigcup_{l=1}^{m_k^i} [\tilde{\alpha}_l^i, \tilde{\beta}_l^i] \quad \text{for some } m_k^i \in \mathbb{N}.$$

Since we have

$$V_\varepsilon(\alpha_k^i, \beta_k^i) = \cup_{l=1}^{m_k^i} V_\varepsilon(\tilde{\alpha}_l^i, \tilde{\beta}_l^i),$$

we get that

$$\liminf_{\varepsilon \to 0^+} \frac{1}{\pi} \int_{V_\varepsilon(\alpha_k^i, \beta_k^i)} \frac{\varepsilon^4 w(x)}{\left(\varepsilon^4 + r^2_{\alpha_k^i, \beta_k^i}(x)\right)^2} \, dx \ge \sum_{l=1}^{m_k^i} \liminf_{\varepsilon \to 0^+} \frac{1}{\pi} \int_{V_\varepsilon(\tilde{\alpha}_l^i, \tilde{\beta}_l^i)} \frac{\varepsilon^4 w(x)}{\left(\varepsilon^4 + r^2_{\tilde{\alpha}_l^i, \tilde{\beta}_l^i}(x)\right)^2} \, dx$$

and we conclude that $\tilde{\ell}_w(\tilde{\mathcal{F}}_i) \le \tilde{\ell}_w(\mathcal{F}_i)$ for $i = 1, 2$. In the collection

$$\left([\tilde{\alpha}_1^1, \tilde{\beta}_1^1], \dots, [\tilde{\alpha}_{\tilde{n}_1}^1, \tilde{\beta}_{\tilde{n}_1}^1], [\tilde{\alpha}_1^2, \tilde{\beta}_1^2], \dots, [\tilde{\alpha}_{\tilde{n}_2}^2, \tilde{\beta}_{\tilde{n}_2}^2] \right),$$

we just have to delete some segments in order to obtain a new element $\mathcal{F}_3 \in \mathcal{P}'(x,y)$ which then satisfies $\tilde{\ell}_w(\mathcal{F}_3) \le \tilde{\ell}_w(\tilde{\mathcal{F}}_1) + \tilde{\ell}_w(\tilde{\mathcal{F}}_2) \le \tilde{\ell}_w(\mathcal{F}_1) + \tilde{\ell}_w(\mathcal{F}_2)$.

From these constructions, we conclude that $\tilde{d}_w(x,y) \leq \tilde{\ell}_w(\mathcal{F}_1) + \tilde{\ell}_w(\mathcal{F}_2)$. Taking the infimum over all $\mathcal{F}_1 \in \mathcal{P}'(x,z)$ and all $\mathcal{F}_2 \in \mathcal{P}'(z,y)$, we derive the triangle inequality.

Step 2. We fix two arbitrary points x_0 and y_0 in Ω and we consider $\zeta : \Omega \to \mathbb{R}$ defined by

$$\zeta(x) = \tilde{d}_w(x, y_0).$$

From the triangle inequality, we get that ζ is 1-Lipschitz with respect to distance $\tilde{d}_w$. Let $z_0 \in \Omega$ and $R > 0$ such that $B_{3R}(z_0) \subset \Omega$ and let $(\rho_n)_{n\in\mathbb{N}}$ be a sequence of smooth mollifiers. For $n > 1/R$, we consider $\zeta_n = \rho_n * \zeta : B_R(z_0) \to \mathbb{R}$. For any $x, y \in B_R(z_0)$ we have

$$\begin{aligned}
|\zeta_n(x) - \zeta_n(y)| &\leq \int_{B_{1/n}} \rho_n(-z)|\zeta(x+z) - \zeta(y+z)|dz \\
&\leq \int_{B_{1/n}} \rho_n(-z)\tilde{d}_w(x+z, y+z)dz \\
&\leq \int_{B_{1/n}} \rho_n(-z)\tilde{\ell}_w\left([x+z, y+z]\right) dz.
\end{aligned}$$

We remark that $V_\varepsilon(x+z, y+z) = z + V_\varepsilon(x,y)$ and that for any $\xi \in V_\varepsilon(x,y)$, we have

$$r_{x,y}(\xi) = r_{x+z,y+z}(\xi + z).$$

Then we obtain for any $z \in B_{1/n}(0)$,

$$\tilde{\ell}_w\left([x+z, y+z]\right) = \liminf_{\varepsilon\to 0^+} \frac{1}{\pi} \int_{V_\varepsilon(x,y)} \frac{\varepsilon^4 w(\xi+z)}{\left(\varepsilon^4 + r_{x,y}^2(\xi)\right)^2} d\xi.$$

Taking an arbitrary sequence $\varepsilon_k \to 0^+$ and using Fatou's lemma, we get that

$$\begin{aligned}
|\zeta_n(x) - \zeta_n(y)| &\leq \liminf_{k\to+\infty} \frac{1}{\pi} \int_{B_{1/n}} \int_{V_{\varepsilon_k}(x,y)} \frac{\varepsilon_k^4\, \rho_n(-z) w(\xi+z)}{\left(\varepsilon_k^4 + r_{x,y}^2(\xi)\right)^2} d\xi dz \\
&\leq \liminf_{k\to+\infty} \frac{1}{\pi} \int_{V_{\varepsilon_k}(x,y)} \frac{\varepsilon_k^4}{\left(\varepsilon_k^4 + r_{x,y}^2(\xi)\right)^2}\, \rho_n * w(\xi)\, d\xi.
\end{aligned}$$

Without loss of generality we may assume that $[x,y] = \{(0,0)\} \times [-R,R]$. Then we have

$$V_\varepsilon(x,y) = \left\{(\xi_1,\xi_2,\xi_3) \in \mathbb{R}^3,\ |\xi_3| \leq R,\ \sqrt{\xi_1^2 + \xi_2^2} \leq \varepsilon\right\}$$

and $r_{x,y}(\xi) = \sqrt{\xi_1^2 + \xi_2^2}$ for $\xi \in V_\varepsilon(x,y)$. Therefore we can write

$$\begin{aligned}
\int_{V_{\varepsilon_k}(x,y)} \frac{\varepsilon_k^4\, \rho_n * w(\xi)}{\left(\varepsilon_k^4 + r_{x,y}^2(\xi)\right)^2} d\xi &= \int_{B_{\varepsilon_k}(0)\times[-R,R]} \frac{\varepsilon_k^4\, \rho_n * w\,(\xi)}{\left(\varepsilon_k^4 + \xi_1^2 + \xi_2^2\right)^2} d\xi \\
&= \int_{B_{\varepsilon_k}(0)\times[-R,R]} \frac{\varepsilon_k^4\, \left(\rho_n * w\,(0,0,\xi_3) + \mathcal{O}_n(\varepsilon_k)\right)}{\left(\varepsilon_k^4 + \xi_1^2 + \xi_2^2\right)^2} d\xi,
\end{aligned}$$

where $\mathcal{O}_n(\varepsilon_k)$ denotes a quantity which tends to 0 as $\varepsilon_k \to 0$ for n fixed. Since we have

$$\int_{B_{\varepsilon_k}(0)} \frac{\varepsilon_k^4}{(\varepsilon_k^4 + \xi_1^2 + \xi_2^2)^2} d\xi = \pi + \mathcal{O}(\varepsilon_k),$$

it follows that

$$|\zeta_n(x) - \zeta_n(y)| \leq \int_{-R}^{R} \rho_n * w(0,0,\xi_3) d\xi_3 = \int_{[x,y]} \rho_n * w(s) ds.$$

As in the proof of Proposition 1.3, we conclude that $|\nabla \zeta| \leq w$ a.e. in $B_R(z_0)$ and since z_0 is arbitrary in Ω, we get that $|\nabla \zeta| \leq w$ a.e. in Ω. According to Proposition 1.3, it implies that for any $x, y \in \Omega$,

$$|\zeta(x) - \zeta(y)| \leq d_w(x,y)$$

which leads to $\tilde{d}_w(x_0, y_0) \leq d_w(x_0, y_0)$ taking $x = x_0$ and $y = y_0$. ∎

1.3.3 Upper bound of the energy

Lemma 1.5. *For any $\delta > 0$, there exists a map $u_\delta \in \mathcal{E}$ such that*

$$\int_\Omega |\nabla u_\delta|^2 w(x) dx \leq 8\pi L_w + \delta.$$

Proof. We relabel the list $(a_i)_{i=1}^N$ as a list of positive points $(p_j)_{j=1}^K$ and a list of negative points $(n_j)_{j=1}^K$ and we may assume that $\sum_j d_w(p_j, n_j) = L_w$. We will construct dipoles between each pair (p_j, n_j) which do not intersect each other. We claim that we can find $\mathcal{F}_1 = ([\alpha_1^1, \beta_1^1], \ldots, [\alpha_{m_1}^1, \beta_{m_1}^1]) \in \mathcal{P}'(p_1, n_1)$ such that

(A.1) $\gamma_1 = \cup_k [\alpha_k^1, \beta_k^1]$ does not contain any $p_j \neq p_1$ and any $n_j \neq n_1$,

(A.2) $\tilde{\ell}_w(\mathcal{F}_1) \leq d_w(p_1, n_1) + \frac{\delta}{8K\pi}$.

Indeed if we define for $x, y \in \Omega_A = \Omega \setminus \{p_j, n_j | \, p_j \neq p_1, n_j \neq n_1\}$,

$$D_w^A(x,y) = \text{Inf}\, \tilde{\ell}_w(\mathcal{F})$$

where the infimum is taken over all $\mathcal{F} = ([\alpha_1, \beta_1], \ldots, [\alpha_m, \beta_m]) \in \mathcal{P}'(x,y)$ such that $\cup_k [\alpha_k, \beta_k] \subset \Omega_A$ then we prove, using the arguments in the proof of Lemma 1.4 that

$$D_w^A(x,y) \leq d_w(x,y) \quad \text{for any } x, y \in \Omega_A.$$

Since $p_1, n_1 \in \Omega_A$, we obtain $D_w^A(p_1, n_1) \leq d_w(p_1, n_1)$ and by the definition of D_w^A, we draw the existence of $\mathcal{F}_1 \in \mathcal{P}'(p_1, n_1)$ satisfying *(A.1)* and *(A.2)*.

Now we will show that we can find some $\mathcal{F}_2 = ([\alpha_1^2, \beta_1^2], \ldots, [\alpha_{m_2}^2, \beta_{m_2}^2])$ in $\mathcal{P}'(p_2, n_2)$ such that

(B.1) $\gamma_2 = \cup_k[\alpha_k^2, \beta_k^2]$ does not contain any $p_j \neq p_2$ and any $n_j \neq n_2$ and does not intersect $\gamma_1 \setminus \{p_1, n_1\}$,

(B.2) $\tilde{\ell}_w(\mathcal{F}_2) \leq d_w(p_2, n_2) + \frac{\delta}{8K\pi}$.

As previously we define

$$\Omega_B = \Omega \setminus \big(\{p_j, n_j | \, p_j \neq p_2, n_j \neq n_2\} \cup (\gamma_1 \setminus \{p_1, n_1\})\big)$$

and

$$D_w^B(x, y) = \text{Inf}\, \tilde{\ell}_w(\mathcal{F}) \quad \text{for } x, y \in \Omega_B$$

where the infimum is taken over all $\mathcal{F} = ([\alpha_1, \beta_1], \ldots, [\alpha_m, \beta_m]) \in \mathcal{P}'(x, y)$ such that $\cup_k[\alpha_k, \beta_k] \subset \Omega_B$. In the same way we infer that for any $x, y \in \Omega_2$,

$$D_w^B(x, y) \leq d_w(x, y)$$

and the existence of $\mathcal{F}_2 \in \mathcal{P}'(p_2, n_2)$ satisfying *(B.1)* and *(B.2)* follows.

Iterating this process, we finally reach the existence of K elements

$$\mathcal{F}_j = ([\alpha_1^j, \beta_1^j], \ldots, [\alpha_{m_j}^j, \beta_{m_j}^j]) \in \mathcal{P}'(p_j, n_j)$$

such that

$$\tilde{\ell}_w(\mathcal{F}_j) \leq d_w(p_j, n_j) + \frac{\delta}{8K\pi},$$

and $\gamma_j = \cup_k[\alpha_k^j, \beta_k^j]$ and $\gamma_i = \cup_k[\alpha_k^i, \beta_k^i]$ do not intersect except maybe at their extremities for $i \neq j$.

From the dipole construction in Lemma 1.3, we construct K maps u_δ^j in $C^1_{\text{loc}}\big(\overline{\Omega} \setminus \{p_j, n_j\}, S^2\big)$ constant outside an arbitrary small open neighborhood $\mathcal{N}_j$ of γ_j and such that $\deg(u_\delta^j, p_j) = +1, \ \deg(u_\delta^j, n_j) = -1$ and

$$\int_\Omega |\nabla u_\delta^j|^2 w(x) dx \leq 8\pi d_w(p_j, n_j) + \frac{\delta}{K}.$$

By construction of the $\mathcal{F}_j$'s, we can choose the $\mathcal{N}_j$ sufficiently small for $\mathcal{N}_j$ and $\mathcal{N}_i$ to not intersect whenever $j \neq i$. Then the map

$$u_\delta(x) = \begin{cases} u_\delta^j(x) & \text{if } x \in \mathcal{N}_j, \\ (0, 0, 1) & \text{if } x \notin \cup_j \mathcal{N}_j, \end{cases}$$

is well defined and satisfies the required properties. ∎

1.4 Some stability and approximation results

1.4.1 Stability results

The stability result below is based on Theorem 3.1 in [34]. It relies on the Γ-convergence of the length functionals (we refer to [41] for the notion of Γ-convergence). In the sequel, we denote by $\mathrm{Lip}\left([0,1],\overline{\Omega}\right)$ the class of all Lipschitz map from $[0,1]$ into $\overline{\Omega}$ and we endow $\mathrm{Lip}\left([0,1],\overline{\Omega}\right)$ with the topology of the uniform convergence on $[0,1]$.

Theorem 1.2. *Let $(w_n)_{n\in\mathbb{N}}$ be a sequence of measurable real functions such that*

$$0 < c_0 \leq w_n \leq C_0 \quad a.e \text{ in } \Omega$$

for some constants c_0 and C_0 independent of $n \in \mathbb{N}$. The following properties are equivalent :

(i) $E_{w_n}\left((a_i,d_i)_{i=1}^N\right) \underset{n\to+\infty}{\longrightarrow} E_w\left((a_i,d_i)_{i=1}^N\right)$ for any configuration $(a_i,d_i)_{i=1}^N$,

(ii) the functionals $\mathbb{L}_{d_{w_n}}$ Γ-converge to $\mathbb{L}_{d_w}$ in $\mathrm{Lip}\left([0,1],\overline{\Omega}\right)$.

In the proof of Theorem 1.2, we will make use of the following lemma.

Lemma 1.6. *Let $(d_n)_{n\in\mathbb{N}}$ be a sequence of geodesic distances on $\overline{\Omega}$ such that*

$$c_0 d_\Omega \leq d_n \leq C_0 d_\Omega \tag{1.29}$$

for some positive constants c_0 and C_0 independent of $n \in \mathbb{N}$. Then there exits a subsequence $(n_k)_{k\in\mathbb{N}}$ and a geodesic distance d' on $\overline{\Omega}$ such that $d_{n_k} \to d'$ as $k \to +\infty$ uniformly on every compact subset of $\overline{\Omega}\times\overline{\Omega}$.

Proof. For $(x_1,y_1),(x_2,y_2) \in \overline{\Omega}\times\overline{\Omega}$ we have

$$\begin{aligned} d_{w_n}(x_1,y_1) - d_{w_n}(x_2,y_2) &\leq d_{w_n}(x_1,x_2) + d_{w_n}(x_2,y_1) - d_{w_n}(x_2,y_2) \\ &\leq d_{w_n}(x_1,x_2) + d_{w_n}(y_1,y_2) \\ &\leq C_0\left(d_\Omega(x_1,x_2) + d_\Omega(y_1,y_2)\right). \end{aligned}$$

Inverting the roles of (x_1,y_1) and (x_2,y_2) we infer that

$$|d_{w_n}(x_1,y_1) - d_{w_n}(x_2,y_2)| \leq C_0\left(d_\Omega(x_1,x_2) + d_\Omega(y_1,y_2)\right).$$

Thus d_{w_n} is C_0-Lipschitz on $\overline{\Omega}\times\overline{\Omega}$ for every $n \in \mathbb{N}$ and we conclude by Ascoli's theorem that we can find a subsequence $(n_k)_{k\in\mathbb{N}}$ and a Lipschitz function d' on $\overline{\Omega}\times\overline{\Omega}$ such that $d_{n_k} \to d'$ as $k \to +\infty$ uniformly on every compact subset of $\overline{\Omega}\times\overline{\Omega}$. We easily check that d' defines a distance on $\overline{\Omega}$ and it remains to prove that d' is geodesic. Since d' satisfies (1.29) as the pointwise limit of $(d_{n_k})_{k\in\mathbb{N}}$, $\overline{\Omega}$ endowed with d' is a complete metric space. By Theorem 1.8 in [55], it suffices to prove that for any $x,y \in \overline{\Omega}$ and $\delta > 0$ there exists $z \in \overline{\Omega}$

such that $\max\left(d'(x,z),d'(z,y)\right) \le \frac{1}{2}\,d'(x,y)+\delta$. We fix $x,y\in\overline{\Omega}$ and $\delta>0$. Since d_{n_k} is of geodesic type, we can find $z_k\in\overline{\Omega}$ such that $\max\left(d_{n_k}(x,z),d_{n_k}(z,y)\right)\le\frac{1}{2}\,d_{n_k}(x,y)+\delta$. Then the sequence (z_k) is bounded and we may assume that $z_k\to z\in\overline{\Omega}$. Since $d_{n_k}\to d'$ uniformly on every compact subset of $\overline{\Omega}\times\overline{\Omega}$, we deduce that $d_{n_k}(x,z_k)\to d'(x,z)$ and $d_{n_k}(z_k,y)\to d'(z,y)$. Letting $k\to+\infty$ in the last inequality we draw that z satisfies the requirement. ■

Proof of Theorem 1.2. Step 1. We prove $(i)\Rightarrow(ii)$. From *(i)* we derive that

$$E_{w_n}(P,N)\to E_w(P,N)$$

in the dipole case for any distinct points $P,N\in\Omega$. By Theorem 1.1 we conclude that $d_{w_n}\to d_w$ pointwise on Ω. As in the proof of Proposition 1.1 we have $c_0 d_\Omega\le d_{w_n}\le C_0 d_\Omega$ in $\overline{\Omega}$. By Lemma 1.6 and the uniqueness of the limit we get that $d_{w_n}\to d_w$ uniformly on every compact subset of $\overline{\Omega}\times\overline{\Omega}$. Using the arguments of the proof of *i) ⇒ ii)* Theorem 3.1 in [34], we infer that $\mathbb{L}_{d_{w_n}}\xrightarrow{\Gamma}\mathbb{L}_{d_w}$ in $\mathrm{Lip}\left([0,1],\overline{\Omega}\right)$.

Step 2. We prove *(ii) ⇒ (i)*. Since we have $c_0 d_\Omega\le d_{w_n}\le C_0 d_{w_n}$ in $\overline{\Omega}$ we draw from Lemma 1.6 that we can find a subsequence $(n_k)_{k\in\mathbb{N}}$ and a geodesic distance d' on $\overline{\Omega}$ such that $d_{w_{n_k}}\to d'$ uniformly on every compact subset of $\overline{\Omega}\times\overline{\Omega}$. As in the previous step, we obtain using the method in [34] that $\mathbb{L}_{d_{w_{n_k}}}\xrightarrow{\Gamma}\mathbb{L}_{d'}$ in $\mathrm{Lip}\left([0,1],\overline{\Omega}\right)$. Then we conclude by assumption *(ii)* that $\mathbb{L}_{d'}\equiv\mathbb{L}_{d_w}$ on $\mathrm{Lip}\left([0,1],\overline{\Omega}\right)$. Since $c_0 d_\Omega\le d'\le C_0 d_\Omega$ as the pointwise limit of $(d_{w_{n_k}})_{k\in\mathbb{N}}$, we can proceed as in Remark 1.1 to prove that for any $x,y\in\overline{\Omega}$ there exists a curve $\gamma\in\mathrm{Lip}\left([0,1],\overline{\Omega}\right)$ such that $d'(x,y)=\mathbb{L}_{d'}(\gamma)$. Since the same property holds for d_w we finally get that $d'\equiv d_w$. The uniqueness of the limit implies the convergence of the full sequence. Then *(i)* follows by Theorem 1.1. ■

In the next proposition, we give some sufficient conditions on a sequence $(w_n)_{n\in\mathbb{N}}$ converging pointwise to w for Property *(i)* in Theorem 1.2 to hold.

Proposition 1.4. *Let $(w_n)_{n\in\mathbb{N}}$ be a sequence of measurable functions such that*

$$0<c_0\le w_n\le C_0\quad a.e\ in\ \Omega$$

for some constants c_0 and C_0 independent of $n\in\mathbb{N}$. Assume that one of the following conditions holds :

(a) $w_n\ge w$ and $w_n\to w$ a.e. in Ω,

(b) $w_n\to w$ in $L^\infty(\Omega)$.

Then Property (i) in Theorem 1.2 holds.

Proof. Step 1. Assume that *(a)* holds. Since $w\le w_n$ a.e. in Ω we infer that

$$E_w\left((a_i,d_i)_{i=1}^N\right)\le E_{w_n}\left((a_i,d_i)_{i=1}^N\right)\quad\text{for any } n\in\mathbb{N}$$

and therefore

$$E_w\left((a_i,d_i)_{i=1}^N\right) \leq \liminf_{n\to+\infty} E_{w_n}\left((a_i,d_i)_{i=1}^N\right). \tag{1.30}$$

Fix some $u \in \mathcal{E}$. Since $w_n \leq C_0$ and $w_n \to w$ a.e. on Ω, we obtain by dominated convergence that

$$\int_\Omega |\nabla u|^2 w_n(x)dx \underset{n\to+\infty}{\to} \int_\Omega |\nabla u|^2 w(x)dx.$$

Then we derive

$$\limsup_{n\to+\infty} E_{w_n}\left((a_i,d_i)_{i=1}^N\right) \leq \int_\Omega |\nabla u|^2 w(x)dx,$$

and since u is arbitrary we conclude

$$\limsup_{n\to+\infty} E_{w_n}\left((a_i,d_i)_{i=1}^N\right) \leq E_w\left((a_i,d_i)_{i=1}^N\right). \tag{1.31}$$

Finally, the announced result follows from (1.30) and (1.31).

Step 2. Assume that *(b)* holds. We set $\delta_n = \| w_n - w \|_{L^\infty(\Omega)}$ and

$$\tilde{w}_n = (1 + c_0^{-1}\delta_n)w_n.$$

By construction we have $\tilde{w}_n \geq w$ and $\tilde{w}_n \to w$ a.e. in Ω. From the previous case we deduce

$$\lim_{n\to+\infty} E_{\tilde{w}_n}\left((a_i,d_i)_{i=1}^N\right) = E_w\left((a_i,d_i)_{i=1}^N\right),$$

which yields the result since

$$E_{\tilde{w}_n}\left((a_i,d_i)_{i=1}^N\right) = (1 + c_0^{-1}\delta_n)E_{w_n}\left((a_i,d_i)_{i=1}^N\right)$$

and $1 + c_0^{-1}\delta_n \to 1$ as $n \to +\infty$. ■

Remark 1.4. The conclusion of Proposition 1.4 case *(b)* may fail if the sequence $\{w_n\}$ converges to w almost everywhere in Ω. Indeed, if one considers a sequence $(w_n)_{n\in\mathbb{N}}$ of smooth functions on $\Omega = B_1(0)$ satisfying

$$w_n(x) = \begin{cases} 1 & \text{if } |x_3| \geq 1/n\,, \\ 1/2 & \text{if } |x_3| = 0\,, \end{cases}$$

and $1/2 \leq w_n \leq 1$ in Ω, one can easily check that $w_n \to 1$ in $L^p(\Omega)$ for any $1 \leq p < +\infty$. Now if we choose two distinct points $P, N \in \{(x_1,x_2,0)\in\Omega\}$, we obtain in the dipole case $E_{w_n}(P,N) = 1/2|P-N|$ for any $n \in \mathbb{N}$ and $E_1(P,N) = |P-N|$. Note that if we consider the sequence of variational problems

$$P_n = \text{Min}\left\{\int_\Omega |\nabla u(x)|^2 w_n(x)dx,\ u \in H_g^1(\Omega,\mathbb{R})\right\},$$

where g denotes some given function in $H^{1/2}(\partial\Omega,\mathbb{R})$, then it follows by classical results (see [41] for instance) that

$$P_n \underset{n\to+\infty}{\longrightarrow} \text{Min}\left\{\int_\Omega |\nabla u(x)|^2 dx,\ u \in H_g^1(\Omega,\mathbb{R})\right\}.$$

1.4.2 Approximation result

In this section, we give an approximation procedure by smooth weights.

Theorem 1.3. *Let $(\rho_n)_{n\in\mathbb{N}}$ be a sequence of smooth mollifiers. Extending w outside Ω by a sufficiently large positive constant and taking $w_n = \rho_n * w$, we have*

$$E_{w_n}\left((a_i,d_i)_{i=1}^N\right) \to E_w\left((a_i,d_i)_{i=1}^N\right) \quad \text{as } n \to +\infty.$$

Proof. Step 1. Assume that $\Omega = \mathbb{R}^3$. Let $(\rho_n)_{n\in\mathbb{N}}$ be a sequence of smooth mollifiers. Fix any function ζ which is 1-Lipschitz with respect to d_w. Using the arguments in the proof of Proposition 1.3, we obtain that the function $\zeta_n = \rho_n * \zeta$ satisfies $|\nabla\zeta_n| \leq \rho_n * w$ on $\mathbb{R}^3$. Then we conclude that ζ_n is 1-Lipschitz with respect to the distance $\delta_{\rho_n * w}$. Relabelling the a_i's as a list of positive and negative points $(p_j, n_j)_{j=1}^K$, we get from formula (1.6) and Theorem 1.1,

$$8\pi\sum_{j=1}^K \zeta_n(p_j) - \zeta_n(n_j) \leq E_{\rho_n * w}\left((a_i,d_i)_{i=1}^N\right).$$

Taking the lim inf as $n \to +\infty$, we obtain

$$8\pi\sum_{j=1}^K \zeta(p_j) - \zeta(n_j) \leq \liminf_{n\to+\infty}\ E_{\rho_n * w}\left((a_i,d_i)_{i=1}^N\right).$$

Since ζ is arbitrary, we deduce from (1.6) and Theorem 1.1 that

$$E_w\left((a_i,d_i)_{i=1}^N\right) \leq \liminf_{n\to+\infty}\ E_{\rho_n * w}\left((a_i,d_i)_{i=1}^N\right). \tag{1.32}$$

Since $\rho_n * w \leq \Lambda$, we obtain by dominated convergence that for any $u \in \mathcal{E}$,

$$\int_\Omega |\nabla u|^2 \rho_n * w(x)dx \underset{n\to+\infty}{\to} \int_\Omega |\nabla u|^2 w(x)dx$$

and therefore

$$\limsup_{n\to+\infty}\ E_{\rho_n * w}\left((a_i,d_i)_{i=1}^N\right) \leq \int_\Omega |\nabla u|^2 w(x)dx.$$

Since u is arbitrary, we infer that

$$\limsup_{n\to+\infty}\ E_{\rho_n * w}\left((a_i,d_i)_{i=1}^N\right) \leq E_w\left((a_i,d_i)_{i=1}^N\right), \tag{1.33}$$

and the result follows from (1.32) and (1.33).

Step 2 : Assume that Ω is a smooth bounded and connected open set. We extend w by setting $w = M$ in $\mathbb{R}^3 \setminus \Omega$ for a large positive constant M that we will choose later. We fix some $\delta > 0$ small enough and consider

$$\Omega_\delta = \left\{x \in \mathbb{R}^3,\ \mathrm{dist}(x,\Omega) < \delta\right\}.$$

We extend to Ω_δ any function ζ which is 1-Lipschitz with respect to d_w by setting

$$\zeta(x) = \zeta(\Pi x) \quad \text{for } x \in \Omega_\delta$$

where Πx denotes the projection of $x \in \Omega_\delta$ on $\overline{\Omega}$. By construction, such a ζ is Lipschitz continuous on Ω_δ and $|\nabla\zeta| \leq C(\Omega, \delta, \Lambda)$ a.e. on $\Omega_\delta \setminus \Omega$ and $|\nabla\zeta| \leq w$ a.e. on Ω. Then we choose $M \geq C(\Omega, \delta, \Lambda)$. Setting $\zeta_n : x \in \Omega \to \rho_n * \zeta(x)$ for $n \geq 1/\delta$, we have $|\nabla\zeta_n| \leq \rho_n * w$ on Ω. Then ζ_n is 1-Lipschitz with respect to the distance $\delta_{\rho_n * w}$ and we can proceed as in Step 1. ■

Remark 1.5. If $(w_n)_{n\in\mathbb{N}}$ denotes the sequence constructed in Theorem 1.3, the previous results show that $d_{w_n} \to d_w$ uniformly on every compact subset of $\overline{\Omega}\times\overline{\Omega}$ and the functionals $\mathbb{L}_{d_{w_n}}$ Γ-converge to $\mathbb{L}_{d_w}$ in $\text{Lip}\left([0,1], \overline{\Omega}\right)$.

1.5 Energy involving a matrix field

In this section, we consider $M = (m_{kl})_{k,l=1}^3$ a continuous map from $\overline{\Omega}$ onto the set of real symmetric 3×3 matrices such that

$$\lambda|\xi|^2 \leq M(x)\xi \cdot \xi \leq \Lambda|\xi|^2 \quad \text{for any } \xi \in \mathbb{R}^3 \text{ and } x \in \overline{\Omega}$$

(here " $\cdot$ " denotes the Euclidean scalar product on $\mathbb{R}^3$) and we investigate on the problem

$$E_M\left((a_i, d_i)_{i=1}^N\right) = \operatorname*{Inf}_{u\in\mathcal{E}} \int_\Omega \sum_{k,l=1}^3 m_{kl}(x) \frac{\partial u}{\partial x_k} \cdot \frac{\partial u}{\partial x_l}\, dx.$$

Under the continuity assumption above, we show that $E_M\left((a_i, d_i)_{i=1}^N\right)$ can also be computed in terms of minimal connections relative to some geodesic distance on $\overline{\Omega}$.

In order to state the result we introduce the following objects. For $x \in \overline{\Omega}$, we denote by $\text{cof}(M(x))$ the cofactor matrix of $M(x)$. For any Lipschitz curve $\gamma : [0,1] \to \overline{\Omega}$, we define the length $\mathbb{L}_M(\gamma)$ by

$$\mathbb{L}_M(\gamma) = \int_0^1 \sqrt{\text{cof}\big(M(\gamma(t))\big)\dot{\gamma}(t) \cdot \dot{\gamma}(t)}\, dt$$

and we construct from $\mathbb{L}_M$ the Riemannian distance d_M on $\overline{\Omega}$ defined by

$$d_M(x, y) = \text{Inf } \mathbb{L}_M(\gamma)$$

where the infimum is taken over all curves $\gamma \in \text{Lip}_{x,y}\left([0,1], \overline{\Omega}\right)$.

Theorem 1.4. *We have*

$$E_M\left((a_i, d_i)_{i=1}^N\right) = 8\pi L_M$$

where L_M is the length of a minimal connection associated to the configuration $(a_i, d_i)_{i=1}^N$ and the distance d_M on $\overline{\Omega}$.

Remark 1.6. One can slightly relax the continuity assumption on M. For example, we can assume that

$$M(x) = \begin{cases} M_1(x) & \textit{if } x \in \Omega_1, \\ M_2(x) & \textit{if } x \in \Omega_2, \end{cases}$$

where Ω_1 and Ω_2 are two open sets of Ω with piecewise smooth boundaries such that $\overline{\Omega_1 \cup \Omega_2} = \overline{\Omega}$, and $x \to M_j(x)$ is continuous on $\overline{\Omega}_j$ for $j = 1, 2$. Hence M is possibly discontinuous on the surface $\Sigma = \overline{\Omega}_1 \cap \overline{\Omega}_2$. Then the conclusion of Theorem 1.4 holds with the geodesic distance d_M constructed from the length $\mathbb{L}_M$ defined by

$$\mathbb{L}_M(\gamma) = \int_0^1 \varphi\left(\gamma(t), \dot{\gamma}(t)\right) dt \quad \text{for } \gamma \in \operatorname{Lip}\left([0,1], \overline{\Omega}\right),$$

where

$$\varphi(x, \nu) = \begin{cases} \sqrt{\operatorname{cof}\left(M(x)\right) \nu \cdot \nu} & \text{if } x \in \overline{\Omega} \setminus \Sigma, \\ \min\left\{\sqrt{\operatorname{cof}\left(M_1(x)\right) \nu \cdot \nu},\ \sqrt{\operatorname{cof}\left(M_2(x)\right) \nu \cdot \nu}\right\} & \text{if } x \in \Sigma. \end{cases}$$

Open Problem . Assuming that the coefficients of M are only in $L^\infty(\Omega)$, is the conclusion of Theorem 1.4 still valid for a certain distance ?

Sketch of the Proof of Theorem 3. The Lower Bound. We follow the strategy in Section 1.3.1. For any $u \in \mathcal{E}$, we have

$$2[\operatorname{cof}(M) D \cdot D]^{1/2} \leq \sum_{k,l=1}^{3} m_{kl}(x) \frac{\partial u}{\partial x_k} \cdot \frac{\partial u}{\partial x_l} \quad \text{a.e. on } \Omega \tag{1.34}$$

where D is the vector field defined by (1.16). Next we infer that

$$\int_\Omega \sum_{k,l=1}^{3} m_{kl}(x) \frac{\partial u}{\partial x_k} \cdot \frac{\partial u}{\partial x_l}\, dx \geq -2 \int_\Omega D \cdot \nabla\zeta = 8\pi \sum_{j=1}^{K} \zeta(p_j) - \zeta(n_j) \tag{1.35}$$

for any Lipschitz function $\zeta : \overline{\Omega} \to \mathbb{R}$ such that

$$\left[\operatorname{cof}(M)^{-1} \nabla\zeta \cdot \nabla\zeta\right]^{1/2} \leq 1 \quad \text{a.e. in } \Omega. \tag{1.36}$$

Since a function ζ satisfies (1.36) if and only if ζ is 1-Lipschitz with respect to the distance d_M, we conclude from (1.35) that

$$E_M\left((a_i, d_i)_{i=1}^N\right) \geq 8\pi \operatorname{Max} \sum_{j=1}^{K} \zeta(p_j) - \zeta(n_j) = 8\pi L_M$$

where the maximum is taken over all functions ζ which are 1-Lipschitz with respect to distance d_M.

The Upper Bound. The proof relies on the dipole construction.

Lemma 1.7. *For any distinct points $P, N \in \Omega$, any smooth simple curve $\gamma \subset \Omega$ running between P and N and $\delta > 0$, there exists a map u_δ in $C^1\left(\overline{\Omega} \setminus \{P, N\}, S^2\right)$ such that* $\deg(u_\delta, P) = +1$, $\deg(u_\delta, N) = -1$ *and*

$$\int_\Omega \sum_{k,l=1}^{3} m_{kl}(x) \frac{\partial u_\delta}{\partial x_k} \cdot \frac{\partial u_\delta}{\partial x_l}\, dx \leq 8\pi \mathbb{L}_M(P, N) + \delta. \tag{1.37}$$

Moreover u_δ is constant outside an arbitrary small neighborhood of γ.

We may assume that $\sum_j d_M(p_j, n_j) = L_M$. Then we choose K smooth simple curves γ_j running between p_j and n_j which do not intersect except at their endpoints and such that $\mathbb{L}_M(p_j, n_j) \leq d_M(p_j, n_j) + \delta$. By Lemma 1.7, we construct K maps u_j constant outside a small neighborhood $\mathcal{N}_j$ of γ_j and $\mathcal{N}_j \cap \mathcal{N}_i = \emptyset$ if $j \neq i$. Letting $u_\delta = u_j$ on $\mathcal{N}_j$ for $j = 1, \ldots, K$ and $u_\delta = (0, 0, 1)$ outside $\cup_j \mathcal{N}_j$, we have $u_\delta \in \mathcal{E}$ and

$$E_M\left((a_i, d_i)_{i=1}^N\right) \leq \int_\Omega \sum_{k,l=1}^{3} m_{kl}(x) \frac{\partial u_\delta}{\partial x_k} \cdot \frac{\partial u_\delta}{\partial x_l}\, dx \leq 8\pi L_M + C\delta.$$

Since δ is arbitrary, we obtain that $E_M\left((a_i, d_i)_{i=1}^N\right) \leq 8\pi L_M$. ■

Sketch of the Proof of Lemma 1.7. Since we can approximate the coefficients of M locally uniformly by smooth coefficients, we just have to prove Lemma 1.7 for M with smooth entries. We construct as in [8] a smooth diffeomorphism Φ from a small neighborhood $\mathcal{V}$ of γ into a small neighborhood of $\{(0, 0)\} \times [-|\gamma|/2, |\gamma|/2]$ such that

$$\Phi(\gamma) = \{(0, 0)\} \times [-|\gamma|/2, |\gamma|/2]$$

(here $|\gamma|$ denotes the Euclidean length of γ) and $\Phi^{-1}(0, 0, \cdot) : [-|\gamma|/2, |\gamma|/2] \to \mathbb{R}^3$ defines a normal parametrization of γ orientating γ from N to P. Then we set for $y_3 \in [-|\gamma|/2, |\gamma|/2]$,

$$B(y_3) = (b_{k,l}(y_3))_{k,l=1}^3 = [\nabla \Phi^{-1}(0, 0, y_3)]^{-1} M(\Phi^{-1}(0, 0, y_3)) \nabla \Phi^{-1}(0, 0, y_3),$$

and

$$\hat{B}(y_3) = (b_{k,l}(y_3))_{k,l=1}^2 .$$

For small $\varepsilon > 0$ and $n \in \mathbb{N}$ large, we consider the map $\tilde{u}_n : \Phi(\mathcal{V}) \to S^2$ defined by

$$\tilde{u}_n(y_1, y_2, y_3) = \omega_\varepsilon \left(\frac{n}{\frac{|\gamma|^2}{4} - y_3^2}\, \hat{B}^{-1/2}(y_3) \cdot (y_1, y_2) \right)$$

where ω_ε is given by (1.20). Then we take

$$u_n(x) = \begin{cases} \tilde{u}_n(\Phi(x)) & \text{if } x \in \mathcal{V}, \\ (0, 0, 1) & \text{if } x \notin \mathcal{V}. \end{cases}$$

Following the computations in [27] and using the properties of Φ, we easily check that $u_n \in W^{1,\infty}_{\text{loc}}\left(\overline{\Omega} \setminus \{P, N\}, S^2\right)$, $\deg(u_n, P) = +1$, $\deg(u_n, N) = -1$. Choosing n sufficiently large and smoothening u_n around γ by the procedure in [16], we get a new map $u_\delta \in \mathcal{E}$ which satisfies (1.37). ■

Chapitre 2

The relaxed energy for S^2-valued maps and measurable weights

2.1 Introduction and main results

Let Ω be a smooth bounded and connected open set of $\mathbb{R}^3$ and let $w : \Omega \to \mathbb{R}$ be a measurable function such that

$$0 < \lambda \leq w \leq \Lambda \quad \text{a.e. in } \Omega \tag{2.1}$$

for some constant λ and Λ. We set $H^1_g(\Omega, S^2) = \{u \in H^1(\Omega, S^2),\ u = g \text{ on } \partial\Omega\}$, where $g : \partial\Omega \to S^2$ is a given smooth boundary data such that $\deg(g) = 0$. Our main goal in this chapter is to obtain an explicit formula for the relaxed functional

$$E_w(u) = \operatorname{Inf}\Big\{\liminf_{n\to+\infty} \int_\Omega |\nabla u_n(x)|^2 w(x)dx,\ u_n \in H^1_g(\Omega, S^2) \cap C^1(\overline{\Omega}),\ u_n \rightharpoonup u \text{ weakly in } H^1\Big\}$$

defined for $u \in H^1_g(\Omega, S^2)$. By a result of F. Bethuel (see [16]), $H^1_g(\Omega, S^2) \cap C^1(\overline{\Omega})$ is sequentially dense for the weak topology in $H^1_g(\Omega, S^2)$ and then the functional E_w is well defined.

In [18], F. Bethuel, H. Brezis and J.M. Coron have proved that for $w \equiv 1$,

$$E_1(u) = \int_\Omega |\nabla u(x)|^2 dx + 8\pi L(u),$$

where $L(u)$ denotes the *length of a minimal connection* relative to the Euclidean geodesic distance d_Ω in $\overline{\Omega}$ connecting the singularities of u (see also M. Giaquinta, G. Modica, J. Souček [53]). If $u \in H^1_g(\Omega, S^2)$ is smooth on $\overline{\Omega}$ except at a finite number of points in Ω, the length of a minimal connection relative to d_Ω connecting the singularities of u is given by

$$L(u) = \operatorname*{Min}_{\sigma \in \mathcal{S}_K} \sum_{i=1}^{K} d_\Omega(P_i, N_{\sigma(i)})$$

where $(P_1, \ldots, P_K)$ and $(N_1, \ldots, N_K)$ are respectively the singularities of positive and negative degree counted according to their multiplicity (since $\deg(g) = 0$, the number of positive singularities is equal to the number of negative ones) and $\mathcal{S}_K$ denotes the set of all permutations of K indices. For the definition of $L(u)$ when u is arbitrary in $H^1_g(\Omega, S^2)$, we refer to (2.6)-(2.7) below. The notion of length of a minimal connection between singularities has its origin in [30]. We also refer to the results of J. Bourgain, H. Brezis, P. Mironescu [23] and H. Brezis, P. Mironescu, A.C. Ponce [32] for similar problems involving S^1-valued maps.

For $u \in H^1(\Omega, S^2)$, the vector field $D(u)$ first introduced in [30] and defined by

$$D(u) = \left(u \cdot \frac{\partial u}{\partial x_2} \wedge \frac{\partial u}{\partial x_3}, u \cdot \frac{\partial u}{\partial x_3} \wedge \frac{\partial u}{\partial x_1}, u \cdot \frac{\partial u}{\partial x_1} \wedge \frac{\partial u}{\partial x_2} \right) \tag{2.2}$$

plays a crucial role. Indeed, if u is smooth except at a finite number of points $(P_i, N_i)_{i=1}^K$ in Ω, then (see [30], Appendix B)

$$\operatorname{div} D(u) = 4\pi \sum_{i=1}^{K} (\delta_{P_i} - \delta_{N_i}) \quad \text{in } \mathcal{D}'(\Omega) \tag{2.3}$$

and if in addition $u_{|\partial\Omega} = g$, we have (since $\deg(g) = 0$, see [30], Section IV)

$$L(u) = \operatorname{Sup} \left\{ \sum_{i=1}^{K} (\zeta(P_i) - \zeta(N_i)) \right\} \tag{2.4}$$

where the supremum is taken over all functions $\zeta : \overline{\Omega} \to \mathbb{R}$ which are 1-Lipschitz with respect to distance d_Ω i.e., $|\zeta(x) - \zeta(y)| \le d_\Omega(x, y)$. Note that for any real Lipschitz function ζ,

$$\sum_{i=1}^{K} \zeta(P_i) - \zeta(N_i) = \frac{1}{4\pi} \int_\Omega \operatorname{div} D(u)\, \zeta = -\frac{1}{4\pi} \int_\Omega D(u) \cdot \nabla \zeta + \frac{1}{4\pi} \int_{\partial\Omega} (D(u) \cdot \nu)\, \zeta \tag{2.5}$$

where ν denotes the outward normal to $\partial\Omega$. We recall that $D(u) \cdot \nu$ is equal to the 2×2 Jacobian determinant of u restricted to $\partial\Omega$ and then it only depends on g. In view of (2.4) and (2.5), $L(u)$ has been defined in [18] for $u \in H^1_g(\Omega, S^2)$ by

$$L(u) = \frac{1}{4\pi} \operatorname{Sup} \left\{ \langle T(u), \zeta \rangle,\, \zeta : \overline{\Omega} \to \mathbb{R} \text{ 1-Lipschitz with respect to } d_\Omega \right\} \tag{2.6}$$

where $T(u) \in \mathcal{D}'(\Omega)$ denotes the distribution defined by its action on real Lipschitz functions through the formula :

$$\langle T(u), \zeta \rangle = \int_\Omega D(u) \cdot \nabla \zeta - \int_{\partial\Omega} (D(u) \cdot \nu)\, \zeta\,. \tag{2.7}$$

In Chapter 1, we have studied the following variational problem : given two distinct points P and N in Ω,

$$E_w(P,N) = \mathrm{Inf}\ \Big\{ \int_\Omega |\nabla v(x)|^2 w(x)dx,\ v \in \mathcal{E}(P,N) \Big\}$$

where

$$\mathcal{E}(P,N) = \big\{ v \in H^1(\Omega,S^2) \cap C^1\big(\overline{\Omega} \setminus \{P,N\}\big), v = \text{const on } \partial\Omega, \\ T(v) = 4\pi(\delta_P - \delta_N) \text{ in } \mathcal{D}'(\Omega) \big\}.$$

In the case $w \equiv 1$, H. Brezis, J.M. Coron and E. Lieb have shown that (see [30])

$$E_1(P,N) = 8\pi d_\Omega(P,N).$$

For an arbitrary function w, we have proved (see Chapter 1) that $E_w(\cdot,\cdot)$ defines a distance function satisfying

$$8\pi\lambda\, d_\Omega(\cdot,\cdot) \le E_w(\cdot,\cdot) \le 8\pi\Lambda\, d_\Omega(\cdot,\cdot). \tag{2.8}$$

From (2.8), we infer that E_w extends to $\overline{\Omega} \times \overline{\Omega}$ into a distance on $\overline{\Omega}$. In what follows, we set for $x, y \in \overline{\Omega}$,

$$d_w(x,y) = \frac{1}{8\pi}\, E_w(x,y).$$

When w is continuous, we also have shown that the distance d_w can be characterized in the following way : for any $x, y \in \overline{\Omega}$,

$$d_w(x,y) = \mathrm{Min} \int_0^1 w(\gamma(t))|\dot\gamma(t)|dt$$

where the minimum is taken over all Lipschitz curve $\gamma : [0,1] \to \overline{\Omega}$ verifying $\gamma(0) = x$ and $\gamma(1) = y$. For an arbitrary measurable function w, the previous formula is meaningless since w is not well defined on curves but a similar characterization of d_w actually holds. We refer to Chapter 1 for more details. We also recall the general result in Chapter 1 :

Theorem 2.1. *Let $(P_i)_{i=1}^K$ and $(N_i)_{i=1}^K$ be two lists of points in Ω and consider*

$$\mathcal{E}\left((P_i,N_i)_{i=1}^K\right) = \Big\{\ v \in H^1(\Omega,S^2) \cap C^1\big(\overline{\Omega} \setminus \{(P_i,N_i)_{i=1}^K\}\big), \\ v = \text{const } \textit{on } \partial\Omega \textit{ and } T(v) = 4\pi \sum_{i=1}^K \delta_{P_i} - \delta_{N_i} \textit{ in } \mathcal{D}'(\Omega) \Big\}.$$

Then we have

$$\mathrm{Inf} \left\{ \int_\Omega |\nabla v(x)|^2 w(x)dx,\ v \in \mathcal{E}\left((P_i,N_i)_{i=1}^K\right) \right\} = 8\pi L_w$$

where L_w is the length of a minimal connection relative to distance d_w connecting the points (P_i) and (N_i), i.e.,

$$L_w = \underset{\sigma \in \mathcal{S}_K}{\mathrm{Min}} \sum_{i=1}^K d_w(P_i, N_{\sigma(i)}).$$

By analogy with the case $w \equiv 1$, we define for $u \in H^1_g(\Omega, S^2)$,

$$L_w(u) = \frac{1}{4\pi} \operatorname{Sup} \left\{ \langle T(u), \zeta \rangle,\ \zeta : \overline{\Omega} \to \mathbb{R} \text{ 1-Lipschitz with respect to } d_w \right\}$$

(note that any real function ζ which is 1-Lipschitz with respect to d_w, is a Lipschitz function with respect to d_Ω since d_w is strongly equivalent to d_Ω and then $\langle T(u), \zeta \rangle$ is well defined). When u is smooth except at a finite number of points $(P_i, N_i)_{i=1}^K$ in Ω, it follows as in [30] that $L_w(u)$ is equal to the length of a minimal connection relative to distance d_w connecting the points (P_i) and (N_i). Our main result is the following.

Theorem 2.2. *For any $u \in H^1_g(\Omega, S^2)$, we have*

$$E_w(u) = \int_\Omega |\nabla u(x)|^2 w(x) dx + 8\pi L_w(u)\,.$$

The proof of Theorem 2.2 is presented in Section 3 and is based on a method similar to the one used in [18] and on a *Dipole Removing Technique* exposed in the next section. This technique is mostly inspired from [16] but involves some tools developed in Chapter 1 in order to treat the problem for a non smooth function w.

In Section 4, we prove a stability property of E_w. More precisely, we give some conditions on a sequence $(w_n)_{n\in\mathbb{N}}$ under which one can conclude that the sequence of functionals $(E_{w_n})_{n\in\mathbb{N}}$ converges pointwise to E_w on $H^1_g(\Omega, S^2)$. The results are obtained using previous ones in Chapter 1. In Section 5, we present similar results for a relaxed type functional in which we do not prescribed any boundary data.

Throughout this chapter, a sequence of smooth mollifiers means any sequence $(\rho_n)_{n\in\mathbb{N}}$ satisfying

$$\rho_n \in C^\infty(\mathbb{R}^3, \mathbb{R}), \quad \operatorname{Supp} \rho_n \subset B_{1/n}, \quad \int_{\mathbb{R}^3} \rho_n = 1, \quad \rho_n \geq 0 \text{ on } \mathbb{R}^3.$$

2.2 The dipole removing technique

In this section, we first give a technical result which will be used for the *dipole removing technique* in Section 2.2.2 .

2.2.1 Preliminaries

Let α and β be two distinct points in Ω. We denote by $p_{\alpha,\beta}(\xi)$ the projection of $\xi \in \mathbb{R}^3$ on the straight line passing by α and β and $r_{\alpha,\beta}(\xi) = \operatorname{dist}(x, [\alpha, \beta])$, where "dist" denotes the Euclidean distance in $\mathbb{R}^3$. For $m \in \mathbb{N}^*$, we set

$$a_m^{\alpha,\beta} = \frac{|\alpha - \beta|}{m} \quad \text{and} \quad s_j^{\alpha,\beta} = j\, a_m^{\alpha,\beta} \quad \text{for } j = 0, \ldots, m.$$

For $\xi \in \mathbb{R}^3$ such that $p_{\alpha,\beta}(\xi) \in [\alpha, \beta]$, we define

$$h_m^{\alpha,\beta}(\xi) = \min_{0 \le j \le m} \left| |p_{\alpha,\beta}(\xi) - \alpha| - s_j^{\alpha,\beta} \right|,$$

and we set

$$\Theta_m([\alpha, \beta]) = \left\{ \xi \in \mathbb{R}^3,\ p_{\alpha,\beta}(\xi) \in [\alpha, \beta] \text{ and } r_{\alpha,\beta}(\xi) \le a_m^{\alpha,\beta} h_m^{\alpha,\beta}(\xi) \right\}.$$

For two points x and y in Ω, we consider the class $\mathcal{Q}(x, y)$ of all finite collections of segments $\mathcal{F} = ([\alpha_k, \beta_k])_{k=1}^{n(\mathcal{F})}$ such that $\beta_k = \alpha_{k+1}$, $\alpha_1 = x$, $\beta_{n(\mathcal{F})} = y$, $[\alpha_k, \beta_k] \subset \Omega$ and $\alpha_k \ne \beta_k$. We define the "length" of an element $\mathcal{F} \in \mathcal{Q}(x, y)$ by

$$\bar{\ell}_w(\mathcal{F}) = \sum_{k=1}^{n(\mathcal{F})} \liminf_{m \to +\infty} \frac{1}{\pi} \int_{\Theta_m([\alpha_k,\beta_k]) \cap \Omega} \varepsilon_{\alpha_k,\beta_k}^m(\xi)\, w(\xi)\, d\xi$$

with

$$\varepsilon_{\alpha_k,\beta_k}^m(\xi) = \frac{\left(h_m^{\alpha_k,\beta_k}(\xi)\right)^2 \left(a_m^{\alpha_k,\beta_k}\right)^4}{\left(\left(h_m^{\alpha_k,\beta_k}(\xi)\right)^2 \left(a_m^{\alpha_k,\beta_k}\right)^4 + r_{\alpha_k,\beta_k}^2(\xi)\right)^2}.$$

We shall use the following Lemma.

Lemma 2.1. *Let $\mathbb{P}$ be a finite collection of distinct points in Ω or $\mathbb{P} = \emptyset$. For any distinct points x_0, y_0 in $\Omega \setminus \mathbb{P}$ and $\delta > 0$, there exists $\mathcal{F}_\delta = ([\alpha_1, \beta_1], \ldots, [\alpha_n, \beta_n]) \in \mathcal{Q}(x_0, y_0)$ such that $(\mathbb{P} \cup \{y_0\}) \cap \left(\cup_{k=1}^{n-1} [\alpha_k, \beta_k] \cup [\alpha_n, \beta_n[\right) = \emptyset$ and*

$$\bar{\ell}_w(\mathcal{F}) \le d_w(x_0, y_0) + \delta.$$

Proof. Step 1. Assume that w is smooth on Ω. We are going to prove that for every element $\mathcal{F} = ([\alpha_1, \beta_1], \ldots, [\alpha_n, \beta_n]) \in \mathcal{Q}(x, y)$, we have

$$\bar{\ell}_w(\mathcal{F}) = \int_{\cup_{k=1}^n [\alpha_k, \beta_k]} w(s) ds.$$

It suffices to prove that for any distinct points $\alpha, \beta \in \Omega$,

$$\lim_{m \to +\infty} \frac{1}{\pi} \int_{\Theta_m([\alpha,\beta]) \cap \Omega} \varepsilon_k^m(\xi)\, w(\xi)\, d\xi = \int_{[\alpha,\beta]} w(s) ds. \tag{2.9}$$

Without loss of generality, we may assume that $[\alpha, \beta] = \{(0, 0)\} \times [0, R]$ and we drop the indices α and β for simplicity. We set for $j = 0, \ldots, m-1$,

$$C_m^{j+} = \left\{ \xi = (\xi_1, \xi_2, \xi_3) \in \Theta_m([\alpha, \beta]),\ \xi_3 \in \left[s_j, s_j + \frac{a_m}{2} \right] \right\},$$

and for $j = 1, \ldots, m$,

$$C_m^{j-} = \left\{ \xi = (\xi_1, \xi_2, \xi_3) \in \Theta_m([\alpha, \beta]),\ \xi_3 \in \left[s_j - \frac{a_m}{2}, s_j \right] \right\}.$$

For $\xi \in C_m^{j+} \cup C_m^{j-}$, we have $h_m(\xi) = |\xi_3 - s_j|$ and we get that for m large enough,

$$\int_{\Theta_m([\alpha,\beta])\cap\Omega} \varepsilon_k^m(\xi)\, w(\xi)\, d\xi = \sum_{j=0}^{m-1} I_m^{j+} + \sum_{j=1}^{m} I_m^{j-} \tag{2.10}$$

with

$$I_m^{j+} = \int_{C_m^{j+}} \frac{|\xi_3 - s_j|^2 a_m^4 w(\xi)}{\left(|\xi_3 - s_j|^2 a_m^4 + r^2(\xi)\right)^2} d\xi \quad \text{for } j = 0, \dots, m-1,$$

$$I_m^{j-} = \int_{C_m^{j-}} \frac{|\xi_3 - s_j|^2 a_m^4 w(\xi)}{\left(|\xi_3 - s_j|^2 a_m^4 + r^2(\xi)\right)^2} d\xi \quad \text{for } j = 1, \dots, m.$$

Using the change of variable $z_1 = \frac{\xi_1}{|\xi_3 - s_j|}$, $z_2 = \frac{\xi_2}{|\xi_3 - s_j|}$ and $z_3 = \xi_3$, we derive that

$$\begin{aligned} I_m^{j+} &= \int_{s_j}^{s_j + \frac{a_m}{2}} \left(\int_{B_{a_m}(0)} \frac{a_m^4 w(|z_3 - s_j| z_1, |z_3 - s_j| z_2, z_3)}{(a_m^4 + z_1^2 + z_2^2)^2} dz_1 dz_2 \right) dz_3 \\ &= \int_{s_j}^{s_j + \frac{a_m}{2}} (w(0,0,z_3) + \mathcal{O}(a_m)) \left(\int_{B_{a_m}(0)} \frac{a_m^4}{(a_m^4 + z_1^2 + z_2^2)^2} dz_1 dz_2 \right) dz_3 \\ &= \pi \int_{s_j}^{s_j + \frac{a_m}{2}} w(0,0,z_3) dz_3 + \mathcal{O}(a_m^2). \end{aligned}$$

By similar computations we get that

$$I_m^{j-} = \pi \int_{s_j - \frac{a_m}{2}}^{s_j} w(0,0,z_3) dz_3 + \mathcal{O}(a_m^2).$$

Combining this equalities with (2.10), we obtain that

$$\int_{\Theta_m([\alpha,\beta])\cap\Omega} \varepsilon_k^m(\xi)\, w(\xi)\, d\xi = \pi \int_0^R w(0,0,z_3) dz_3 + \mathcal{O}(a_m)$$

which ends the proof of (2.9).

Step 2. We fix two distinct points $x_0, y_0 \in \Omega \setminus \mathbb{P}$. For any points x, y in $\Omega \setminus (\mathbb{P} \cup \{y_0\})$, let $\mathcal{Q}'(x,y)$ be the class of elements $\mathcal{F} = ([\alpha_1, \beta_1], \dots, [\alpha_n, \beta_n]) \in \mathcal{Q}(x,y)$ such that

$$\cup_{k=1}^n [\alpha_k, \beta_k] \subset \Omega \setminus (\mathbb{P} \cup \{y_0\}).$$

We consider the function $\mathcal{D}_w : \Omega \setminus (\mathbb{P} \cup \{y_0\}) \times \Omega \setminus (\mathbb{P} \cup \{y_0\}) \to \mathbb{R}_+$ defined by

$$\mathcal{D}_w(x,y) = \operatorname*{Inf}_{\mathcal{F} \in \mathcal{Q}'(x,y)} \overline{\ell}(\mathcal{F}).$$

We are going to show that $\mathcal{D}_w$ defines a distance function which can be extended to $\overline{\Omega} \times \overline{\Omega}$. Let $x, y \in \Omega \setminus (\mathbb{P} \cup \{y_0\})$ and let $\mathcal{F} = ([\alpha_1, \beta_1], \dots, [\alpha_n, \beta_n])$ be an element of $\mathcal{Q}'(x,y)$. Assumption (2.1) and similar computations to those in Step 1 lead to

$$\lambda \sum_{k=1}^n |\alpha_k - \beta_k| \le \overline{\ell}_w(\mathcal{F}) \le \Lambda \sum_{k=1}^n |\alpha_k - \beta_k|.$$

Taking the infimum over all $\mathcal{F} \in \mathcal{Q}'(x,y)$, we infer that

$$\lambda\, d_\Omega(x,y) \leq \mathcal{D}_w(x,y) \leq \Lambda\, d_\Omega(x,y). \tag{2.11}$$

From (2.11), we deduce that $\mathcal{D}_w(x,y)=0$ if and only if $x=y$. Let us now prove that $\mathcal{D}_w$ is symmetric. Let $x,y \in \Omega \setminus (\mathbb{P} \cup \{y_0\})$ and $\delta > 0$ arbitrary small. By definition, we can find $\mathcal{F}_\delta = ([\alpha_1,\beta_2],\dots,[\alpha_n,\beta_n])$ in $\mathcal{Q}'(x,y)$ satisfying

$$\bar{\ell}_w\left(\mathcal{F}_\delta\right) \leq \mathcal{D}_w(x,y)+\delta.$$

Then for $\mathcal{F}'_\delta = ([\beta_n,\alpha_n],\dots,[\beta_1,\alpha_1]) \in \mathcal{Q}'(y,x)$, we have

$$\mathcal{D}_w(y,x) \leq \bar{\ell}_w\left(\mathcal{F}'_\delta\right) = \bar{\ell}_w\left(\mathcal{F}_\delta\right) \leq \mathcal{D}_w(x,y)+\delta.$$

Since δ is arbitrary, we obtain $\mathcal{D}_w(y,x) \leq \mathcal{D}_w(x,y)$ and then, inverting the roles of x and y, we conclude that $\mathcal{D}_w(y,x)=\mathcal{D}_w(x,y)$. The triangle inequality is immediate since the juxtaposition of $\mathcal{F}_1 \in \mathcal{Q}'(x,z)$ with $\mathcal{F}_2 \in \mathcal{Q}'(z,y)$ is an element of $\mathcal{Q}'(x,y)$. Hence $\mathcal{D}_w$ defines a distance on $\Omega \setminus (\mathbb{P} \cup \{y_0\})$ verifying (2.11). Therefore distance $\mathcal{D}_w$ extends uniquely to $\overline{\Omega} \times \overline{\Omega}$ into a distance function that we still denote by $\mathcal{D}_w$. By continuity, $\mathcal{D}_w$ satisfies (2.11) for any $x,y \in \overline{\Omega}$.

Step 3. We consider the function $\zeta : \overline{\Omega} \to \mathbb{R}$ defined by

$$\zeta(x) = \mathcal{D}_w(x,x_0).$$

Note that function ζ is 1-Lipschitz with respect to distance $\mathcal{D}_w$ and therefore Λ-Lipschitz with respect to the Euclidean geodesic distance on $\overline{\Omega}$ by (2.11). We fix an arbitrary point $z_0 \in \Omega \setminus (\mathbb{P} \cup \{y_0\})$ and some $R>0$ such that $B_{3R}(z_0) \subset \Omega \setminus (\mathbb{P} \cup \{y_0\})$. Let $(\rho_n)_{n\in\mathbb{N}}$ be a sequence of smooth mollifiers. For $n > 1/R$, we consider the smooth function $\zeta_n = \rho_n * \zeta : B_R(z_0) \to \mathbb{R}$. We write

$$\zeta_n(x) = \int_{B_{1/n}} \rho_n(-z)\zeta(x+z)dz$$

and therefore for any $x,y \in B_R(z_0)$,

$$\begin{aligned}
|\zeta_n(x)-\zeta_n(y)| &\leq \int_{B_{1/n}} \rho_n(-z)\,|\zeta(x+z)-\zeta(y+z)|\,dz \\
&\leq \int_{B_{1/n}} \rho_n(-z)\,\mathcal{D}_w(x+z,y+z)dz \\
&\leq \int_{B_{1/n}} \rho_n(-z)\,\bar{\ell}_w\left([x+z,y+z]\right)dz.
\end{aligned}$$

We remark that $\Theta_m\left([x+z,y+z]\right) = z + \Theta_m\left([x,y]\right)$. Whenever m is large enough, we have $z+\Theta_m\left([x,y]\right) \subset B_{3R}(z_0)$ and then

$$\varepsilon^m_{x+z,y+z}\left(\xi+z\right) = \varepsilon^m_{x,y}\left(\xi\right) \quad \text{for any vector } \xi \in \Theta_m\left([x,y]\right).$$

Hence we obtain for any $z \in B_{1/n}(0)$,

$$\overline{\ell}_w\left([x+z, y+z]\right) = \liminf_{m\to+\infty} \frac{1}{\pi} \int_{\Theta_m([x,y])} \varepsilon^m_{x,y}(\xi)\, w(\xi+z)\, d\xi.$$

Using Fatou's lemma, we get that

$$\begin{aligned} |\zeta_n(x) - \zeta_n(y)| &\leq \int_{B_{1/n}} \rho_n(-z) \left(\liminf_{m\to+\infty} \frac{1}{\pi} \int_{\Theta_m([x,y])} \varepsilon^m_{x,y}(\xi)\, w(\xi+z) d\xi \right) dz \\ &\leq \liminf_{m\to+\infty} \frac{1}{\pi} \int_{B_{1/n}} \int_{\Theta_m([x,y])} \rho_n(-z)\, \varepsilon^m_{x,y}(\xi)\, w(\xi+z) d\xi dz. \end{aligned}$$

For each $m \in \mathbb{N}$ sufficiently large we have

$$\frac{1}{\pi} \int_{B_{1/n}} \int_{\Theta_m([x,y])} \rho_n(-z)\, \varepsilon^m_{x,y}(\xi)\, w(\xi+z) d\xi dz = \frac{1}{\pi} \int_{\Theta_m([x,y])} \varepsilon^m_{x,y}(\xi)\, \rho_n * w(\xi) d\xi,$$

and since $\rho_n * w$ is smooth, we obtain as in Step 1,

$$\frac{1}{\pi} \int_{\Theta_m([x,y])} \varepsilon^m_{x,y}(\xi)\, \rho_n * w(\xi) d\xi \to \int_{[x,y]} \rho_n * w(s) ds \quad \text{as } m \to +\infty.$$

Thus for each $x, y \in B_R(z_0)$ we have

$$|\zeta_n(x) - \zeta_n(y)| \leq \int_{[x,y]} \rho_n * w(s) ds.$$

Then for $x \in B_R(z_0)$, $h \in S^2$ fixed and $\delta > 0$ small, we infer that

$$\frac{|\zeta_n(x+\delta h) - \zeta_n(x)|}{\delta} \leq \frac{1}{\delta} \int_{[x,x+\delta h]} \rho_n * w(s) ds \underset{\delta\to 0^+}{\longrightarrow} \rho_n * w(x)$$

and we conclude, letting $\delta \to 0$, that $|\nabla\zeta_n(x) \cdot h| \leq \rho_n * w(x)$ for each $x \in B_R(z_0)$ and $h \in S^2$ which implies that $|\nabla\zeta_n| \leq \rho_n * w$ in $B_R(z_0)$. Since $\nabla\zeta_n \to \nabla\zeta$ and $\rho_n * w \to w$ a.e. on $B_R(z_0)$ as $n \to +\infty$, we deduce that $|\nabla\zeta| \leq w$ a.e. in $B_R(z_0)$. Since z_0 is arbitrary in $\Omega \setminus (\mathbb{P} \cup \{y_0\})$, we derive

$$|\nabla\zeta| \leq w \quad \text{a.e. in } \Omega.$$

By Proposition 1.3 in Chapter 1, it follows that $|\zeta(x) - \zeta(y)| \leq d_w(x,y)$ for any $x, y \in \overline{\Omega}$ and in particular, we obtain choosing $y = x_0$,

$$\mathcal{D}_w(x, x_0) \leq d_w(x, x_0) \quad \text{for any } x \in \overline{\Omega}.$$

Step 4. End of the Proof. Let $\delta > 0$ be given. We choose some $\tilde{y}_0 \in \Omega \setminus (\mathbb{P} \cup \{y_0\})$ such that $[\tilde{y}_0, y_0] \subset \Omega \setminus \mathbb{P}$ and $|\tilde{y}_0 - y_0| \leq \frac{\delta}{3\Lambda}$. By the previous step, we can find an element $\mathcal{F}' = ([\alpha_1, \beta_1], \ldots, [\alpha_n, \beta_n]) \in \mathcal{Q}'(x_0, \tilde{y}_0)$ verifying

$$\overline{\ell}_w\left(\mathcal{F}'\right) \leq d_w(x_0, \tilde{y}_0) + \frac{\delta}{3}.$$

Then we consider $\mathcal{F} = ([\alpha_1, \beta_1], \ldots, [\alpha_n, \beta_n], [\tilde{y}_0, y_0]) \in \mathcal{Q}(x_0, y_0)$. We have

$$\begin{aligned}\bar{\ell}_w(\mathcal{F}) \le \bar{\ell}_w(\mathcal{F}') + \Lambda|\tilde{y}_0 - y_0| &\le d_w(x_0, \tilde{y}_0) + \frac{2\delta}{3} \\ &\le d_w(x_0, y_0) + d_w(y_0, \tilde{y}_0) + \frac{2\delta}{3} \\ &\le d_w(x_0, y_0) + \delta\end{aligned}$$

and then $\mathcal{F}$ satisfies the requirement. ■

2.2.2 The dipole removing technique

We first present the *dipole removing technique* for a single dipole. We then treat the case of several point singularities.

Lemma 2.2. *Let P and N be two distinct points in Ω and $u \in H^1(\Omega, S^2) \cap C^1\left(\overline{\Omega} \setminus \{P, N\}\right)$ with* $\deg(u, P) = +1$ *and* $\deg(u, N) = -1$. *Let $\mathcal{F} = ([\alpha_1, \beta_1], \ldots, [\alpha_n, \beta_n])$ be an element of $\mathcal{Q}(P, N)$ such that $N \notin \cup_{k=1}^{n-1}[\alpha_k, \beta_k] \cup [\alpha_n, \beta_n[$. Then for any $\delta > 0$ small enough, there exists a map $u_\delta \in C^1\left(\overline{\Omega}, S^2\right)$ such that :*

$$\int_\Omega |\nabla u_\delta(x)|^2 w(x)dx \le \int_\Omega |\nabla u(x)|^2 w(x)dx + 8\pi\bar{\ell}_w(\mathcal{F}) + \delta$$

and u_δ coincides with u outside a δ-neighborhood of $\cup_{k=1}^{n}[\alpha_k, \beta_k]$ included in Ω.

Proof. Let $\mathcal{F} = ([\alpha_1, \beta_1], \ldots, [\alpha_n, \beta_n]) \in \mathcal{Q}(P, N)$ such that $N \notin \cup_{k=1}^{n-1}[\alpha_k, \beta_k] \cup [\alpha_n, \beta_n[$ and fix some $\delta > 0$ small. We proceed in several steps.

Step 1. We consider a small $0 < r_0 < \delta$ verifying $B_{r_0}(\alpha_1) \subset \Omega \setminus \{N\}$. By Lemma A.1 in [16], we can find $v \in C^1\left(\overline{\Omega} \setminus \{\alpha_1, N\}, S^2\right) \cap H^1(\Omega)$ (recall that $\alpha_1 = P$) satisfying

$$v(x) = \begin{cases} u(x) & \text{on } \Omega \setminus B_{r_0}(\alpha_1), \\ R\left(\dfrac{x - \alpha_1}{|x - \alpha_1|}\right) & \text{on } B_{r_0}(\alpha_1), \end{cases} \tag{2.12}$$

for some rotation R and

$$\int_\Omega |\nabla v(x)|^2 w(x)dx \le \int_\Omega |\nabla u(x)|^2 w(x)dx + \delta. \tag{2.13}$$

Let $W = \{x \in \mathbb{R}^3, \operatorname{dist}(x, [\alpha_1, \beta_1]) < \delta\}$. For δ small enough, we have $\overline{W} \subset \Omega \setminus \{N\}$. We set $d = |\alpha_1 - \beta_1|$. We choose normal coordinates such that $\alpha_1 = (0, 0, 0)$ and $\beta_1 = (0, 0, d)$. Let $0 < r < \frac{r_0}{2}$. Since v is smooth on $W \setminus B_{r_0}(\alpha_1)$, we can find a constant $\sigma(r)$ such that $|\nabla v| \le \sigma(r)$ on $W \setminus B_{r_0}(\alpha_1)$. For $m \in \mathbb{N}^*$, we consider

$$K_m = \left[-\frac{a_m^{\alpha_1,\beta_1}}{2}, \frac{a_m^{\alpha_1,\beta_1}}{2}\right]^2 \times \left[-\frac{a_m^{\alpha_1,\beta_1}}{2}, d + \frac{a_m^{\alpha_1,\beta_1}}{2}\right].$$

For m large enough, we have $\Theta_m([\alpha_1,\beta_1]) \subset K_m \subset W$. As in [16], we are going to construct in the next step a map $v_1 \in C^1\left(\overline{W}\setminus\{\beta_1\}, S^2\right) \cap H^1(W)$ verifying $v_1 = v$ in a neighborhood of ∂W and $\deg(v_1,\beta_1) = +1$. For simplicity, we drop the indices α_1 and β_1.

Step 2. We divide K_m in $m+1$ cubes Q_m^j defined by

$$Q_m^j = \left[-\frac{a_m}{2}, \frac{a_m}{2}\right]^2 \times \left[\left(j-\frac{1}{2}\right)a_m, \left(j+\frac{1}{2}\right)a_m\right] \quad \text{for } j = 0, \dots, m.$$

Arguing as in [16], we infer from (2.12) that

$$\sum_{j=0}^{m} \int_{\partial Q_m^j} |\nabla v|^2 \leq C\left(\frac{r}{a_m} + m\,\sigma(r)^2 a_m^2\right). \tag{2.14}$$

We are going to make use of a map $\omega_m : B_{a_m}^2(0) \subset \mathbb{R}^2 \to S^2$ defined by

$$\omega_m(x_1,x_2) = \frac{2a_m^2}{a_m^4 + x_1^2 + x_2^2}\left(x_1, x_2, -a_m^2\right) + (0,0,1)$$

(ω_m was first introduced in [29] and we refer to the proof of Lemma 2 in [29] for its main properties). For $j = 1, \dots, m$, we choose an orthonormal direct basis (e_1^j, e_2^j, e_3^j) of $\mathbb{R}^3$ such that

$$v(0,0,(j-1/2)a_m) = (0,0,1) \quad \text{in the basis } (e_1^j, e_2^j, e_3^j),$$

and we define the map $v_1^m : \cup_{j=0}^m \partial Q_m^j \to S^2$ by

1) for $(x_1,x_2,x_3) \in (\cup_{j=0}^m \partial Q_m^j) \setminus (\cup_{j=1}^m B_{a_m^2}^2(0) \times \{(j-1/2)a_m\})$,

$$v_1^m(x_1,x_2,x_3) = v(x_1,x_2,x_3),$$

2) for $j = 1, \dots, m$ and $(x_1,x_2,x_3) \in B_{\frac{a_m^2}{2}}^2(0) \times \{(j-1/2)a_m\}$,

$$v_1^m(x_1,x_2,x_3) = \omega_m\left(\frac{2x_1}{a_m}, \frac{2x_2}{a_m}\right) \quad \text{in the basis } (e_1^j, e_2^j, e_3^j),$$

3) for $j = 1, \dots, m$, for $(x_1,x_2,x_3) \in (B_{a_m^2}^2(0) \setminus B_{\frac{a_m^2}{2}}^2(0)) \times \{(j-1/2)a_m\}$ and using cylindrical coordinates $(x_1,x_2,x_3) = (\rho\cos\theta, \rho\sin\theta, z)$,

$$v_1^m(x_1,x_2,x_3) = \left(A_1\rho + B_1, A_2\rho + B_2, \sqrt{1-(A_1\rho+B_1)^2-(A_2\rho+B_2)^2}\right)$$

in the basis (e_1^j, e_2^j, e_3^j), where A_1, A_2, B_1, B_2 are determined to make v_1^m continuous. More precisely, if we write $v = v_1 e_1^j + v_2 e_2^j + v_3 e_3^j$ then

$$\begin{cases} a_m^2 A_i(\theta) + B_i(\theta) = v_i(a_m^2\cos\theta, a_m^2\sin\theta, (j-1/2)a_m) \quad i = 1,2, \\[2mm] \dfrac{a_m^2}{2}A_1(\theta) + B_1(\theta) = \dfrac{2a_m^3}{a_m^4 + a_m^2}\cos\theta, \\[2mm] \dfrac{a_m^2}{2}A_2(\theta) + B_2(\theta) = \dfrac{2a_m^3}{a_m^4 + a_m^2}\sin\theta. \end{cases}$$

The map v_1^m satisfies by construction $v_1^m = v$ on ∂K_m. Moreover, it follows exactly as in the proof of Lemma 2 in [16] that $\deg(v_1^m, \partial Q_m^j) = 0$ for $j = 0, \dots, m-1$ and $\deg(v_1^m, \partial Q_m^m) = +1$. Then we extend v_1^m on each cube Q_m^j by setting

$$v_1^m(x) = v_1^m\left(\frac{a_m(x-b_j)}{2\|x-b_j\|_\infty} + b_j\right) \quad \text{on } Q_m^j \ \text{ for } j = 0, \dots, m,$$

where $b_j = (0, 0, s_j)$ is the barycenter of Q_m^j and $\|x - b_j\|_\infty = \max(|x_1|, |x_2|, |x_3 - s_j|)$. We easily check that $v_1^m \in H^1(K_m, S^2)$, $v_1^m = v$ on ∂K_m, v_1^m is continuous except at the points b_j and Lipschitz continuous outside any small neighborhood of the points b_j. We also get that

$$\deg(v_1^m, b_m) = +1 \quad \text{and} \quad \deg(v_1^m, b_j) = 0 \quad \text{for } j = 0, \dots, m-1\,. \tag{2.15}$$

We remark that if we set

$$D_m^j = B^2_{\frac{a_m^2}{2}}(0) \times \{(j-1/2)a_m\} \cup B^2_{\frac{a_m^2}{2}}(0) \times \{(j+1/2)a_m\} \quad \text{for } j = 1, \dots, m-1,$$

$$D_m^0 = B^2_{\frac{a_m^2}{2}}(0) \times \{1/2\, a_m\} \quad \text{and} \quad D_m^m = B^2_{\frac{a_m^2}{2}}(0) \times \{(m-1/2)a_m\}\,,$$

then we have

$$\bigcup_{j=0}^m \left\{ x \in Q_m^j,\ \frac{a_m(x-b_j)}{2\|x-b_j\|_\infty} + b_j \in D_m^j \text{ if } x \neq b_j \text{ or } x = b_j \text{ otherwise} \right\} = \Theta_m\left([\alpha_1, \beta_1]\right)$$

and if $x \in Q_m^j \cap \Theta_m([\alpha_1, \beta_1])$ for some $j \in \{0, \dots, m\}$ then

$$h_m(x) = |x_3 - s_j| = \|x - b_j\|_\infty \quad \text{and} \quad r(x) = \sqrt{x_1^2 + x_2^2}\,. \tag{2.16}$$

Some classical computations (see [16] and [29]) lead to, for $j = 0, \dots, m$,

$$\int_{(\partial Q_m^j)\setminus D_m^j} |\nabla v_1^m|^2 \le \int_{\partial Q_m^j} |\nabla v|^2 + \mathcal{O}(a_m^2)$$

and therefore

$$\int_{Q_m^j \setminus \Theta_m([\alpha_1,\beta_1])} |\nabla v_1^m(x)|^2 w(x) dx \le C_1 \Lambda\, a_m \int_{\partial Q_m^j} |\nabla v|^2 + C_2 \Lambda\, a_m^3.$$

Adding these inequalities for $j = 0, \dots, m$ and combining with (2.14) we obtain

$$\int_{K_m \setminus \Theta_m([\alpha_1,\beta_1])} |\nabla v_1^m(x)|^2 w(x) dx \le C\Lambda\left(r + m\sigma(r)^2 a_m^3 + a_m^2\right). \tag{2.17}$$

For $x \in Q_m^j \cap \Theta_m([\alpha_1, \beta_1])$ for some $j \in \{0, \dots, m\}$, we have

$$v_1^m(x) = \begin{cases} \omega_m\left(\dfrac{x_1}{|x_3 - s_j|}, \dfrac{x_2}{|x_3 - s_j|}\right) & \text{in the basis } (e_1^{j+1}, e_2^{j+1}, e_3^{j+1}) \text{ if } x_3 - s_j > 0, \\[2ex] \omega_m\left(\dfrac{x_1}{|x_3 - s_j|}, \dfrac{x_2}{|x_3 - s_j|}\right) & \text{in the basis } (e_1^j, e_2^j, e_3^j) \text{ otherwise.} \end{cases}$$

Following the computations in [27], we infer that

$$|\nabla v_1^m(x)|^2 \le \frac{1+Ca_m^2}{|x_3-s_j|^2}\left|\nabla\omega_m\left(\frac{x_1}{|x_3-s_j|},\frac{x_2}{|x_3-s_j|}\right)\right|^2 \quad \text{in } Q_m^j\cap\Theta_m([\alpha_1,\beta_1]).$$

Since we have (see [29])

$$\left|\nabla\omega_m\left(\frac{x_1}{|x_3-s_j|},\frac{x_2}{|x_3-s_j|}\right)\right|^2 = \frac{8|x_3-s_j|^4a_m^4}{(|x_3-s_j|^2a_m^4+x_1^2+x_2^2)^2},$$

we derive that

$$\int_{Q_m^j\cap\Theta_m([\alpha_1,\beta_1])}|\nabla v_1^m(x)|^2w(x)dx \le \int_{Q_m^j\cap\Theta_m([\alpha_1,\beta_1])}\frac{8|x_3-s_j|^2a_m^4w(x)}{(|x_3-s_j|^2a_m^4+x_1^2+x_2^2)^2}dx + C\Lambda a_m^3.$$

Summing these inequalities for $j=0,\dots,m$ and using (2.16) we obtain that

$$\int_{\Theta_m([\alpha_1,\beta_1])}|\nabla v_1^m(x)|^2w(x)dx \le 8\int_{\Theta_m([\alpha_1,\beta_1])}\varepsilon^m_{\alpha_1,\beta_1}(x)w(x)dx + C\Lambda a_m^2 \tag{2.18}$$

Combining (2.17) with (2.18) we conclude that

$$\int_{K_m}|\nabla v_1^m(x)|^2w(x)dx \le 8\int_{\Theta_m([\alpha_1,\beta_1])}\varepsilon^m_{\alpha_1,\beta_1}(x)w(x)dx + C\Lambda\left(r+m\sigma(r)^2a_m^3+a_m^2\right).$$

Taking the lim inf in m, we derive that we can find $m_1\in\mathbb{N}$ large and r small enough such that

$$\int_{K_{m_1}}|\nabla v_1^{m_1}(x)|^2w(x)dx \le 8\liminf_{m\to+\infty}\int_{\Theta_m([\alpha_1,\beta_1])}\varepsilon^m_{\alpha_1,\beta_1}(x)w(x)dx + \delta. \tag{2.19}$$

Since $v_1^{m_1}=v$ on ∂K_{m_1}, we may extend $v_1^{m_1}$ to W by setting $v_1^{m_1}=v$ on $W\setminus K_{m_1}$. Now we recall that $v_1^{m_1}$ is singular only at the points b_j, $j=0,\dots,m$ (we also recall that $b_m=\beta_1$). From (2.15) and the results in [16, 17, 22], we infer that exists a map v_1 in $C^1\left(\overline{W}\setminus\{\beta_1\},S^2\right)\cap H^1(W)$ satisfying $v_1=v$ in a neighborhood of ∂W, $\deg(v_1,\beta_1)=+1$ and

$$\int_{W_1}|\nabla v_1(x)|^2w(x)dx \le \int_{W_1}|\nabla v_1^{m_1}(x)|^2w(x)dx + \delta. \tag{2.20}$$

Since $v=u$ in a neighborhood of ∂W, we may extend v_1 to $\overline{\Omega}$ by setting $v_1=u$ in $\overline{\Omega}\setminus W$. Then we conclude that $v_1\in C^1\left(\overline{\Omega}\setminus\{\beta_1,N\},S^2\right)\cap H^1(\Omega)$, $\deg(v_1,\beta_1)=+1$, $\deg(v_1,N)=-1$ and by (2.13)-(2.19)-(2.20),

$$\int_\Omega|\nabla v_1(x)|^2w(x)dx \le \int_\Omega|\nabla u(x)|^2w(x)dx + 8\liminf_{m\to+\infty}\int_{\Theta_m([\alpha_1,\beta_1])}\varepsilon^m_{\alpha_1,\beta_1}(x)w(x)dx + C\delta.$$

Step 3. Applying Step 1 and Step 2 to v_1 instead of u and replacing (α_1,β_1) by (α_2,β_2) (recall that $\beta_1=\alpha_2$), we obtain a map $v_2\in C^1\left(\overline{\Omega}\setminus\{\beta_2,N\},S^2\right)\cap H^1(\Omega)$ satisfying $v_2=v_1$ outside a δ-neighborhood of $[\alpha_2,\beta_2]$ included in Ω, $\deg(v_2,\beta_2)=+1$, $\deg(v_2,N)=-1$ and

$$\int_\Omega|\nabla v_2(x)|^2w(x)dx \le \int_\Omega|\nabla v_1(x)|^2w(x)dx + 8\liminf_{m\to+\infty}\int_{\Theta_m([\alpha_2,\beta_2])}\varepsilon^m_{\alpha_2,\beta_2}(x)w(x)dx + C\delta.$$

Iterating this process, we finally obtain a map $v_{n-1} \in C^1\left(\overline{\Omega} \setminus \{\alpha_n, \beta_n\}, S^2\right) \cap H^1(\Omega)$ (recall that $\beta_n = N$) verifying $v_{n-1} = u$ outside a δ-neighborhood of $\cup_{k=1}^{n-1}[\alpha_k, \beta_k]$ included in Ω, $\deg(v_{n-1}, \alpha_n) = +1$, $\deg(v_{n-1}, \beta_n) = -1$ and

$$\int_\Omega |\nabla v_{n-1}(x)|^2 w(x)dx \leq \int_\Omega |\nabla u(x)|^2 w(x)dx + 8\sum_{k=1}^{n-1} \liminf_{m\to+\infty} \int_{\Theta_m([\alpha_k,\beta_k])} \varepsilon^m_{\alpha_k,\beta_k}(x)w(x)dx + C\delta.$$

As in Step 1, we consider $0 < r_0 < \delta$ such that

$$B_{r_0}(\alpha_n) \cap B_{r_0}(\beta_n) = \emptyset \quad \text{and} \quad B_{r_0}(\alpha_n) \cup B_{r_0}(\beta_n) \subset \Omega$$

and we construct, using Lemma A1 in [16], a map $\tilde{v} \in C^1\left(\overline{\Omega} \setminus \{\alpha_n, \beta_n\}, S^2\right) \cap H^1(\Omega)$ satisfying

$$\tilde{v}(x) = \begin{cases} u(x) & \text{on } \Omega \setminus B_{r_0}(\alpha_n), \\ R_+\left(\dfrac{x-\alpha_n}{|x-\alpha_n|}\right) & \text{on } B_{r_0}(\alpha_n), \\ -R_-\left(\dfrac{x-\beta_n}{|x-\beta_n|}\right) & \text{on } B_{r_0}(\beta_n), \end{cases}$$

for some rotations R_+ and R_- and

$$\int_\Omega |\nabla \tilde{v}(x)|^2 w(x)dx \leq \int_\Omega |\nabla v_{n-1}(x)|^2 w(x)dx + \delta.$$

Applying the construction in Step 2 starting from $\tilde{v}$, we obtain a new map $\tilde{v}_n^{m_n}$ (for some large $m_n \in \mathbb{N}$) defined on δ-neighborhood W' of $[\alpha_n, \beta_n]$ included in Ω, which coincide with $\tilde{v}$ near $\partial W'$, which then has only point singularities of degree zero (since $\deg(\tilde{v}, \beta_n) = -1$) and satisfying

$$\int_{W'} |\nabla v_n^{m_n}(x)|^2 w(x)dx \leq \int_{W'} |\nabla \tilde{v}(x)|^2 w(x)dx + 8\liminf_{m\to+\infty} \int_{\Theta_m([\alpha_n,\beta_n])} \varepsilon^m_{\alpha_n,\beta_n}(x)w(x)dx + C\delta.$$

Since the degree of each singularities of $v_n^{m_n}$ is zero, we can construct a map v_n in $C^1(\overline{W}', S^2)$ (see [17, 22]) verifying $v_n = \tilde{v}$ in a neighborhood of $\partial W'$ and

$$\int_{W'} |\nabla v_n(x)|^2 w(x)dx \leq \int_{W'} |\nabla v_n^{m_n}(x)|^2 w(x)dx + \delta.$$

Then we define $u_\delta : \overline{\Omega} \to S^2$ by

$$u_\delta(x) = \begin{cases} v_{n-1}(x) & \text{if } x \in \overline{\Omega} \setminus W', \\ v_n(x) & \text{if } x \in \overline{W}'. \end{cases}$$

Since $v_{n-1} = \tilde{v}$ and $\tilde{v} = v_{n-1}$ near $\partial W'$, we deduce that $u_\delta \in C^1(\overline{\Omega}, S^2)$. Moreover it follows by construction that $u_\delta = u$ outside a δ-neighborhood of $\cup_{k=1}^{n}[\alpha_k, \beta_k]$ included in Ω and

$$\int_\Omega |\nabla u_\delta(x)|^2 w(x)dx \leq \int_\Omega |\nabla u(x)|^2 w(x)dx + 8\pi\overline{\ell}(\mathcal{F}) + C\delta,$$

which ends the proof since δ is arbitrary small. ■

Lemma 2.3. *Let $(P_i, N_i)_{i=1}^K$ be $2K$ distinct points in Ω and consider $u \in H^1(\Omega, S^2) \cap C^1(\overline{\Omega} \setminus \cup_{i=1}^K \{P_i, N_i\})$ such that $\deg(u, P_i) = +1$ and $\deg(u, N_i) = -1$ for $i = 1, \dots, K$. There exists a sequence of maps $(u_n)_{n\in\mathbb{N}} \subset C^1(\overline{\Omega}, S^2)$ satisfying $u_{n|\partial\Omega} = u_{|\partial\Omega}$,*

$$\int_\Omega |\nabla u_n(x)|^2 w(x)dx \le \int_\Omega |\nabla u(x)|^2 w(x)dx + 8\pi L_w(u) + 2^{-n},$$

and

$$\text{meas}\left(\{x \in \Omega,\ u_n(x) \ne u(x)\}\right) \le 2^{-n}.$$

Proof. Without loss of generality we may assume that $\sum_i d_w(P_i, N_i)$ is equal to the length of a minimal connection relative to d_w between the points (P_i) and (N_i). As in [16], we are going to "remove" each dipole (P_i, N_i). More precisely, for each $n \in \mathbb{N}$, we construct successively K maps $(u_n^i)_{i=1}^K$ satisfying

(a) $u_n^i \in H^1(\Omega, S^2) \cap C^1\left(\overline{\Omega} \setminus \bigcup_{i+1\le j\le K}\{P_j, N_j\}\right)$ for $i = 1, \dots, K$,

(b) $u_n^1 = u$ on $\overline{\Omega} \setminus W_n^1$ and $u_n^i = u_n^{i-1}$ on $\overline{\Omega} \setminus W_n^i$ for $i = 2, \dots, K$ where W_n^i is is strictly included in $\Omega \setminus \bigcup_{i+1\le j\le K}\{P_j, N_j\}$ and $|W_n^i| \le 2^{-n}/K$,

(c) $\displaystyle\int_\Omega |\nabla u_n^1(x)|^2 w(x)dx \le \int_\Omega |\nabla u(x)|^2 w(x)dx + 8\pi d_w(P_1, N_1) + \frac{2^{-n}}{K}$ and

$$\int_\Omega |\nabla u_n^i(x)|^2 w(x)dx \le \int_\Omega |\nabla u_n^{i-1}(x)|^2 w(x)dx + 8\pi d_w(P_i, N_i) + \frac{2^{-n}}{K} \text{ for } i = 2, \dots, K.$$

We easily check that the sequence $(u_n^K)_{n\in\mathbb{N}}$ then satisfies the requirement since we have $L_w(u) = \sum_i d_w(P_i, N_i)$. We start with the construction of u_n^1.

Construction of u_n^1. By Lemma 2.1, we can find $\mathcal{F}_1 = ([\alpha_1, \beta_1], \dots, [\alpha_l, \beta_l]) \in \mathcal{Q}(P_1, N_1)$ satisfying

$$\left(\cup_{i=2}^K \{P_i, N_i\} \cup \{N_1\}\right) \cap \left(\cup_{k=2}^l [\alpha_k, \beta_k] \cup [\alpha_1, \beta_1[\right) = \emptyset, \tag{2.21}$$

and

$$\overline{\ell}_w(\mathcal{F}_1) \le d_w(P_1, N_1) + \frac{2^{-(n+1)}}{8K\pi}.$$

From (2.21), we infer that we can find $\delta > 0$ small enough such that

$$W_\delta^1 = \{x \in \mathbb{R}^3,\ \text{dist}(x, \cup_{k=1}^l [\alpha_k, \beta_k]) \le \delta\} \subset \Omega \setminus \cup_{i=2}^K \{P_i, N_i\} \quad \text{and} \quad |W_\delta^1| \le \frac{2^{-n}}{K}.$$

By the method described in the proof of Lemma 2.2, we construct a map $u_n^1 \in H^1(\Omega, S^2) \cap C^1(\overline{\Omega} \setminus \cup_{i=2}^K \{P_i, N_i\})$ verifying $u_n^1 = u$ outside W_δ^1 and

$$\begin{aligned}\int_\Omega |\nabla u_n^1(x)|^2 w(x)dx &\le \int_\Omega |\nabla u(x)|^2 w(x)dx + 8\pi\overline{\ell}_w(\mathcal{F}_1) + \frac{2^{-(n+1)}}{K}\\ &\le \int_\Omega |\nabla u(x)|^2 w(x)dx + 8\pi d_w(P_1, N_1) + \frac{2^{-n}}{K}.\end{aligned}$$

Construction of u_n^i, $i = 2, \dots, K$. We iterate the previous process, i.e., we proceed as for the construction of u_n^1 but starting from u_n^{i-1} instead of u. ■

2.3 Proof of Theorem 2.2

2.3.1 Lower bound of the energy

In this section, we denote by F_w the functional defined for maps $u \in H_g^1(\Omega, S^2)$ by

$$F_w(u) = \int_\Omega |\nabla u(x)|^2 w(x)dx + 8\pi L_w(u).$$

Proposition 2.1. *The functional F_w is sequentially lower semi-continuous on $H_g^1(\Omega, S^2)$ for the weak H^1-topology.*

Proof. We follow the method in [18]. Since the supremum of a family of sequentially lower semi-continuous functionals is sequentially lower semi-continuous, it suffices to show that for any function $\zeta : \overline{\Omega} \to \mathbb{R}$ which is 1-Lipschitz with respect to d_w, the functional

$$u \in H_g^1 \mapsto \int_\Omega |\nabla u(x)|^2 w(x)dx + 2\int_\Omega D(u) \cdot \nabla\zeta \, dx$$

is sequentially lower semi-continuous for the weak H^1-topology (the term $\int_{\partial\Omega}(D(u)\cdot\nu)\zeta$ only depends on g and ζ). Consider $(u_n)_{n\in\mathbb{N}} \subset H_g^1(\Omega, S^2)$ and $u \in H_g^1(\Omega, S^2)$ such that $u_n \rightharpoonup u$ weakly in H^1. Setting $v_n = u_n - u$, we have

$$\int_\Omega |\nabla u_n(x)|^2 w(x)dx = \int_\Omega |\nabla u(x)|^2 w(x)dx + \int_\Omega |\nabla v_n(x)|^2 w(x)dx + o(1),$$

and writing

$$2\int_\Omega D(u_n) \cdot \nabla\zeta \, dx = A_n + B_n + C_n$$

with

$$A_n = 2\int_\Omega u_n \cdot \left(\frac{\partial u}{\partial x_2} \wedge \frac{\partial u}{\partial x_3}\frac{\partial \zeta}{\partial x_1} + \frac{\partial u}{\partial x_3} \wedge \frac{\partial u}{\partial x_1}\frac{\partial \zeta}{\partial x_3} + \frac{\partial u}{\partial x_1} \wedge \frac{\partial u}{\partial x_2}\frac{\partial \zeta}{\partial x_3} \right),$$

$$B_n = 2\int_\Omega u_n \cdot \left(\frac{\partial v_n}{\partial x_2} \wedge \frac{\partial u}{\partial x_3} + \frac{\partial u}{\partial x_2} \wedge \frac{\partial v_n}{\partial x_3} \right) \frac{\partial \zeta}{\partial x_1} + 2\int_\Omega u_n \cdot \left(\frac{\partial v_n}{\partial x_3} \wedge \frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_3} \wedge \frac{\partial v_n}{\partial x_1} \right) \frac{\partial \zeta}{\partial x_2}$$
$$+ 2\int_\Omega u_n \cdot \left(\frac{\partial v_n}{\partial x_1} \wedge \frac{\partial u}{\partial x_2} + \frac{\partial u}{\partial x_1} \wedge \frac{\partial v_n}{\partial x_2} \right) \frac{\partial \zeta}{\partial x_3},$$

$$C_n = 2\int_\Omega u_n \cdot \left(\frac{\partial v_n}{\partial x_2} \wedge \frac{\partial v_n}{\partial x_3}\frac{\partial \zeta}{\partial x_1} + \frac{\partial v_n}{\partial x_3} \wedge \frac{\partial v_n}{\partial x_1}\frac{\partial \zeta}{\partial x_3} + \frac{\partial v_n}{\partial x_1} \wedge \frac{\partial v_n}{\partial x_2}\frac{\partial \zeta}{\partial x_3} \right).$$

We easily obtain that $A_n \to 2\int_\Omega D(u)\cdot\nabla\zeta$ as $n\to+\infty$ since $u_n \rightharpoonup u$ weak$\star$ in L^∞ and that $B_n \to 0$ since $v_n \rightharpoonup 0$ weakly in L^2 and $u_n \to u$ strongly in L^2. Now we set

$$V_n = \left(u_n\cdot\frac{\partial v_n}{\partial x_2}\wedge\frac{\partial v_n}{\partial x_3},\, u_n\cdot\frac{\partial v_n}{\partial x_3}\wedge\frac{\partial v_n}{\partial x_1},\, u_n\cdot\frac{\partial v_n}{\partial x_1}\wedge\frac{\partial v_n}{\partial x_2}\right).$$

We have

$$|C_n| = 2\left|\int_\Omega V_n\cdot\nabla\zeta\right| \le 2\int_\Omega |V_n||\nabla\zeta|.$$

By Lemma 1 in [18], we know that $2|V_n|\le|\nabla v_n|^2$ and by Proposition 1.3 in Chapter 1, any $\zeta:\overline{\Omega}\to\mathbb{R}$ which 1-Lipschitz with respect to d_w satisfies $|\nabla\zeta|\le w$ a.e. in Ω. Then we obtain

$$|C_n| \le \int_\Omega |\nabla v_n(x)|^2 w(x)dx$$

and we conclude that

$$\int_\Omega |\nabla u_n(x)|^2 w(x)dx + 2\int_\Omega D(u_n)\cdot\nabla\zeta\,dx \ge \int_\Omega |\nabla u(x)|^2 w(x)dx + 2\int_\Omega D(u)\cdot\nabla\zeta\,dx + o(1)$$

which clearly implies the result. ■

Proof of "≥" in Theorem 2.2. Let $u\in H^1_g(\Omega,S^2)$ and consider an arbitrary sequence $(u_n)_{n\in\mathbb{N}}\subset H^1_g(\Omega,S^2)\cap C^1(\overline{\Omega})$ such that $u_n\rightharpoonup u$ weakly in H^1. Since u_n is smooth in Ω, we have $T(u_n)\equiv 0$ and then $L_w(u_n)=0$. We conclude by Proposition 2.1 that

$$\liminf_{n\to+\infty}\int_\Omega |\nabla u_n(x)|^2 w(x)dx = \liminf_{n\to+\infty} F_w(u_n) \ge F_w(u) = \int_\Omega |\nabla u(x)|^2 w(x)dx + 8\pi L_w(u).$$

Since the sequence $(u_n)_{n\in\mathbb{N}}$ is arbitrary, we get the announced result. ■

2.3.2 Upper bound of the energy

Proposition 2.2. *Let $u\in H^1_g(\Omega,S^2)$. There exists a sequence $(u_n)_{n\in\mathbb{N}}\subset H^1_g(\Omega,S^2)\cap C^1(\overline{\Omega})$ such that $u_n\rightharpoonup u$ weakly in H^1 and*

$$\limsup_{n\to+\infty}\int_\Omega |\nabla u_n(x)|^2 w(x)dx \le \int_\Omega |\nabla u(x)|^2 w(x)dx + 8\pi L_w(u).$$

End of the proof of Theorem 2.2. Let $u\in H^1_g(\Omega,S^2)$ and let $(u_n)_{n\in\mathbb{N}}$ be the sequence of maps given by Proposition 2.2. By definition of $E_w(u)$ and Proposition 2.2, we have

$$E_w(u) \le \liminf_{n\to+\infty}\int_\Omega |\nabla u_n(x)|^2 w(x)dx \le \int_\Omega |\nabla u(x)|^2 w(x)dx + 8\pi L_w(u),$$

which ends the proof of Theorem 2.2. ■

To prove Proposition 2.2, we need the following Lemma. We postpone its proof at the end of this section.

Lemma 2.4. *For any $u, v \in H^1_g(\Omega, S^2)$, we have*

$$|L_w(u) - L_w(v)| \leq C\Lambda \left(\|\nabla u\|_{L^2(\Omega)} + \|\nabla v\|_{L^2(\Omega)}\right) \|\nabla u - \nabla v\|_{L^2(\Omega)}, \tag{2.22}$$

for a constant C independent of w.

Proof of Proposition 2.2. Let $u \in H^1_g(\Omega, S^2)$. By the result in [16, 22], we can find a sequence of maps $(v_n)_{n\in\mathbb{N}} \subset H^1_g(\Omega, S^2)$ such that $v_n \in C^1(\overline{\Omega} \setminus \cup_{i=1}^{K_n}\{P_i, N_i\})$ for some $2K_n$ distinct points (P_i, N_i) in Ω, $\deg(v_n, P_i) = +1$ and $\deg(v_n, N_i) = -1$ for $i = 1, \ldots, K_n$ and such that

$$\|\nabla(v_n - u)\|_{L^2(\Omega)} \leq 2^{-n}. \tag{2.23}$$

From this inequality we infer that

$$\text{meas}\left(\{x \in \Omega,\ |v_n(x) - u(x)| < 2^{-n/2}\}\right) \leq C\, 2^{-n}. \tag{2.24}$$

Applying Lemma 2.3 to v_n, we find a map $u_n \in C^1(\overline{\Omega}, S^2)$ satisfying $u_{n|\partial\Omega} = g$,

$$\int_\Omega |\nabla u_n(x)|^2 w(x)dx \leq \int_\Omega |\nabla v_n(x)|^2 w(x)dx + 8\pi L_w(v_n) + 2^{-n} \tag{2.25}$$

and

$$\text{meas}\left(\{x \in \Omega,\ u_n(x) \neq v_n(x)\}\right) \leq 2^{-n}. \tag{2.26}$$

From (2.23) and Lemma 2.4 we deduce that $L_w(v_n) \to L_w(u)$ as $n \to +\infty$ and then it follows that $(u_n)_{n\in\mathbb{N}}$ is bounded in H^1. Moreover we obtain from (2.24) and (2.26) that $u_n \to u$ a.e. in Ω and we conclude that $u_n \rightharpoonup u$ weakly in H^1. Letting $n \to +\infty$ in (2.25) leads to

$$\limsup_{n\to+\infty} \int_\Omega |\nabla u_n(x)|^2 w(x)dx \leq \int_\Omega |\nabla u(x)|^2 w(x)dx + 8\pi L_w(u),$$

which completes the proof. ■

Proof of Lemma 2.4. To prove Lemma 2.4, we follow the method in [18]. For u and v in $H^1_g(\Omega, S^2)$, we set

$$L_w(u, v) = \text{Sup}\left\{\int_\Omega (D(u) - D(v)) \cdot \nabla\zeta\,,\ \zeta : \overline{\Omega} \to \mathbb{R} \text{ 1-Lipschitz with respect to } d_w\right\}.$$

Since $D(u) \cdot \nu = D(v) \cdot \nu$ on $\partial\Omega$ (it only depends on g), we have

$$\int_\Omega D(u)\cdot\nabla\zeta - \int_{\partial\Omega} (D(u)\cdot\nu)\,\zeta = \int_\Omega D(v)\cdot\nabla\zeta - \int_{\partial\Omega} (D(v)\cdot\nu)\,\zeta + \int_\Omega (D(u) - D(v))\cdot\nabla\zeta\,,$$

and we easily derive that

$$|L_w(u) - L_w(v)| \leq L_w(u, v).$$

Similar computations to those in [18], proof of Theorem 1, lead to

$$\left|\int_\Omega (D(u) - D(v)) \cdot \nabla\zeta\right| \leq C\left(\|\nabla u\|_{L^2(\Omega)} + \|\nabla v\|_{L^2(\Omega)}\right) \|\nabla u - \nabla v\|_{L^2(\Omega)} \|\nabla\zeta\|_{L^\infty(\Omega)}.$$

By Proposition 1.3 in Chapter 1, any real function ζ which is 1-Lipschitz with respect to d_w satisfies $|\nabla\zeta| \leq w$ a.e. in Ω. We deduce that (2.22) holds since $w \leq \Lambda$ a.e. in Ω. ■

2.4 Stability and approximation properties

2.4.1 A stability property

Before stating the result, we need to recall some previous ones obtained in Chapter 1. For any real measurable function w satisfying assumption (2.1), we may associate to distance d_w the length functional $\mathbb{L}_{d_w}$ defined by

$$\mathbb{L}_{d_w}(\gamma) = \mathrm{Sup}\left\{\sum_{k=0}^{m-1} d_w\left(\gamma(t_k), \gamma(t_{k+1})\right), \, 0 = t_0 < t_1 < \ldots < t_m = 1, \, m \in \mathbb{N}^\star \right\},$$

where $\gamma : [0,1] \to \overline{\Omega}$ is any continuous curve. In Chapter 1, we have proved that for any $x, y \in \overline{\Omega}$,

$$d_w(x,y) = \mathrm{Inf}\left\{\mathbb{L}_{d_w}(\gamma), \, \gamma \in \mathrm{Lip}([0,1], \overline{\Omega}), \, \gamma(0) = x \text{ and } \gamma(1) = y\right\} \qquad (2.27)$$

where $\mathrm{Lip}([0,1], \overline{\Omega})$ denotes the class of all Lipschitz maps from $[0,1]$ into $\overline{\Omega}$. We have also shown that the infimum in (2.27) is in fact achieved.

The following stability result relies on the Γ-convergence of the length functionals (we refer to [41] for the notion of Γ-convergence). In the sequel, we endow $\mathrm{Lip}([0,1], \overline{\Omega})$ with the topology of the uniform convergence on $[0,1]$.

Theorem 2.3. *Let $(w_n)_{n\in\mathbb{N}}$ be a sequence of measurable real functions such that*

$$0 < c_0 \leq w_n \leq C_0 \quad \textit{a.e. in } \Omega \qquad (2.28)$$

for some constants c_0 and C_0 independent of $n \in \mathbb{N}$. The following properties are equivalent :

(i) the functionals $\mathbb{L}_{d_{w_n}}$ Γ-converge to $\mathbb{L}_{d_w}$ in $\mathrm{Lip}([0,1], \overline{\Omega})$ *and*

$$\int_\Omega |\nabla\varphi(x)|^2 w_n(x)dx \underset{n\to+\infty}{\longrightarrow} \int_\Omega |\nabla\varphi(x)|^2 w(x)dx \quad \textit{for any } \varphi \in H^1(\Omega, \mathbb{R}), \qquad (2.29)$$

(ii) for every smooth boundary data $g : \partial\Omega \to S^2$ such that $\deg(g) = 0$*,*

$$E_{w_n}(u) \underset{n\to+\infty}{\longrightarrow} E_w(u) \quad \textit{for any } u \in H^1_g(\Omega, S^2).$$

Proof. (i)⇒(ii). We fix a smooth boundary data $g : \Omega \to S^2$ such that $\deg(g) = 0$. Clearly (2.29) implies that

$$\int_\Omega |\nabla u(x)|^2 w_n(x)dx \underset{n\to+\infty}{\longrightarrow} \int_\Omega |\nabla u(x)|^2 w(x)dx \quad \text{for any } u \in H^1_g(\Omega, S^2),$$

and by Theorem 2.2, it remains to prove that

$$L_{w_n}(u) \underset{n\to+\infty}{\longrightarrow} L_w(u) \quad \text{for any } u \in H^1_g(\Omega, S^2). \qquad (2.30)$$

Consider $u \in H^1_g(\Omega, S^2)$. By the result in [16, 22], there exits a sequence of maps $(v_k)_{k\in\mathbb{N}} \subset H^1_g(\Omega, S^2)$ such that $v_k \in C^1(\overline{\Omega} \setminus \cup_{j=1}^{M_k}\{P_j, N_j\}, S^2)$ for some $2M_k$ points (P_j, N_j) in Ω, $\deg(v_k, P_j) = +1$ and $\deg(v_k, N_j) = -1$ for $j = 1, \dots, M_k$, and $v_k \to u$ strongly in H^1. We have

$$L_{w_n}(v_k) = \operatorname*{Min}_{\sigma\in\mathcal{S}_{M_k}} \sum_{j=1}^{M_k} d_{w_n}(P_j, N_{\sigma(j)}) \quad \text{and} \quad L_w(v_k) = \operatorname*{Min}_{\sigma\in\mathcal{S}_{M_k}} \sum_{j=1}^{M_k} d_w(P_j, N_{\sigma(j)})$$

Since the functionals $\mathbb{L}_{d_{w_n}}$ Γ-converge to $\mathbb{L}_{d_w}$ in $\mathrm{Lip}([0,1], \overline{\Omega})$, we deduce from Theorem 1.2 in Chapter 1 that for every $k \in \mathbb{N}$, $L_{w_n}(v_k) \to L_w(v_k)$ as $n \to +\infty$. Now we fix a small $\delta > 0$. Since $v_k \to u$ strongly in H^1, we derive from Lemma 2.4 and (2.28) that exists $k_0 \in \mathbb{N}$ which only depends on u, δ and C_0 such that

$$L_{w_n}(v_k) - \delta \le L_{w_n}(u) \le L_{w_n}(v_k) + \delta \quad \text{for any } n \in \mathbb{N} \text{ and } k \ge k_0.$$

Letting $n \to +\infty$ in this inequality, we get that

$$L_w(v_k) - \delta \le \liminf_{n\to+\infty} L_{w_n}(u) \le \limsup_{n\to+\infty} L_{w_n}(u) \le L_w(v_k) + \delta \quad \text{for } k \ge k_0.$$

Passing to the limit in k and using Lemma 2.4, we obtain

$$L_w(u) - \delta \le \liminf_{n\to+\infty} L_{w_n}(u) \le \limsup_{n\to+\infty} L_{w_n}(u) \le L_w(u) + \delta,$$

which leads to the result since δ is arbitrary small.

(ii)⇒(i). First we prove (2.29) for $\varphi \in C^\infty(\overline{\Omega}, \mathbb{R})$. Let $\varphi \in C^\infty(\overline{\Omega}, \mathbb{R})$ and consider the smooth map $g : \partial\Omega \to S^2$ defined by $g(x) = \big(\cos(\varphi(x)), \sin(\varphi(x)), 0\big)$. We easily check that $\deg(g) = 0$. Now consider the map u defined for $x \in \overline{\Omega}$ by

$$u(x) = (\cos(\varphi(x)), \sin(\varphi(x)), 0).$$

We have $u \in H^1_g(\Omega, S^2) \cap C^\infty(\overline{\Omega})$ and then $L_{w_n}(u) = L_w(u) = 0$ for any $n \in \mathbb{N}$. Since $|\nabla u|^2 = |\nabla\varphi|^2$, we derive from assumption *(ii)* and Theorem 2.2 that

$$\int_\Omega |\nabla\varphi(x)|^2 w_n(x)dx \underset{n\to+\infty}{\longrightarrow} \int_\Omega |\nabla\varphi(x)|^2 w(x)dx.$$

Let us now prove (2.29) for any $\varphi \in H^1(\Omega, \mathbb{R})$. Let $\varphi \in H^1(\Omega, \mathbb{R})$ and consider a sequence $(\varphi_k)_{k\in\mathbb{N}} \subset C^\infty(\overline{\Omega}, \mathbb{R})$ such that $\varphi_k \to \varphi$ strongly in H^1. We fix a small $\delta > 0$. From assumption (2.28), we infer that exists $k_0 \in \mathbb{N}$ which only depends on φ, δ and C_0 such that for any $n \in \mathbb{N}$ and $k \ge k_0$,

$$\int_\Omega |\nabla\varphi_k(x)|^2 w_n(x)dx - \delta \le \int_\Omega |\nabla\varphi(x)|^2 w_n(x)dx \le \int_\Omega |\nabla\varphi_k(x)|^2 w_n(x)dx + \delta.$$

Since φ_k is smooth, letting $n \to +\infty$ we obtain for $k \geq k_0$,

$$\int_\Omega |\nabla \varphi_k(x)|^2 w(x)dx - \delta \leq \liminf_{n\to+\infty} \int_\Omega |\nabla \varphi(x)|^2 w_n(x)dx$$
$$\leq \limsup_{n\to+\infty} \int_\Omega |\nabla \varphi(x)|^2 w_n(x)dx \leq \int_\Omega |\nabla \varphi_k(x)|^2 w(x)dx + \delta.$$

Passing to the limit in k and then $\delta \to 0$, we conclude

$$\lim_{n\to+\infty} \int_\Omega |\nabla \varphi(x)|^2 w_n(x)dx = \int_\Omega |\nabla \varphi(x)|^2 w(x)dx.$$

It remains to prove that the functionals $\mathbb{L}_{d_{w_n}}$ Γ-converge to $\mathbb{L}_{d_w}$ in $\mathrm{Lip}([0,1],\overline{\Omega})$. Let P and N be two distinct points in Ω. We take $g \equiv (0,0,1)$ and consider $u \in H^1_g(\Omega, S^2) \cap C^1(\overline{\Omega} \setminus \{P, N\})$ (such a map is constructed for instance in [27, 30]). By Theorem 2.2, we have

$$E_{w_n}(u) = \int_\Omega |\nabla u(x)|^2 w_n(x)dx + 8\pi d_{w_n}(P, N)$$

and

$$E_w(u) = \int_\Omega |\nabla u(x)|^2 w(x)dx + 8\pi d_w(P, N).$$

From (2.29) we get that $\int_\Omega |\nabla u(x)|^2 w_n(x)dx \to \int_\Omega |\nabla u(x)|^2 w(x)dx$ and from assumption *(ii)* we deduce that

$$d_{w_n}(P, N) \to d_w(P, N) \quad \text{as } n \to +\infty.$$

Since the points P and N are arbitrary in Ω, we derive that d_{w_n} converges to d_w pointwise in $\Omega \times \Omega$ and the conclusion follows by the results in Chapter 1, Section 1.4.1. ■

In the next proposition, we give some sufficient conditions on a sequence $(w_n)_{n\in\mathbb{N}}$ converging pointwise a.e. to w for property *(ii)* in Theorem 2.3 to hold.

Proposition 2.3. *Let $(w_n)_{n\in\mathbb{N}}$ be a sequence of measurable real functions satisfying (2.28) and assume that one of the following conditions holds :*

(a) $w_n \geq w$ and $w_n \to w$ a.e. in Ω,

(b) $w_n \to w$ in $L^\infty(\Omega)$.

Then property (ii) in Theorem 2.3 holds.

Proof. By Proposition 1.4 and Theorem 1.2 in Chapter 1, *(a)* or *(b)* implies that the functionals $\mathbb{L}_{d_{w_n}}$ Γ-converge to $\mathbb{L}_{d_w}$ in $\mathrm{Lip}([0,1],\overline{\Omega})$. We also check that *(a)* or *(b)* implies (2.29) by dominated convergence. Then the conclusion follows from Theorem 2.3. ■

Remark 2.1. The conclusion of Proposition 2.3 may fails if we only assumes that $w_n \to w$ a.e. in Ω (see Remark 1.4 in Chapter 1).

2.4.2 Approximation property

In this section, we show that the functional E_w can be obtain as pointwise limit of a sequence $(E_{w_n})_{n\in\mathbb{N}}$ in which the weight function w_n is smooth.

Proposition 2.4. *Let $(\rho_n)_{n\in\mathbb{N}}$ be a sequence of smooth mollifiers. Extending w by a sufficiently large constant and setting $w_n = \rho_n * w$, we have*

$$E_{w_n}(u) \underset{n\to+\infty}{\to} E_w(u) \quad \textit{for any } u \in H^1_g(\Omega, S^2).$$

Proof. By construction, (2.29) clearly holds. Then property *(i)* in Theorem 2.3 follows from Theorem 1.2 and Theorem 1.3 in Chapter 1 which leads to the result by Theorem 2.3. ∎

2.5 The relaxed energy without prescribed boundary data

In this section, we consider the relaxed type functional

$$\tilde{E}_w(u) = \operatorname{Inf}\left\{\liminf_{n\to+\infty}\int_\Omega |\nabla u_n(x)|^2 w(x)dx,\ u_n \in C^1(\overline{\Omega}, S^2),\ u_n \rightharpoonup u \text{ weakly in } H^1\right\}$$

defined for $u \in H^1(\Omega, S^2)$. We recall that F. Bethuel has also proved (see [16]) that $C^1(\overline{\Omega}, S^2)$ is sequentially dense in $H^1(\Omega, S^2)$ for the weak H^1 topology and then $\tilde{E}_w$ is well defined.

As in [18], there is also a notion of length of a minimal connection relative to d_w defined for any $u \in H^1(\Omega, S^2)$:

$$\tilde{L}_w(u) = \frac{1}{4\pi}\operatorname{Sup}\big\{\langle T(u), \zeta\rangle,\ \zeta : \overline{\Omega} \to \mathbb{R} \text{ 1-Lipschitz with respect to } d_w \text{ and } \zeta = 0 \text{ on } \partial\Omega\big\}.$$

Since no assumptions are made on $u_{|\partial\Omega}$, it may happen that $\deg(u_{|\partial\Omega}) \neq 0$ or that $\deg(u_{|\partial\Omega})$ is not well defined. But clearly $\tilde{L}_w(u)$ always makes sense. When u is smooth except at a finite number of point in Ω, $\tilde{L}_w(u)$ is equal to the length of a minimal connection relative to d_w between the singularities of u and some virtual singularities on the boundary (see [30]). More precisely, one adds some virtual singularities on the boundary in such a way that the new configuration has the same number of positive and negative points and one consider the length of a minimal connection relative to d_w for this configuration. Then $\tilde{L}_w(u)$ corresponds to the infimum of these quantities when one varies the position and the number of the boundary points. There is the variant of Theorem 2.2 for $\tilde{E}_w$.

Theorem 2.4. *For any $u \in H^1(\Omega, S^2)$, we have*

$$\tilde{E}_w(u) = \int_\Omega |\nabla u(x)|^2 w(x)dx + 8\pi\tilde{L}_w(u).$$

2.5.1 Proof of Theorem 2.4

The inequality "$\geq$" in Theorem 2.4 can be proved using a method similar to the one used in Section 2.3.1 and we omit it. We obtain "$\leq$" as in Section 2.3.2 using Proposition 2.5 and Lemma 2.5 below instead of Proposition 2.2 and Lemma 2.4. The proof of Lemma 2.5 is almost identical to the proof of Lemma 2.4 and we also omit it (note that all the boundary integrals vanish since $\zeta = 0$ on $\partial\Omega$).

Proposition 2.5. *Let $u \in H^1(\Omega, S^2)$. There exists a sequence $(u_n)_{n\in\mathbb{N}} \subset C^1(\overline{\Omega}, S^2)$ such that*

$$u_n \rightharpoonup u \quad \textit{weakly in } H^1$$

and

$$\limsup_{n\to+\infty} \int_\Omega |\nabla u_n(x)|^2 w(x)dx \leq \int_\Omega |\nabla u(x)|^2 w(x)dx + 8\pi \tilde{L}_w(u).$$

Lemma 2.5. *For any $u, v \in H^1(\Omega, S^2)$, we have*

$$\left|\tilde{L}_w(u) - \tilde{L}_w(v)\right| \leq C\Lambda \left(\|\nabla u\|_{L^2(\Omega)} + \|\nabla v\|_{L^2(\Omega)}\right) \|\nabla u - \nabla v\|_{L^2(\Omega)}, \tag{2.31}$$

for a constant C independent of w.

Proof of Proposition 2.5. Let $u \in H^1(\Omega, S^2)$. By the result in [16, 22], we can find a sequence $(v_n)_{n\in\mathbb{N}} \subset H^1(\Omega, S^2)$ such that $v_n \in C^1(\overline{\Omega} \setminus \{(a_i)_{i=1}^{N_n}\})$ for some N_n distinct points $a_1, \ldots, a_{N_n}$ in Ω and

$$\|u - v_n\|_{H^1(\Omega)} \leq 2^{-n}. \tag{2.32}$$

Since we are dealing with an approximating sequence, we may assume that (see [16])

$$|\deg(v_n, a_i)| = 1 \quad \text{for } i = 1, \ldots, N_n.$$

Since v_n is smooth except at a finite number of point in Ω, the length of a minimal connection $\tilde{L}_w(v_n)$ is computed as follows (see [30], part II). We pair each singularity a_i either to another singularity in Ω of opposite degree or to a virtual singularity on the boundary with opposite degree. In other words, we allow connections to the boundary of Ω. Pairing all the singularities in this way, we take a configuration that minimizes the sum of the distances between the paired singularities, computing the distances with d_w. We relabel all the singularities (the a_i's and the virtual singularities on the boundary), according to their multiplicity for those on the boundary, as a list of positive and negative points say $(P_1, \ldots, P_{K_n})$ and $(N_1, \ldots, N_{K_n})$ such that

$$\tilde{L}_w(v_n) = \sum_{j=1}^{K_n} d_w(P_j, N_j).$$

Using Lemma 2 bis in [16], we can find $\tilde{v}_n \in H^1(\Omega, S^2) \cap C^1(\overline{\Omega} \setminus \cup_{j=1}^{K_n}\{\tilde{P}_j, \tilde{N}_j\})$ for $2K_n$ distinct points $(\tilde{P}_j, \tilde{N}_j)$ in Ω such that $\tilde{v}_n = v_n$ outside a small neighborhood of $\partial\Omega$,

$\deg(\tilde{v}_n, \tilde{P}_j) = +1$ and $\deg(\tilde{v}_n, \tilde{N}_j) = -1$ for $j = 1, \dots, K_n$, $\tilde{P}_j = P_j$ (respectively $\tilde{N}_j = N_j$) if $P_j \in \Omega$ (respectively if $N_j \in \Omega$) and $|\tilde{P}_j - P_j| \leq \frac{2^{-n}}{K_n}$ otherwise (respectively $|\tilde{N}_j - N_j| \leq \frac{2^{-n}}{K_n}$), and

$$\|\tilde{v}_n - v_n\|_{H^1(\Omega)} \leq 2^{-n}. \tag{2.33}$$

Note that, for each pair (P_j, N_j), we necessarily have $\tilde{P}_j = P_j$ or $\tilde{N}_j = N_j$ and then

$$\left| \sum_{j=1}^{K_n} d_w(P_j, N_j) - \sum_{j=1}^{K_n} d_w(\tilde{P}_j, \tilde{N}_j) \right| \leq C\, 2^{-n}, \tag{2.34}$$

and from (2.32) and (2.33), we infer that

$$\text{meas}\left(\{x \in \Omega,\ |u(x) - \tilde{v}_n(x)| < 2^{-n/2}\} \right) \leq C\, 2^{-n}. \tag{2.35}$$

Applying Lemma 2.3 to $\tilde{v}_n$, we find a map $u_n \in C^1(\overline{\Omega}, S^2)$ satisfying

$$\int_\Omega |\nabla u_n(x)|^2 w(x) dx \leq \int_\Omega |\nabla \tilde{v}_n(x)|^2 w(x) dx + 8\pi \sum_{j=1}^{K_n} d_w(\tilde{P}_j, \tilde{N}_j) + 2^{-n} \tag{2.36}$$

and

$$\text{meas}\left(\{x \in \Omega,\ u_n(x) \neq \tilde{v}_n(x)\} \right) \leq 2^{-n}. \tag{2.37}$$

From (2.34) and (2.36), we derive that

$$\int_\Omega |\nabla u_n(x)|^2 w(x) dx \leq \int_\Omega |\nabla v_n(x)|^2 w(x) dx + 8\pi \tilde{L}_w(v_n) + C\, 2^{-n}. \tag{2.38}$$

Since $v_n \to u$ strongly in H^1, we deduce from Lemma 2.5 that $\tilde{L}_w(v_n) \to \tilde{L}_w(u)$ as $n \to +\infty$ which implies that $(u_n)_{n\in\mathbb{N}}$ is bounded in H^1. From (2.33) and (2.37) we obtain $u_n \to u$ a.e. in Ω and then we conclude that $u_n \rightharpoonup u$ weakly in H^1. Passing to the limit in (2.38) leads to

$$\limsup_{n\to+\infty} \int_\Omega |\nabla u_n(x)|^2 w(x) dx \leq \int_\Omega |\nabla u(x)|^2 w(x) dx + 8\pi \tilde{L}_w(u)$$

and the proof is complete. ∎

2.5.2 Stability and approximation properties for $\tilde{E}_w$

We present in this section the variants for $\tilde{E}_w$ of the results in Section 2.4.

Theorem 2.5. *Let $(w_n)_{n\in\mathbb{N}}$ be a sequence of measurable real functions satisfying (2.28) and assume that (i) in Theorem 2.3 holds. Then we have*

$$\tilde{E}_{w_n}(u) \underset{n\to+\infty}{\longrightarrow} \tilde{E}_w(u) \quad \text{for any } u \in H^1(\Omega, S^2). \tag{2.39}$$

Proof. Assumption (2.29) clearly implies that

$$\int_\Omega |\nabla u(x)|^2 w_n(x)dx \underset{n\to+\infty}{\to} \int_\Omega |\nabla u(x)|^2 w(x)dx \quad \text{for any } u \in H^1(\Omega, S^2),$$

and by Theorem 2.4, we just have to prove that

$$\tilde{L}_{w_n}(u) \underset{n\to+\infty}{\to} \tilde{L}_w(u) \quad \text{for any } u \in H^1(\Omega, S^2). \tag{2.40}$$

Consider $u \in H^1(\Omega, S^2)$. By the result in [16, 22], we can find a sequence $(v_k)_{k\in\mathbb{N}} \subset H^1(\Omega, S^2)$ such that $v_k \in C^1(\overline{\Omega} \setminus \cup_{i=1}^{M_k}\{a_j\}, S^2)$ for some M_k points (a_i) in Ω and $v_k \to u$ strongly in H^1. We easily check that a minimal connection for v_k relative to distance d_{w_n} does not allow more than $\sum_{i=1}^{M_k} |\deg(v_k, a_i)|$ connections to the boundary. Therefore, extracting a subsequence $(n_l)_{l\in\mathbb{N}}$, we can relabel the singularities of v_k and the virtual singularities on the boundary given by a minimal connection relative to $d_{w_{n_l}}$, as a list of positive points $(P_1^l, \dots, P_{K_k}^l)$ and a list of negative points $(N_1^l, \dots, N_{K_k}^l)$ with K_k independent of l and such that

$$\tilde{L}_{w_{n_l}}(v_k) = \underset{\sigma\in\mathcal{S}_{K_k}}{\text{Min}} \sum_{j=1}^{K_k} d_{w_{n_l}}(P_j^l, N_{\sigma(j)}^l) = \sum_{j=1}^{K_k} d_{w_{n_l}}(P_j^l, N_{\sigma_l(j)}^l)$$

for some permutation $\sigma_l \in \mathcal{S}_{K_k}$. Extracting another subsequence if necessary, we may assume that $\sigma_l = \sigma_\star$ is independent of $l \in \mathbb{N}$ and that $P_j^l \underset{l\to+\infty}{\to} P_j$ and $N_j^l \underset{l\to+\infty}{\to} N_j$ for $j = 1, \dots, K_k$. From the results in Chapter 1 Section 1.4.1, we know that assumption *(i)* implies that d_{w_n} converges to d_w uniformly on $\overline{\Omega} \times \overline{\Omega}$ and then we have

$$\tilde{L}_{w_{n_l}}(v_k) = \sum_{j=1}^{K_k} d_{w_{n_l}}(P_j^l, N_{\sigma_\star(j)}^l) \underset{l\to+\infty}{\to} \sum_{j=1}^{K_k} d_w(P_j, N_{\sigma_\star(j)})$$

By definition of $\tilde{L}_w(v_k)$, we obtain that

$$\tilde{L}_w(v_k) \le \lim_{l\to+\infty} \tilde{L}_{w_{n_l}}(v_k).$$

On the other hand, we can also relabel the singularities of v_k and the virtual singularities on the boundary given by a minimal connection relative to d_w, as a list of positive points $(\overline{P}_1, \dots, \overline{P}_{\overline{K}})$ and a list of negative points $(\overline{N}_1, \dots, \overline{N}_{\overline{K}})$ such that

$$\tilde{L}_w(v_k) = \sum_{j=1}^{\overline{K}} d_w(\overline{P}_j, \overline{N}_j).$$

As previously, we have for any $l \in \mathbb{N}$,

$$\tilde{L}_{w_{n_l}}(v_k) \le \sum_{j=1}^{\overline{K}} d_{w_{n_l}}(\overline{P}_j, \overline{N}_j).$$

Letting $l \to +\infty$, we obtain

$$\lim_{l \to +\infty} \tilde{L}_{w_{n_l}}(v_k) \leq \sum_{j=1}^{\overline{K}} d_w(\overline{P}_j, \overline{N}_j)$$

and then we conclude that $\lim\limits_{l \to +\infty} \tilde{L}_{w_{n_l}}(v_k) = \tilde{L}_w(v_k)$. By uniqueness of the limit, we deduce that the convergence holds for the full sequence, i.e.,

$$\tilde{L}_{w_n}(v_k) \underset{n \to +\infty}{\longrightarrow} \tilde{L}_w(v_k).$$

At this stage, we can proceed as in the proof of Theorem 2.4 *(i)⇒(ii)* using Lemma 2.5 instead of Lemma 2.4. ■

We also obtain the following variants of Proposition 2.3 and Proposition 2.4 using Theorem 2.5 instead of Theorem 2.3.

Proposition 2.6. *Let $(w_n)_{n\in\mathbb{N}}$ be a sequence of measurable real functions satisfying (2.28) and assume that (a) or (b) in Proposition 2.3 holds. Then (2.39) holds.*

Proposition 2.7. *Let $(\rho_n)_{n\in\mathbb{N}}$ be a sequence of smooth mollifiers. Extending w by a sufficiently large constant and setting $w_n = \rho_n * w$, then (2.39) holds.*

Chapitre 3

Vortices in a two dimensional rotating Bose-Einstein condensate

3.1 Introduction

The phenomenon of Bose-Einstein condensation has given rise to an intense research, both experimentally and theoretically, since its first realization in alkali gases in 1995. One of the most beautiful experiments was carried out by the ENS group and consisted of rotating the trap holding the atoms [68, 69] (see also [1]). Since a Bose-Einstein condensate (BEC) is a quantum gas, it can be described by a single complex-valued wave function (order parameter) and it rotates as a superfluid : above a critical velocity, it rotates through the existence of vortices, i.e., zeroes of the wave function around which there is a circulation of phase. Then the number of vortices increases as the angular speed gets larger and the vortices arrange themselves in a regular pattern around the center of the condensate.

A two-dimensional model for a rotating BEC was used by Y. Castin and R. Dum [40]. This model corresponds to a harmonic trap that confines strongly the atoms in the direction of the rotation axis, so that the system becomes effectively two-dimensional (see [77]). After the nondimensionalization of the energy (see [4]), the wave function u_ε minimizes the Gross-Pitaevskii energy

$$\int_{\mathbb{R}^2} \left\{ \frac{1}{2}|\nabla u|^2 + \frac{1}{2\varepsilon^2}(x_1^2 + \lambda^2 x_2^2)|u|^2 + \frac{1}{4\varepsilon^2}|u|^4 - \Omega\, x^\perp \cdot (iu, \nabla u) \right\} dx \tag{3.1}$$

under the mass constraint

$$\int_{\mathbb{R}^2} |u|^2 = 1 \tag{3.2}$$

where $\varepsilon > 0$ is small and represents a ratio of two characteristic lengths, $0 < \lambda \leq 1$ and $\Omega = \Omega(\varepsilon) \geq 0$ denotes the rotational velocity. The term $x_1^2 + \lambda^2 x_2^2$ in (3.1) models the trapping potential. In [40], the equilibrium configurations are studied by looking for the minimizers in a reduced class of functions and some numerical simulations are presented.

In this chapter, our main goal is to study the number and the location of vortices according to the value of the angular speed $\Omega(\varepsilon)$ as $\varepsilon \to 0$. We consider the situation in which the trap is axisymmetric, i.e. $\lambda = 1$, and Ω is at most of order $|\ln \varepsilon|$. Using (3.2), we rewrite the energy (3.1) in the equivalent form

$$F_\varepsilon(u) = \int_{\mathbb{R}^2} \left\{ \frac{1}{2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}\big[(|u|^2 - a(x))^2 - (a^-(x))^2\big] - \Omega x^\perp \cdot (iu, \nabla u) \right\} dx \qquad (3.3)$$

where $a(x) = a_0 - |x|^2$ and a_0 is determined by $\int_{\mathbb{R}^2} a^+(x) = 1$ so that $a_0 = \sqrt{2/\pi}$. Here a^+ and a^- represent respectively the positive and the negative part of a. We will see that in the limit $\varepsilon \to 0$, the minimization of F_ε strongly forces $|u_\varepsilon|^2$ to be close to a^+ which means that the resulting density is asymptotically localized in

$$\mathcal{D} := \big\{x \in \mathbb{R}^2,\ a(x) > 0\big\} = B(0, \sqrt{a_0}\,).$$

We will also prove that $|u_\varepsilon|$ decays exponentially outside $\mathcal{D}$. We will seek vortices only inside the domain $\mathcal{D}$ and compute an asymptotic expansion of $F_\varepsilon(u_\varepsilon)$ in order to

a) determine the critical velocity Ω_d for which the dth vortex becomes energetically favourable,

b) express the part of the energy governing the location of the vortices (the so called "*renormalized energy*").

Let us now recall some related works. In [20], F. Bethuel, H. Brezis and F. Hélein have developed the main tools for studying vortices in "Ginzburg-Landau type" problems. We also refer to L. Almeida and F. Bethuel [7], F. Bethuel and T. Rivière [21], E. Sandier [73] and E. Sandier and S. Serfaty [75, 74, 76] for additional techniques. A similar functional to (3.3) was considered by S. Serfaty in [80] where $a(x) \equiv 1$ and $\mathbb{R}^2$ is replaced by a disc. She proves the existence of local minimizers having vortices for different ranges of rotational velocity. In [4], A. Aftalion and Q. Du follow the strategy in [80] for the study of global minimizers of the Gross-Pitaevskii energy (3.3) where $\mathbb{R}^2$ is replaced by $\mathcal{D}$. In [2], A. Aftalion, S. Alama and L. Bronsard analyze the global minimizers of (3.3) for potentials of different nature leading to an annular region of confinement. We finally refer to [5, 6, 63] for mathematical studies on 3D models.

We emphasize that we tackle here the problem which corresponds exactly to the physical model. In particular, we minimize F_ε under the mass constraint (3.2) and the admissible configurations are defined on the whole space $\mathbb{R}^2$. Several difficulties arise especially in the proof of the existence results and the construction of test functions. We point out that we do not assume any implicit bound on the number of vortices. The singular and degenerate behavior of $\sqrt{a^+}$ near $\partial\mathcal{D}$ induces a cost of order $|\ln \varepsilon|$ in the energy and requires specific tools to detect vortices in the boundary region.

We now start to describe our main results. We introduce the functional space in which we perform the minimization

$$\mathcal{H} := \big\{u \in H^1(\mathbb{R}^2, \mathbb{C}),\ \int_{\mathbb{R}^2} |x|^2|u|^2 < \infty\big\}. \qquad (3.4)$$

When $\Omega = 0$, $F_\varepsilon(u) = E_\varepsilon(u)$ where

$$E_\varepsilon(u) = \int_{\mathbb{R}^2} \left\{ \frac{1}{2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}\left[(|u|^2 - a(x))^2 - (a^-(x))^2\right] \right\} dx. \tag{3.5}$$

We shall prove that for ε small enough, the minimization problem

$$\text{Min } \left\{ E_\varepsilon(\eta),\, \eta \in \mathcal{H},\, \|\eta\|_{L^2(\mathbb{R}^2)} = 1 \right\} \tag{3.6}$$

admits a unique solution $\tilde{\eta}_\varepsilon$ (up to a complex multiplier of modulus one) which is a real positive function. Moreover $\tilde{\eta}_\varepsilon$ converges to $\sqrt{a^+}$ in $L^\infty(\mathbb{R}^2)$ as $\varepsilon \to 0$.

The notion of vortex that we consider here, is similar to the one used in [7, 78, 79, 80] and will be specified later. Defining for any integer $d \geq 1$, the critical velocities

$$\Omega_d = \frac{2}{a_0}|\ln \varepsilon| + \frac{2(d-1)}{a_0} \ln|\ln \varepsilon|, \tag{3.7}$$

our main theorem can be stated as follows :

Theorem 3.1. *Let u_ε be any minimizer of F_ε in $\mathcal{H}$ under the mass constraint* (3.2) *and let $0 < \delta \ll 1$ be any small constant.*

(i) If $\Omega \leq \Omega_1 - \delta \ln|\ln \varepsilon|$, then for any $R_0 < \sqrt{a_0}$, there exists $\varepsilon_{R_0} > 0$ such that for any $\varepsilon < \varepsilon_{R_0}$, u_ε is vortex free in B_{R_0}, i.e., u_ε does not vanish in B_{R_0}. In addition,

$$F_\varepsilon(u_\varepsilon) = E_\varepsilon(\tilde{\eta}_\varepsilon) + o(1).$$

(ii) If $\Omega_d + \delta \ln|\ln \varepsilon| \leq \Omega \leq \Omega_{d+1} - \delta \ln|\ln \varepsilon|$ for some integer $d \geq 1$, then for any $R_0 < \sqrt{a_0}$, there exists $\varepsilon_{R_0} > 0$ such that for any $\varepsilon < \varepsilon_{R_0}$, u_ε has exactly d vortices $x_1^\varepsilon, \ldots, x_d^\varepsilon$ of degree one in B_{R_0}. Moreover, we have that $|x_j^\varepsilon| \leq C\Omega^{-1/2}$ for any $j = 1, \ldots, d$ and $|x_i^\varepsilon - x_j^\varepsilon| \geq C\Omega^{-1/2}$ for any $i \neq j$ for some constant $C > 0$ independent of ε. Setting $\tilde{x}_j^\varepsilon = \sqrt{\Omega}\, x_j^\varepsilon$, the configuration $(\tilde{x}_1^\varepsilon, \ldots, \tilde{x}_d^\varepsilon)$ tends to minimize in $\mathbb{R}^{2d}$ the renormalized energy

$$w(b_1, \ldots, b_d) = -\pi a_0 \sum_{i \neq j} \ln|b_i - b_j| + \frac{\pi a_0}{2} \sum_{j=1}^{d} |b_j|^2. \tag{3.8}$$

In addition,

$$F_\varepsilon(u_\varepsilon) = E_\varepsilon(\tilde{\eta}_\varepsilon) - \frac{\pi a_0^2}{2}\, d(\Omega - \Omega_1) + \frac{\pi a_0}{2}\,(d^2 - d) \ln|\ln \varepsilon| + \underset{b \in \mathbb{R}^{2d}}{\text{Min}}\, w(b) + Q_d + o(1) \tag{3.9}$$

where Q_d is an explicit constant depending only on d.

These results are in agreement with theoretical predictions on Bose-Einstein condensates. More precisely : the critical angular velocity Ω_1 coincides with the one found in [4, 40] and the vortices are concentrated around the origin at a scale $\sqrt{\Omega}$. The minimizing configurations for the renormalized energy $w(\cdot)$ has been studied by S. Gueron and

I. Shafrir in [56]. They prove that for $d \leq 6$, regular polygons centered at the origin and "stars" are local minimizers. For larger d, they numerically found minimizers with a shape of concentric polygons and then triangular lattices as d increases. These figures are exactly the ones observed in physical experiments (see [68, 69]).

We now describe briefly the content of this chapter. Section 2 is devoted to the study of the density profile $\tilde{\eta}_\varepsilon$ defined by (3.6). We first introduce the real positive minimizer η_ε of E_ε, i.e.,

$$E_\varepsilon(\eta_\varepsilon) = \min_{\eta \in \mathcal{H}} E_\varepsilon(\eta). \tag{3.10}$$

We show the existence and uniqueness of η_ε (see Theorem 3.2) and we have that $\eta_\varepsilon \to \sqrt{a^+}$ in $L^\infty(\mathbb{R}^2) \cap C^1_{\text{loc}}(\mathcal{D})$ as $\varepsilon \to 0$ (see Proposition 3.1). Then we explicitly characterize the link between η_ε and $\tilde{\eta}_\varepsilon$ in Theorem 3.3 and we prove that $|E_\varepsilon(\eta_\varepsilon) - E_\varepsilon(\tilde{\eta}_\varepsilon)| = o(\varepsilon)$. We point out that the mass of η_ε may not be equal to 1 in general. Therefore, we shall use the profile $\tilde{\eta}_\varepsilon$ as a test function.

In Section 3, we prove the existence of minimizers u_ε under the mass constraint (3.2) (see Proposition 3.2) and some general results about their behavior : $E_\varepsilon(u_\varepsilon) \leq C|\ln\varepsilon|^2$, $|\nabla u_\varepsilon| \leq C_K \varepsilon^{-1}$ and $|u_\varepsilon| \lesssim \sqrt{a^+}$ in any compact $K \subset \mathcal{D}$, u_ε decreases exponentially quickly to 0 outside $\mathcal{D}$ (see Proposition 3.3). Using a method introduced by L. Lassoued and P. Mironescu [65], we show that $F_\varepsilon(u_\varepsilon)$ splits into two independent pieces (see Lemme 3.4) : the energy $E_\varepsilon(\eta_\varepsilon)$ and a reduced energy $\mathcal{F}_\varepsilon^{\eta_\varepsilon}$ of $v_\varepsilon = u_\varepsilon/\eta_\varepsilon$, i.e.,

$$F_\varepsilon(u_\varepsilon) = E_\varepsilon(\eta_\varepsilon) + \mathcal{F}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon) \tag{3.11}$$

where

$$\mathcal{F}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon) = \mathcal{E}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon) - \mathcal{R}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon), \tag{3.12}$$

$$\mathcal{E}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon) = \int_{\mathbb{R}^2} \frac{\eta_\varepsilon^2}{2}|\nabla v_\varepsilon|^2 + \frac{\eta_\varepsilon^4}{4\varepsilon^2}(|v_\varepsilon|^2 - 1)^2 \quad \text{and} \quad \mathcal{R}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon) = \Omega \int_{\mathbb{R}^2} \eta_\varepsilon^2 x^\perp \cdot (iv_\varepsilon, \nabla v_\varepsilon). \tag{3.13}$$

In (3.11), $E_\varepsilon(\eta_\varepsilon)$ carries the energy of the singular layer near $\partial\mathcal{D}$ and hence, we may detect vortices by the reduced energy $\mathcal{F}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon)$. We study the vortex structure of u_ε via the map v_ε applying the Ginzburg-Landau techniques to the weighted energy $\mathcal{E}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon)$; the difficulty will arise in the region where η_ε is small. We notice that v_ε inherits the following properties : $\mathcal{E}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon) \leq C|\ln\varepsilon|^2$, $|\nabla v_\varepsilon| \leq C_K\varepsilon^{-1}$ and $|v_\varepsilon| \lesssim 1$ in any compact $K \subset \mathcal{D}$. Using (3.11) and $\tilde{\eta}_\varepsilon$ as a test function, we obtain in Proposition 3.5 an important upper bound of the reduced energy inside $\mathcal{D}$:

$$\mathcal{F}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon, \mathcal{D}) \leq o(1). \tag{3.14}$$

In Section 4, we compute a first lower bound of $\mathcal{E}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon)$ using a method due to E. Sandier and S. Serfaty (see [75, 76]). We start with a first construction of small vortex balls $\{B(p_i,r_i)\}_{i\in I_\varepsilon}$ in a domain $\mathcal{D}_\varepsilon$ slightly smaller than $\mathcal{D}$: outside these balls $|v_\varepsilon|$ is close to 1, so that v_ε carries a degree d_i on $\partial B(p_i,r_i)$ and (see Proposition 3.7)

$$\mathcal{E}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon,\mathcal{D}_\varepsilon) \geq \sum_{i\in I_\varepsilon} \mathcal{E}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon, B(p_i,r_i)) \gtrsim \pi \sum_{i\in I_\varepsilon} a(p_i)|d_i|\,|\ln\varepsilon|. \tag{3.15}$$

Then we prove an asymptotic expansion of the rotational energy outside the vortex balls $\{B(p_i,r_i)\}_{i\in I_\varepsilon}$ (see Proposition 3.8),

$$\mathcal{R}_\varepsilon^{\eta_\varepsilon}\big(v_\varepsilon,\mathcal{D}_\varepsilon\setminus\cup_{i\in I_\varepsilon}B_i\big) \approx \frac{\pi\Omega}{2}\sum_{i\in I_\varepsilon} a^2(p_i)\,d_i. \tag{3.16}$$

Estimates (3.15) and (3.16) yield a first lower bound of $\mathcal{F}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon,\mathcal{D})$ that we match with (3.14) in order to derive the first critical angular velocity Ω_1 and to prove the absence of vortices for velocities strictly less than Ω_1 (see Proposition 3.9). We also obtain that for $\Omega \leq \Omega_1 + \mathcal{O}(\ln|\ln\varepsilon|)$, the number of vortex balls with nonzero degree is uniformly bounded in ε (see Proposition 3.10). We conclude by two fundamental energy estimates (see Proposition 3.11)

$$\mathcal{E}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon,\mathcal{D}_\varepsilon) = \mathcal{O}(|\ln\varepsilon|) \quad\text{and}\quad \mathcal{E}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon,\mathcal{D}_\varepsilon\setminus B_{2|\ln\varepsilon|^{-1/6}}) = o(|\ln\varepsilon|). \tag{3.17}$$

In Section 5, we give a finer description of the vortex structure inside $B_R \subset\subset \mathcal{D}$ using the method of “bad discs” introduced by F. Bethuel, H. Brezis and F. Hélein [20]. We find that the number of bad discs is uniformly bounded, all of them remaining close to the origin (see Theorem 3.4). The main ingredients are the energy estimates (3.17) and a local version of the Pohozaev identity. Using a “clustering” method presented in [7], we obtain a new family of modified bad discs $\big\{B(x_j^\varepsilon,\rho)\big\}_{j\in\tilde{J}_\varepsilon}$ such that $\rho\sim\varepsilon^\alpha$, $|v_\varepsilon|\geq 1/2$ outside these discs and v_ε has a non zero degree D_j on each $\partial B(x_j^\varepsilon,\rho)$ (see Proposition 3.15). We identify a *vortex* with the center of a modified bad disc $B(x_j^\varepsilon,\rho)$.

In Section 6, we establish some lower estimates of the energy taking into account the interaction between vortices. Following similar methods to [20], we evaluate separately the energy carried by each vortex (see Lemma 3.9)

$$\mathcal{E}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon, B(x_j^\varepsilon,\rho)) \geq \pi a(x_j^\varepsilon)|D_j|\,\ln\frac{\rho}{\varepsilon} + \mathcal{O}(1) \tag{3.18}$$

and the energy away from the vortices (see Proposition 3.16)

$$\mathcal{E}_\varepsilon^{\eta_\varepsilon}\big(v_\varepsilon, B_R\setminus\cup_{j\in\tilde{J}_\varepsilon}B(x_j^\varepsilon,\rho)\big) \geq \pi\sum_{j\in\tilde{J}_\varepsilon} D_j^2\,a(x_j^\varepsilon)|\ln\rho| + W_{R,\varepsilon}\big((x_j^\varepsilon,D_j)_{j\in\tilde{J}_\varepsilon}\big) + \mathcal{O}_R(1). \tag{3.19}$$

Here, the radius $R\in(\frac{\sqrt{a_0}}{2},\sqrt{a_0})$ is fixed and the error term $\mathcal{O}_R(1)$ is computed in function of R. The quantity $W_{R,\varepsilon}$ is similar to the renormalized energy in [20] and involves the interaction between the vortices. As for (3.16), we find an asymptotic expansion of the rotational energy outside the modified bad discs $\{B(x_j^\varepsilon,\rho)\}_{j\in\tilde{J}_\varepsilon}$ and it yields (see (3.155) in the proof of Lemma 3.10)

$$\mathcal{F}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon,\mathcal{D}_\varepsilon) \geq \mathcal{E}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon,B_R) - \frac{\pi\Omega}{2}\sum_{j\in\tilde{J}_\varepsilon} a^2(x_j^\varepsilon)\, D_j + o_R(1). \tag{3.20}$$

Using (3.14), (3.18), (3.19) and (3.20), we prove in Section 7 that each vortex is of degree 1, i.e., $D_j = 1$ (see Lemma 3.11). Then it allows us to improve the above estimates and to obtain the result in the subcritical case *(i)* in Theorem 3.1. The rest of the proof requires an upper bound of $\mathcal{F}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon)$ which is proved at the end of the chapter adapting a method due to N. André and I. Shafrir [12] (see Theorem 3.5). We are then led to the following expansion (see Proposition 3.18)

$$\mathcal{F}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon) = -\frac{\pi n a_0^2(\Omega-\Omega_1)}{2} + \frac{\pi a_0}{2}(n^2-n)\ln|\ln\varepsilon| + \mathcal{O}(1) \quad \text{where} \quad n = \mathrm{Card}\,\tilde{J}_\varepsilon.$$

If $\Omega_d + \delta\ln|\ln\varepsilon| \leq \Omega \leq \Omega_{d+1} - \delta\ln|\ln\varepsilon|$ for any small $\delta > 0$, this expansion yields the exact number of vortices $\Omega : n = d$ (see Proposition 3.19). Moreover, we find that the vortices are uniformly distributed at a scale $\Omega^{-1/2}$ around the origin (see Lemma 3.13). Then we compute an asymptotic formula of the energy $W_{R,\varepsilon}$ given in (3.19) as $\varepsilon \to 0$ (see (3.175) in the proof of Proposition 3.20) :

$$\lim_{\varepsilon\to 0}\Big(W_{R,\varepsilon}(x_1^\varepsilon,\ldots,x_d^\varepsilon) + \pi a_0\sum_{i\neq j}\ln|x_i^\varepsilon - x_j^\varepsilon| \Big) = -\frac{\pi a_0 d^2}{2} + \frac{\pi a_0 d^2}{2}\ln a_0 + \mathcal{O}(|R-\sqrt{a_0}|). \tag{3.21}$$

We derive from (3.18), (3.19), (3.20) and (3.21) the lower estimate of $\mathcal{F}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon)$ (see (3.176) in the proof of Proposition 3.20) :

$$\liminf_{\varepsilon\to 0}\Big(\mathcal{F}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon) + \frac{\pi a_0^2}{2}\, d(\Omega-\Omega_1) - \frac{\pi a_0}{2}\,(d^2-d)\ln|\ln\varepsilon| - w(\tilde{x}_1^\varepsilon,\ldots,\tilde{x}_d^\varepsilon) \Big) \geq \geq Q_d + \mathcal{O}(|R-\sqrt{a_0}|) \tag{3.22}$$

(the constant Q_d is explicitly given in Proposition 3.20 and w is the *renormalized energy* given by (3.8)). Since the left hand side in (3.22) does not depend on R, we can pass to the limit $R \to \sqrt{a_0}$ on the right hand side. Using the upper bound of $\mathcal{F}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon)$ given by test functions (see Theorem 3.5), we find the expansion of the energy (3.9) and we conclude that the rescaled configuration $(\tilde{x}_1^\varepsilon,\ldots,\tilde{x}_d^\varepsilon)$ tends to minimize the renormalized energy w (see Proposition 3.20).

We now list some open problems about the 2d model (3.3). The first one concerns the non-existence of vortices in the whole space $\mathbb{R}^2$ for Ω small ($\Omega = \mathcal{O}(1)$). For Ω larger, vortices may exist in the region where u_ε is small. Therefore, a natural problem is to investigate the vortex structure close to the boundary $\partial\mathcal{D}$ and outside the domain $\mathcal{D}$ for $\Omega \sim \Omega_1$. One can also ask if our results hold for the case of asymmetric trapping potentials, i.e., $a(x) = a_0 - x_1^2 - \lambda x_2^2$ with $0 < \lambda < 1$, or even for some functions $a(x)$ positive in a domain which is not simply connected.

Notations. Throughout this chapter, we denote by C a positive constant independent of ε and we use the subscript to point out a possible dependence on the argument. For $\mathcal{A} \subset \mathbb{R}^2$, we write

$$e_\varepsilon(u) = \frac{1}{2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}\left((|u|^2 - a(x))^2 - (a^-(x))^2\right),$$

$$e_\varepsilon^\eta(v) = \frac{1}{2}\eta^2|\nabla v|^2 + \frac{\eta^4}{4\varepsilon^2}(1-|v|^2)^2,$$

$$E_\varepsilon(u,\mathcal{A}) = \int_{\mathcal{A}} e_\varepsilon(u), \quad R_\varepsilon(u,\mathcal{A}) = \Omega\int_{\mathcal{A}} x^\perp \cdot (iu, \nabla u), \quad F_\varepsilon(u,\mathcal{A}) = E_\varepsilon(u,\mathcal{A}) - R_\varepsilon(u,\mathcal{A}),$$

$$\mathcal{E}_\varepsilon^\eta(v,\mathcal{A}) = \int_{\mathcal{A}} e_\varepsilon^\eta(v), \ \mathcal{R}_\varepsilon^\eta(v,\mathcal{A}) = \Omega\int_{\mathcal{A}} \eta^2 x^\perp \cdot (iv, \nabla v), \ \mathcal{F}_\varepsilon^\eta(v,\mathcal{A}) = \mathcal{E}_\varepsilon^\eta(v,\mathcal{A}) - \mathcal{R}_\varepsilon^\eta(v,\mathcal{A}),$$

where η denotes one of the functions a, η_ε or $\tilde{\eta}_\varepsilon$. We do not write the dependence on $\mathcal{A}$ when $\mathcal{A} = \mathbb{R}^2$.

3.2 Analysis of the density profiles

In this section, we establish some preliminary results on η_ε and $\tilde{\eta}_\varepsilon$ defined respectively by (3.10) and (3.6). We will show that the shapes of η_ε and $\tilde{\eta}_\varepsilon$ are similar.

We notice that the space $\mathcal{H}$ is the set of finiteness for E_ε, i.e.,

$$\mathcal{H} = \left\{u \in H^1(\mathbb{R}^2, \mathbb{C}),\ E_\varepsilon(u) < +\infty\right\}.$$

In the sequel, we endow $\mathcal{H}$ with the scalar product

$$\langle u, v\rangle_{\mathcal{H}} = \int_{\mathbb{R}^2} \nabla u \cdot \nabla v + (1+|x|^2)(u\cdot v) \quad \text{for } u, v \in \mathcal{H},$$

and then $(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})$ is a Hilbert space.

3.2.1 The free profile

We start by proving the existence and uniqueness for small ε of η_ε defined as the real positive solution of (3.10). Hence η_ε has to satisfy the associated Euler-Lagrange equation

$$\begin{cases} \varepsilon^2\Delta\eta_\varepsilon + (a(x) - \eta_\varepsilon^2)\eta_\varepsilon = 0 & \text{in } \mathbb{R}^2, \\ \eta_\varepsilon > 0 & \text{in } \mathbb{R}^2. \end{cases} \tag{3.23}$$

We have the following result.

Theorem 3.2. *For any $0 < \varepsilon < \frac{a_0}{2}$, there exists a unique classical solution η_ε of (3.23). Moreover, η_ε is radial, $\eta_\varepsilon \leq \sqrt{a_0}$ and η_ε is the unique minimizer of E_ε in $\mathcal{H}$ up to a complex multiplier of modulus one.*

The method that we use for solving (3.23) involves several classical arguments generally used for a bounded domain. The main difficulty here is due to the fact that the equation is posed in the entire space $\mathbb{R}^2$ without any condition at infinity. We start with the construction of the *minimal solution* : we consider the solution $\eta_{R,\varepsilon}$ of the same equation posed in a ball of large radius R with homogeneous Dirichlet boundary conditions and then we pass to the limit in R. We prove the uniqueness by estimating the ratio between the constructed solution and any other solution. A crucial point in the proof is an L^∞-bound of any weak solution.

Before proving Theorem 3.2, we present the asymptotic properties of η_ε as ε goes to 0. We show that η_ε decays exponentially fast outside $\mathcal{D}$ and that η_ε^2 tends uniformly to a^+. The following estimates will be essential at several steps of our analysis.

Proposition 3.1. *For ε sufficiently small, we have*

3.1.a) $E_\varepsilon(\eta_\varepsilon) \leq C|\ln \varepsilon|$,

3.1.b) $0 < \eta_\varepsilon(x) \leq C\varepsilon^{1/3} \exp\left(-\dfrac{|x|^2 - a_0}{4\varepsilon^{2/3}}\right)$ *in* $\mathbb{R}^2 \setminus \mathcal{D}$,

3.1.c) $0 \leq \sqrt{a(x)} - \eta_\varepsilon(x) \leq C\varepsilon^{1/3}\sqrt{a(x)}$ *for* $x \in \mathcal{D}$ *with* $\operatorname{dist}(x, \partial\mathcal{D}) \geq \varepsilon^{1/3}$,

3.1.d) $\|\nabla\eta_\varepsilon\|_{L^\infty(\mathbb{R}^2)} \leq C\varepsilon^{-1}$,

3.1.e) $\|\eta_\varepsilon - \sqrt{a}\,\|_{C^1(K)} \leq C_K\varepsilon^2$ *for any compact subset* $K \subset \mathcal{D}$.

Remark 3.1. As a direct consequence of *3.1.a)*, we obtain

$$\int_{\mathbb{R}^2\setminus\mathcal{D}} |\eta_\varepsilon|^4 + 2a^-(x)|\eta_\varepsilon|^2 + \int_{\mathcal{D}} (a(x) - |\eta_\varepsilon|^2)^2 \leq C\varepsilon^2|\ln\varepsilon|. \tag{3.24}$$

Proof of Theorem (3.2). Step 1 : Existence. For $0 < \varepsilon < \frac{a_0}{2}$ and $R > 0$, we consider the equation

$$\begin{cases} \varepsilon^2\Delta\eta_R + (a(x) - \eta_R^2)\eta_R = 0 & \text{in } B_R, \\ \eta_R > 0 & \text{in } B_R, \\ \eta_R = 0 & \text{on } \partial B_R. \end{cases} \tag{3.25}$$

By a result of H. Brezis and L. Oswald (see [33]), we have the existence and uniqueness of weak solutions of (3.25) if and only if the following first eigenvalue condition holds

$$\operatorname{Inf}\left\{\int_{B_R} |\nabla\phi|^2 - \frac{a(x)|\phi|^2}{\varepsilon^2}\,,\ \phi \in H_0^1(B_R) \text{ with } \|\phi\|_{L^2(B_R)} = 1\right\} < 0$$

or equivalently

$$\operatorname{Inf}\left\{\int_{B_R} |\nabla\phi|^2 + \frac{|x|^2|\phi|^2}{\varepsilon^2}\,,\ \phi \in H_0^1(B_R) \text{ with } \|\phi\|_{L^2(B_R)} = 1\right\} < \frac{a_0}{\varepsilon^2}\,. \tag{3.26}$$

We claim that for R sufficiently large, this condition is fulfilled. Indeed, setting for $x \in \mathbb{R}^2$,

$$\psi(x) = \frac{1}{\sqrt{\varepsilon\pi}} \exp\left(-\frac{|x|^2}{2\varepsilon}\right),$$

we define for any integer $n \geq 1$,

$$\psi_n(x) = c_n \, \zeta\left(\frac{|x|}{n}\right) \psi(x),$$

where $\zeta : \mathbb{R} \to \mathbb{R}$ denotes the "cut-off" type function given by

$$\zeta(t) = \begin{cases} 1 & \text{if } t \leq 1, \\ 2 - t & \text{if } t \in (1, 2), \\ 0 & \text{if } t \geq 2, \end{cases} \tag{3.27}$$

and the constant c_n is chosen such that $\|\psi_n\|_{L^2(\mathbb{R}^2)} = 1$. We easily check that

$$\int_{B_{2n}} \left(|\nabla \psi_n|^2 + \frac{|x|^2}{\varepsilon^2}|\psi_n|^2\right) \underset{n\to+\infty}{\longrightarrow} \int_{\mathbb{R}^2} \left(|\nabla \psi|^2 + \frac{|x|^2}{\varepsilon^2}|\psi|^2\right) = \frac{2}{\varepsilon},$$

and we deduce that for $R \geq 2n$,

$$\operatorname*{Inf}_{\substack{\phi \in H_0^1(B_R) \\ \|\phi\|_{L^2(B_R)}=1}} \int_{B_R} \left(|\nabla \phi|^2 + \frac{|x|^2}{\varepsilon^2}|\phi|^2\right) \leq \int_{B_{2n}} \left(|\nabla \psi_n|^2 + \frac{|x|^2}{\varepsilon^2}|\psi_n|^2\right) = \frac{2}{\varepsilon} + o(1),$$

where $o(1)$ denotes a quantity which tends to 0 as $n \to +\infty$. Hence there exists $R_\varepsilon > 0$ such that for every $R > R_\varepsilon$, condition (3.26) is fulfilled and equation (3.25) admits a unique weak solution $\eta_{R,\varepsilon}$. By standard methods, it results that $\eta_{R,\varepsilon}$ is a radial classical solution of (3.25). We notice that, for any $R_\varepsilon < R < \tilde{R}$, $\eta_{\tilde{R},\varepsilon}$ is a supersolution of (3.25) in B_R and thus

$$\eta_{R,\varepsilon} \leq \eta_{\tilde{R},\varepsilon} \quad \text{in } B_R$$

by the uniqueness of $\eta_{R,\varepsilon}$. By the maximum principle, we have

$$\eta_{R,\varepsilon} \leq \sqrt{a_0} \quad \text{in } \mathbb{R}^2.$$

For every $R > R_\varepsilon$, we extend $\eta_{R,\varepsilon}$ by 0 on $\mathbb{R}^2 \setminus B_R$. Since the function $R \to \eta_{R,\varepsilon}(x)$ is non-decreasing for any $x \in \mathbb{R}^2$, we may define for $x \in \mathbb{R}^2$,

$$\eta_\varepsilon(x) = \lim_{R\to+\infty} \eta_{R,\varepsilon}(x).$$

From the properties of $\eta_{R,\varepsilon}$, we deduce that η_ε is radial and satisfies

$$0 < \eta_\varepsilon \leq \sqrt{a_0}$$

and

$$\varepsilon^2\Delta\eta_\varepsilon + (a(x) - \eta_\varepsilon^2)\eta_\varepsilon = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^2).$$

Since $\eta_\varepsilon \in L^\infty(\mathbb{R}^2)$, we derive by standard methods that η_ε is smooth and then it defines a classical solution of (3.2).

Step 2. L^∞-bound for solutions of (3.23). The result in this step is due to A. Farina (see [51]) and relies on a result of H. Brezis (see [26]). We present the proof for convenience. Let η be any weak solution of (3.23) in $L^3_{\text{loc}}(\mathbb{R}^2)$. We claim that

$$\eta \leq \sqrt{a_0} \quad \text{a.e. in } \mathbb{R}^2.$$

Indeed, if we consider $w = \varepsilon^{-1}(\eta - \sqrt{a_0})$, then $w \in L^3_{\text{loc}}(\mathbb{R}^2)$ and since η satisfies (3.23), we infer that $\Delta w \in L^1_{\text{loc}}(\mathbb{R}^2)$. By Kato's inequality, we have

$$\Delta(w^+) \geq \operatorname{sgn}^+(w)\Delta w \geq \frac{\operatorname{sgn}^+(w)}{\varepsilon^3}(\eta^2 - a_0)\eta = \frac{1}{\varepsilon^2}\, w^+(\varepsilon w + 2\sqrt{a_0})(\varepsilon w + \sqrt{a_0}) \geq (w^+)^3.$$

Therefore $w^+ \in L^3_{\text{loc}}(\mathbb{R}^2)$ and w^+ satisfies

$$-\Delta(w^+) + (w^+)^3 \leq 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^2).$$

By Lemma 2 in [26], it leads to $w^+ \leq 0$ a.e. in $\mathbb{R}^2$ and thus $w^+ \equiv 0$.

Step 3. Uniqueness. Let η_ε be the solution constructed at Step 1 and let η be any weak solution of (3.23) in $L^3_{\text{loc}}(\mathbb{R}^2)$. By the previous step, $\eta \in L^\infty(\mathbb{R}^2)$ and using standard arguments, we derive that η is a classical solution of (3.23). We remark that η is a super-solution of (3.25) for every $R > R_\varepsilon$. Hence (recall that we extend $\eta_{R,\varepsilon}$ by 0 outside B_R),

$$\eta_{R,\varepsilon} \leq \eta \quad \text{in } \mathbb{R}^2.$$

Passing to the limit in R, we get that $0 < \eta_\varepsilon \leq \eta$ in $\mathbb{R}^2$. Thus, η_ε is the *minimal solution* of (3.23) and we can define the L^∞-function $\rho : \mathbb{R}^2 \to \mathbb{R}$ by

$$\rho = \frac{\eta_\varepsilon}{\eta}.$$

The function ρ is smooth, takes values in $(0, 1]$ and satisfies

$$\operatorname{div}(\eta^2\nabla\rho) + \frac{\eta^4}{\varepsilon^2}(1 - \rho^2)\rho = 0 \quad \text{in } \mathbb{R}^2. \tag{3.28}$$

For every integer $n \geq 1$, we set $\zeta_n(x) = \zeta\left(n^{-1}|x|\right)$, where ζ is given by (3.27). Multiplying (3.28) by $(1 - \rho)\zeta_n^2$ and integrating by parts, we derive

$$\int_{\mathbb{R}^2} \left(\frac{\eta^4}{\varepsilon^2}\,\rho(1 - \rho)^2(1 + \rho)\zeta_n^2 + \eta^2\zeta_n^2|\nabla\rho|^2\right) = 2\int_{\mathbb{R}^2} \eta^2(1 - \rho)\zeta_n(\nabla\rho \cdot \nabla\zeta_n). \tag{3.29}$$

Since ρ is bounded, the Cauchy-Schwarz inequality yields

$$\begin{aligned}\int_{\mathbb{R}^2} \eta^2(1-\rho)\zeta_n(\nabla\rho\cdot\nabla\zeta_n) &= \int_{B_{2n}\setminus B_n} \eta^2(1-\rho)\zeta_n(\nabla\rho\cdot\nabla\zeta_n)\\ &\le \left(\int_{B_{2n}} \eta^2(1-\rho)^2|\nabla\zeta_n|^2\right)^{1/2}\left(\int_{B_{2n}\setminus B_n} \eta^2\zeta_n^2|\nabla\rho|^2\right)^{1/2}\\ &\le 2\sqrt{\pi}\,\|\eta\|_{L^\infty(\mathbb{R}^2)}\left(\int_{\mathbb{R}^2\setminus B_n} \eta^2\zeta_n^2|\nabla\rho|^2\right)^{1/2}.\end{aligned}$$

Using (3.29) and the bound on η obtained in Step 2, we infer that

$$\int_{\mathbb{R}^2} \eta^2\zeta_n^2|\nabla\rho|^2 \le 4\sqrt{\pi a_0}\left(\int_{\mathbb{R}^2\setminus B_n} \eta^2\zeta_n^2|\nabla\rho|^2\right)^{1/2}. \tag{3.30}$$

It then follows

$$16\pi a_0 \ge \int_{\mathbb{R}^2} \eta^2\zeta_n^2|\nabla\rho|^2 \underset{n\to+\infty}{\longrightarrow} \int_{\mathbb{R}^2} \eta^2|\nabla\rho|^2$$

by monotone convergence. Since $\eta^2|\nabla\rho|^2 \in L^1(\mathbb{R}^2)$, the right hand side in (3.30) tends to 0 as $n \to +\infty$ and we deduce

$$\int_{\mathbb{R}^2} \eta^2|\nabla\rho|^2 = 0.$$

Hence ρ is constant in $\mathbb{R}^2$ and by (3.29), we necessarily have $\rho = 1$ i.e., $\eta = \eta_\varepsilon$.

Step 4. End of the proof. The existence of a minimizer η of E_ε in $\mathcal{H}$ is standard. Since $E_\varepsilon(|\tilde\eta|) \le E_\varepsilon(\tilde\eta)$ for any $\tilde\eta \in \mathcal{H}$, we infer that $\tilde\eta := |\eta|$ is also a minimizer and therefore $\tilde\eta$ satisfies the equation

$$\begin{cases}\varepsilon^2\Delta\tilde\eta + (a(x)-\tilde\eta^2)\tilde\eta = 0 & \text{in } \mathbb{R}^2,\\ \tilde\eta \ge 0 & \text{in } \mathbb{R}^2.\end{cases}$$

By the maximum principle, it follows that either $\tilde\eta > 0$ in $\mathbb{R}^2$ or $\tilde\eta \equiv 0$. Let us prove that $\tilde\eta > 0$. For $0 < \varepsilon < \frac{a_0}{2}$ and $R > 0$ sufficiently large, we consider the unique solution $\eta_{R,\varepsilon}$ of (3.25). By the results in [33], $\eta_{R,\varepsilon}$ is the unique non-negative minimizer of $E_\varepsilon(\cdot, B_R)$ in $H_0^1(B_R, \mathbb{R})$. Extending $\eta_{R,\varepsilon}$ by 0 outside B_R, we have

$$E_\varepsilon(\tilde\eta) \le E_\varepsilon(\eta_{R,\varepsilon}) = E_\varepsilon(\eta_{R,\varepsilon}, B_R) < E_\varepsilon(0, B_R) = E_\varepsilon(0)$$

which implies that $\tilde\eta$ is not identically equal to 0. Then $\tilde\eta$ solves (3.23) and by Step 3, we conclude that $|\eta| = \tilde\eta = \eta_\varepsilon$. From the equality $E_\varepsilon(|\eta|) = E_\varepsilon(\eta)$, we easily deduce that exists a real constant α such that $\eta = |\eta|e^{i\alpha} = \eta_\varepsilon e^{i\alpha}$. ■

Remark 3.2. The range of $\varepsilon \in (0, \frac{a_0}{2})$ where we have existence and uniqueness of positive minimizers η_ε is optimal. This is due to an argument based on the first eigenvalue condition (given in [33]) for problem (3.25). In particular, for ε larger than $\frac{a_0}{2}$, zero is the unique minimizer of E_ε.

Remark 3.3. We emphasize that from the proof of Theorem 3.2, it follows that any smooth function η satisfying

$$\begin{cases} -\varepsilon^2\Delta\eta \geq (a(x)-|\eta|^2)\eta & \text{in } \mathbb{R}^2, \\ \eta > 0 & \text{in } \mathbb{R}^2, \end{cases}$$

verifies $\eta \geq \eta_\varepsilon$ in $\mathbb{R}^2$.

Proof of Proposition 3.1. Proof of 3.1.a). We construct an explicit test function $\xi \in H^1(\mathbb{R}^2)$ such that $E_\varepsilon(\xi) \leq C|\ln\varepsilon|$. Since η_ε minimizes E_ε, we deduce

$$E_\varepsilon(\eta_\varepsilon) \leq E_\varepsilon(\xi) \leq C|\ln\varepsilon|.$$

We construct ξ as follows. We consider for $s \in \mathbb{R}$,

$$\gamma(s) = \begin{cases} \sqrt{s} & \text{if } s \geq \varepsilon^{2/3} \\ \dfrac{s}{\varepsilon^{1/3}} & \text{otherwise} \end{cases}$$

and we set $\xi(x) = \gamma(a^+(x))$ for $x \in \mathbb{R}^2$. We obtain exactly as in [63] that

$$\int_{\mathbb{R}^2} |\nabla\xi|^2 \leq C|\ln\varepsilon| \quad \text{and} \quad \int_{\mathbb{R}^2} (a^+ - \gamma(a^+)^2)^2 \leq C\varepsilon^2$$

for a positive constant C independent of ε.

Proof of 3.1.b). We construct a radial supersolution $\overline{\eta}$ of (3.23) of the form :

$$\overline{\eta}(x) = \overline{\eta}(|x|) = \begin{cases} \sqrt{a_0 - |x|^2} & \text{if } |x| \leq \sqrt{a_0-\delta}\,, \\ \dfrac{-|x|\sqrt{a_0-\delta}+a_0}{\sqrt{\delta}} & \text{if } \sqrt{a_0-\delta} \leq |x| \leq r_\delta, \\ \beta\exp(-|x|^2/2\sigma) & \text{otherwise,} \end{cases} \tag{3.31}$$

where $\delta > 0$ will be determined later,

$$r_\delta = \frac{a_0}{2\sqrt{a_0-\delta}} + \frac{\sqrt{a_0}}{2},$$

and β, σ are chosen such that $\overline{\eta} \in C^1(\mathbb{R}^2)$ i.e.,

$$\beta = \frac{a_0 - \sqrt{a_0(a_0-\delta)}}{2\sqrt{\delta}}\exp(r_\delta^2/2\sigma) \quad \text{and} \quad \sigma = \frac{a_0\delta}{4(a_0-\delta)}.$$

A straightforward computation shows that for $\delta = 4a_0^{1/3}\varepsilon^{2/3}$, $\overline{\eta}$ is a supersolution of (3.23) and then

$$r_\delta - \sqrt{a_0} = \mathcal{O}(\varepsilon^{2/3}), \quad \sigma = \mathcal{O}(\varepsilon^{2/3}) \quad \text{and} \quad \beta = \mathcal{O}(\varepsilon^{1/3}e^{a_0/2\sigma}).$$

By Remark 3.3, it results that $\eta_\varepsilon \leq \overline{\eta}$ in $\mathbb{R}^2$ which leads to *3.1.b)*. Note that we also obtain

$$\eta_\varepsilon(x) \leq \sqrt{a(x)} \text{ for } |x| \leq \sqrt{a_0 - \delta} \quad \text{and} \quad \eta_\varepsilon(x) \leq C\varepsilon^{1/3} \text{ for } \sqrt{a_0-\delta} \leq |x| \leq \sqrt{a_0}. \tag{3.32}$$

Proof of 3.1.c). Here, the proof is similar to that of Proposition 2.1 in [2]. Let $x_0 \in \mathcal{D}$ be such that

$$\sqrt{a_0} - |x_0| \geq \varepsilon^{1/3} \tag{3.33}$$

and set

$$\alpha = \min_{B(x_0,\,\varepsilon^{2/3})} a = a_0 - (|x_0| + \varepsilon^{2/3})^2 = \mathcal{O}(\varepsilon^{1/3}).$$

We want to construct a subsolution in $B_\delta(x_0)$. For $\tilde{\varepsilon} = \varepsilon^{1/3}/\sqrt{\alpha}$, we denote by $\tilde{w}$ the unique solution of

$$\begin{cases} -\Delta\tilde{w} + \dfrac{1}{\tilde{\varepsilon}^2}\,(\tilde{w}^2 - 1)\tilde{w} = 0 & \text{in } B_1, \\ \tilde{w} > 0 & \text{in } B_1, \\ \tilde{w} = 0 & \text{on } \partial B_1. \end{cases} \tag{3.34}$$

From Proposition 2.1 in [10], we know that

$$0 \leq 1 - \tilde{w}(x) \leq C \exp\left(-\frac{1-|x|^2}{2\tilde{\varepsilon}}\right).$$

Then we map $\tilde{w}$ to $B(x_0, \varepsilon^{2/3})$, namely

$$w(x) = \sqrt{\alpha}\,\tilde{w}\left(\frac{x-x_0}{\varepsilon^{2/3}}\right).$$

From (3.34) we derive

$$-\Delta w + \frac{1}{\varepsilon^2}(w^2 - a(x))w \leq -\Delta w + \frac{1}{\varepsilon^2}(w^2 - \alpha)w = 0 \quad \text{in} \quad B(x_0, \varepsilon^{2/3}).$$

Since $\eta_\varepsilon > 0$ on $\partial B(x_0, \varepsilon^{2/3})$, by the uniqueness of $\tilde{w}$, we deduce that $w \leq \eta_\varepsilon$ in $B(x_0, \varepsilon^{2/3})$. The decay estimate on $\tilde{w}$ implies $0 \leq \sqrt{\alpha} - w(x_0) \leq C\sqrt{\alpha}\,\exp\left(-\frac{\sqrt{\alpha}}{2\varepsilon^{1/3}}\right) \ll C\sqrt{\alpha}\,\varepsilon^{1/3}$. By (3.33), we have

$$\sqrt{a(x_0)} - \sqrt{\alpha} \leq C\sqrt{a(x_0)}\varepsilon^{1/3}.$$

Then (3.32) yields

$$0 \leq \frac{\sqrt{a(x_0)} - \eta_\varepsilon(x_0)}{\sqrt{a(x_0)}} \leq \frac{\sqrt{a(x_0)} - w(x_0)}{\sqrt{a(x_0)}} = \frac{\sqrt{a(x_0)} - \sqrt{\alpha}}{\sqrt{a(x_0)}} + \frac{\sqrt{\alpha} - w(x_0)}{\sqrt{a(x_0)}} \leq C\varepsilon^{1/3},$$

for a constant C independent of x_0.

Proof of 3.1.d). Let $x_0 \in \mathbb{R}^2$ arbitrary. We are going to show that $|\nabla \eta_\varepsilon| \le C\varepsilon^{-1}$ in $B(x_0, \varepsilon)$ with a constant C independent of x_0. We define $\theta : B_2(0) \to \mathbb{R}$ by $\theta(y) = \eta_\varepsilon(x_0 + \varepsilon y)$. From *3.1.b)* and *3.1.c)*, we infer that $|\Delta\theta| = |\,(a(x_0+\varepsilon y) - \theta^2)\,\theta| \le C$ in $B_2(0)$ for a constant C independent of x_0. By elliptic regularity, we deduce that for any $1 \le p < \infty$, $\|\theta\|_{W^{2,p}(B_1(0))} \le C_p$ for a constant C_p independent of x_0. Taking some $p > 2$, it implies that

$$\|\nabla\theta\|_{L^\infty(B_1(0))} \le C$$

for a constant C independent of x_0 which leads to the result.

Proof of 3.1.e). The idea of the proof is due to I. Shafrir. Suppose that $K \subset B_r \subset B_R \subset \mathcal{D}$ for some $0 < r < R < \sqrt{a_0}$. First we prove that

$$|\eta_\varepsilon - \sqrt{a}| \le C_{R,r}\,\varepsilon^2 \quad \text{in } B_r. \tag{3.35}$$

From (3.23) we infer that

$$-\varepsilon^2\Delta(\eta_\varepsilon - \sqrt{a}\,) + \eta_\varepsilon(\eta_\varepsilon + \sqrt{a}\,)(\eta_\varepsilon - \sqrt{a}\,) = \varepsilon^2\Delta(\sqrt{a}\,) = \mathcal{O}(\varepsilon^2) \quad \text{in } B_R.$$

By *3.1.c)*, for ε small, we have $|\eta_\varepsilon - \sqrt{a}| \le \frac{\sqrt{a}}{2}$ in B_R and then, $\eta_\varepsilon(\eta_\varepsilon + \sqrt{a}) \ge d_0 > 0$ in B_R for some constant d_0 which only depends on R. Then estimate (3.35) comes immediately by the following result (which is a slight modification of Lemma 2 in [19]) :

Lemma 3.1. *Assume that $d_0 > 0$ and $0 < r < R$. Let w_ε be a smooth function such that*

$$\begin{cases} -\varepsilon^2\Delta w_\varepsilon + d_0 w_\varepsilon \le 0 & \text{in } B_R,\\ w_\varepsilon \le 1 & \text{on } \partial B_R. \end{cases}$$

Then $w_\varepsilon \le e^{-C\varepsilon^{-1}}$ in B_r with $C = C(R, r, d_0)$.

From (3.23) and (3.35), we deduce that η_ε is uniformly bounded in $W^{2,p}(B_r)$ for any $1 \le p < \infty$. In particular, it implies

$$\|\nabla\eta_\varepsilon\|_{L^\infty(K)} \le C_r\,. \tag{3.36}$$

We now use the same argument to prove *3.1.e)*. We denote

$$z_\varepsilon = \frac{\partial\eta_\varepsilon}{\partial x_j} \quad \text{and} \quad z_0 = \frac{\partial\sqrt{a}}{\partial x_j} \quad \text{for } j \in \{1, 2\}.$$

Obviously, we can assume that (3.35) and (3.36) holds in B_R. Then we have that

$$\begin{aligned} &-\varepsilon^2\Delta z_0 + (3\eta_\varepsilon^2 - a)z_0 = 2az_0 + \mathcal{O}(\varepsilon^2) = \sqrt{a}\frac{\partial a}{\partial x_j} + \mathcal{O}(\varepsilon^2) = \eta_\varepsilon\frac{\partial a}{\partial x_j} + \mathcal{O}(\varepsilon^2) \text{ in } B_R,\\ &-\varepsilon^2\Delta z_\varepsilon + (3\eta_\varepsilon^2 - a)z_\varepsilon = \eta_\varepsilon\frac{\partial a}{\partial x_j} \quad \text{in } B_R. \end{aligned}$$

Therefore,

$$-\varepsilon^2\Delta(z_\varepsilon - z_0) + (3\eta_\varepsilon^2 - a)(z_\varepsilon - z_0) = \mathcal{O}(\varepsilon^2)$$

and we conclude by Lemma 3.1. ■

We now state a result that we will require in Section 2.2. We follow here a technique introduced by M. Struwe (see [81]).

Lemma 3.2. *Let $I : (0, \frac{a_0}{2}) \mapsto \mathbb{R}_+$ be the function defined by*

$$I(\varepsilon) = \text{Min}\left\{E_\varepsilon(\eta),\, \eta \in \mathcal{H}\right\}. \tag{3.37}$$

Then $I(\cdot)$ is locally Lipschitz continuous and non-increasing in $(0, \frac{a_0}{2})$. Moreover,

$$|I'(\varepsilon)| \leq \frac{C|\ln\varepsilon|}{\varepsilon} \quad \textit{for almost every } \varepsilon \in (0, \frac{a_0}{2}). \tag{3.38}$$

Proof. First we infer from *3.1.b)* in Proposition 3.1 that we can find $R > \sqrt{a_0}$ such that for any $0 < \varepsilon < \frac{a_0}{2}$,

$$\int_{\mathbb{R}^2\setminus B_R} |\eta_\varepsilon|^4 + 2a^-(x)|\eta_\varepsilon|^2 \leq C\varepsilon^3. \tag{3.39}$$

Let us now fix some $\varepsilon_0 \in (0, \frac{a_0}{2})$ and $0 < h < \frac{\varepsilon_0}{2}$. We have

$$E_{\varepsilon_0+h}(\eta_{\varepsilon_0+h}) = I(\varepsilon_0 + h) \leq E_{\varepsilon_0+h}(\eta_{\varepsilon_0-h}) \leq E_{\varepsilon_0-h}(\eta_{\varepsilon_0-h}) = I(\varepsilon_0 - h) \leq E_{\varepsilon_0-h}(\eta_{\varepsilon_0+h})$$

and hence

$$E_{\varepsilon_0-h}(\eta_{\varepsilon_0-h}) - E_{\varepsilon_0+h}(\eta_{\varepsilon_0-h}) \leq I(\varepsilon_0 - h) - I(\varepsilon_0 + h) \leq E_{\varepsilon_0-h}(\eta_{\varepsilon_0+h}) - E_{\varepsilon_0+h}(\eta_{\varepsilon_0+h})$$

By (3.39), it leads to

$$\frac{I(\varepsilon_0 + h) - I(\varepsilon_0 - h)}{2h} \geq \frac{-\varepsilon_0}{2(\varepsilon_0 + h)^2(\varepsilon_0 - h)^2}\int_{B_R}\left[(a(x) - |\eta_{\varepsilon_0+h}|^2)^2 - (a^-(x))^2\right] - C \tag{3.40}$$

and

$$\frac{I(\varepsilon_0 + h) - I(\varepsilon_0 - h)}{2h} \leq \frac{-\varepsilon_0}{2(\varepsilon_0 + h)^2(\varepsilon_0 - h)^2}\int_{B_R}\left[(a(x) - |\eta_{\varepsilon_0-h}|^2)^2 - (a^-(x))^2\right] + C. \tag{3.41}$$

which proves with (3.24) that $I(\cdot)$ is locally Lipschitz continuous in $(0, \frac{a_0}{2})$. Therefore $I(\cdot)$ is differentiable almost everywhere in $(0, \frac{a_0}{2})$. We easily check using standard arguments that $\eta_{\varepsilon_0-h} \to \eta_{\varepsilon_0}$ and $\eta_{\varepsilon_0+h} \to \eta_{\varepsilon_0}$ in $L^2(B_R)$ and $L^4(B_R)$ as $h \to 0$. Assuming that ε_0 is a point of differentiability of $I(\cdot)$, we obtain letting $h \to 0$ in (3.40) and (3.41),

$$I'(\varepsilon_0) = \frac{-1}{2\varepsilon_0^3}\int_{B_R}\left[(a(x) - |\eta_{\varepsilon_0}|^2)^2 - (a^-(x))^2\right] + \mathcal{O}(1). \tag{3.42}$$

Then we deduce (3.38) combining (3.24) and (3.42). ■

3.2.2 The profile under the mass constraint

In this section, we study the minimization problem (3.6). The main motivation here is to construct admissible test functions for our model. The result is stated as follows :

Theorem 3.3. *For ε sufficiently small, problem (3.6) admits a unique solution $\tilde{\eta}_\varepsilon$ up to a complex multiplier of modulus one. Moreover, denoting by $k_\varepsilon \in \mathbb{R}$ the Lagrange multiplier associated to the constraint $\|\tilde{\eta}_\varepsilon\|_{L^2(\mathbb{R}^2)} = 1$, we have*

$$|k_\varepsilon| \leq C|\ln \varepsilon| \tag{3.43}$$

and $\tilde{\eta}_\varepsilon$ is characterized by

$$\tilde{\eta}_\varepsilon(x) = \frac{\sqrt{a_0 + k_\varepsilon \varepsilon^2}}{\sqrt{a_0}}\, \eta_{\tilde{\varepsilon}}\Big(\frac{\sqrt{a_0}\, x}{\sqrt{a_0 + k_\varepsilon \varepsilon^2}}\Big) \quad \textit{with} \quad \tilde{\varepsilon} = \frac{a_0 \varepsilon}{a_0 + k_\varepsilon \varepsilon^2}\,. \tag{3.44}$$

In addition,

$$\big|E_\varepsilon(\tilde{\eta}_\varepsilon) - E_\varepsilon(\eta_\varepsilon)\big| \leq C\varepsilon^2 |\ln \varepsilon|^2. \tag{3.45}$$

Remark 3.4. Identity (3.44) gives us automatically the asymptotic properties of $\tilde{\eta}_\varepsilon$ from those of η_ε by a simple change of scale and hence we obtain the analogue of Proposition 3.1 for $\tilde{\eta}_\varepsilon$.

Proof of Theorem 3.3. Step 1 : Existence. Let $(\eta_n)_{n\in\mathbb{N}}$ be a minimizing sequence for (3.6). Extracting a subsequence if necessary, we may assume that $\eta_n \rightharpoonup \tilde{\eta}_\varepsilon$ weakly in $\mathcal{H}$ and strongly in $L^2_{\text{loc}}(\mathbb{R}^2)$ as $n \to \infty$. We easily check that E_ε is lower semi-continuous on $\mathcal{H}$ with respect to the weak $\mathcal{H}$-topology and therefore

$$E_\varepsilon(\tilde{\eta}_\varepsilon) \leq \liminf_{n\to\infty} E_\varepsilon(\eta_n).$$

To conclude that $\tilde{\eta}_\varepsilon$ is a solution of (3.6), it remains to prove that $\|\tilde{\eta}_\varepsilon\|_{L^2(\mathbb{R}^2)} = 1$. Writing $\eta_n = \tilde{\eta}_\varepsilon + \rho_n$, we have $\rho_n \rightharpoonup 0$ weakly in $\mathcal{H}$ and therefore,

$$1 = \int_{\mathbb{R}^2} |\eta_n|^2 = \int_{\mathbb{R}^2} |\tilde{\eta}_\varepsilon|^2 + \int_{\mathbb{R}^2} |\rho_n|^2 + o(1). \tag{3.46}$$

Obviously, $\rho_n \to 0$ in $L^2_{\text{loc}}(\mathbb{R}^2)$ and $\int_{\mathbb{R}^2} |x|^2 |\rho_n|^2 \leq C$. For any $R > 0$, we have

$$R^2 \limsup_{n\to\infty} \int_{\mathbb{R}^2 \setminus B_R} |\rho_n|^2 \leq \limsup_{n\to\infty} \int_{\mathbb{R}^2} |x|^2 |\rho_n|^2 \leq C.$$

Letting $R \to +\infty$ in this inequality, we conclude that $\rho_n \to 0$ strongly in $L^2(\mathbb{R}^2)$. Then we derive from (3.46) that $\|\tilde{\eta}_\varepsilon\|_{L^2(\mathbb{R}^2)} = 1$. Since $E_\varepsilon(|\tilde{\eta}_\varepsilon|) = E_\varepsilon(\tilde{\eta}_\varepsilon)$, we infer that $\tilde{\eta}_\varepsilon = |\tilde{\eta}_\varepsilon| e^{i\alpha}$ for some constant α. Hence we may assume that $\tilde{\eta}_\varepsilon$ is $\mathbb{R}$-valued and $\tilde{\eta}_\varepsilon \geq 0$ in $\mathbb{R}^2$.

Step 2 : Energy bound. We now prove that

$$E_\varepsilon(\tilde{\eta}_\varepsilon) \leq C|\ln \varepsilon|. \tag{3.47}$$

Setting $\hat{\eta}_\varepsilon = \|\eta_\varepsilon\|^{-1}_{L^2(\mathbb{R}^2)}\eta_\varepsilon$, it suffices to check that $E_\varepsilon(\hat{\eta}_\varepsilon) \leq C|\ln\varepsilon|$ by the minimizing property of $\tilde{\eta}_\varepsilon$. First we show that $\|\eta_\varepsilon\|_{L^2(\mathbb{R}^2)}$ remains close to 1 as $\varepsilon \to 0$. Since $\int_{\mathbb{R}^2} a^+ = 1$, we have

$$\int_{\mathbb{R}^2} |\eta_\varepsilon|^2 = 1 + \int_{\mathbb{R}^2} (|\eta_\varepsilon|^2 - a^+(x))$$

and by (3.24),

$$\int_{\mathbb{R}^2} ||\eta_\varepsilon|^2 - a^+(x)| \leq \int_{\{a^- \geq 1/2\}} 2a^-(x)|\eta_\varepsilon|^2 + C\left(\int_{\{a^- \leq 1/2\}} (|\eta_\varepsilon|^2 - a^+(x))^2\right)^{1/2} \leq C\varepsilon|\ln\varepsilon|^{1/2}.$$

Hence $\|\eta_\varepsilon\|^2_{L^2(\mathbb{R}^2)} = 1 + \mathcal{O}(\varepsilon|\ln\varepsilon|^{1/2})$. Then we derive from *3.1.a)* in Proposition 3.1,

$$\int_{\mathbb{R}^2} |\nabla\hat{\eta}_\varepsilon|^2 = \|\eta_\varepsilon\|^{-2}_{L^2(\mathbb{R}^2)} \int_{\mathbb{R}^2} |\nabla\eta_\varepsilon|^2 \leq \int_{\mathbb{R}^2} |\nabla\eta_\varepsilon|^2 + C\varepsilon|\ln\varepsilon|^{3/2} \leq C|\ln\varepsilon|.$$

Using (3.24), we deduce that

$$\begin{aligned}\frac{1}{\varepsilon^2}\int_{\mathbb{R}^2} (a(x) - |\hat{\eta}_\varepsilon|^2)^2 - (a^-(x))^2 &= \frac{1}{\varepsilon^2}\int_{\mathcal{D}} (a(x) - |\hat{\eta}_\varepsilon|^2)^2 + \frac{\|\eta_\varepsilon\|^{-4}_{L^2(\mathbb{R}^2)}}{\varepsilon^2}\int_{\mathbb{R}^2\setminus\mathcal{D}} |\eta_\varepsilon|^4 \\ &\quad + \frac{\|\eta_\varepsilon\|^{-2}_{L^2(\mathbb{R}^2)}}{\varepsilon^2}\int_{\mathbb{R}^2\setminus\mathcal{D}} 2a^-(x)|\eta_\varepsilon|^2 \\ &\leq \frac{1}{\varepsilon^2}\int_{\mathcal{D}} (a(x) - |\hat{\eta}_\varepsilon|^2)^2 + C|\ln\varepsilon|\end{aligned}$$

and

$$\begin{aligned}\frac{1}{\varepsilon^2}\int_{\mathcal{D}} (a(x) - |\hat{\eta}_\varepsilon|^2)^2 &= \frac{1}{\varepsilon^2}\int_{\mathcal{D}} (a(x) - |\eta_\varepsilon|^2)^2 + \frac{2(1 - \|\eta_\varepsilon\|^{-2}_{L^2(\mathbb{R}^2)})}{\varepsilon^2}\int_{\mathcal{D}} (a(x) - |\eta_\varepsilon|^2)|\eta_\varepsilon|^2 \\ &\quad + \frac{(1 - \|\eta_\varepsilon\|^{-2}_{L^2(\mathbb{R}^2)})^2}{\varepsilon^2}\int_{\mathcal{D}} |\eta_\varepsilon|^4 \\ &\leq C|\ln\varepsilon| + C\left(\frac{1}{\varepsilon^2}\int_{\mathcal{D}} (a(x) - |\eta_\varepsilon|^2)^2\right)^{1/2} \\ &\leq C|\ln\varepsilon|.\end{aligned}$$

Therefore $E_\varepsilon(\hat{\eta}_\varepsilon) \leq C|\ln\varepsilon|$ and (3.47) holds.

Step 3 : First bound on the Lagrange multiplier. Since $\tilde{\eta}_\varepsilon$ is a solution of (3.6), there exists $k_\varepsilon \in \mathbb{R}$ such that $\tilde{\eta}_\varepsilon$ satisfies

$$-\Delta\tilde{\eta}_\varepsilon = \frac{1}{\varepsilon^2}(a(x) - |\tilde{\eta}_\varepsilon|^2)\tilde{\eta}_\varepsilon + k_\varepsilon\tilde{\eta}_\varepsilon \quad \text{in } \mathbb{R}^2. \tag{3.48}$$

Multiplying this equation by $\tilde{\eta}_\varepsilon$, integrating by parts and using that $\int_{\mathbb{R}^2} |\tilde{\eta}_\varepsilon|^2 = 1$, we obtain that

$$k_\varepsilon = \int_{\mathbb{R}^2} |\nabla\tilde{\eta}_\varepsilon|^2 + \frac{1}{\varepsilon^2}\int_{\mathbb{R}^2} \big(|\tilde{\eta}_\varepsilon|^2 - a(x)\big)|\tilde{\eta}_\varepsilon|^2.$$

From (3.47) we derive

$$\left|\int_{\mathbb{R}^2} |\nabla\tilde{\eta}_\varepsilon|^2 + \frac{1}{\varepsilon^2}\int_{\mathbb{R}^2\setminus\mathcal{D}} \big(|\tilde{\eta}_\varepsilon|^2 - a(x)\big)|\tilde{\eta}_\varepsilon|^2\right| \le C|\ln\varepsilon|$$

and

$$\begin{aligned}\left|\frac{1}{\varepsilon^2}\int_{\mathcal{D}}(|\tilde{\eta}_\varepsilon|^2 - a(x))|\tilde{\eta}_\varepsilon|^2\right| &\le \frac{1}{\varepsilon^2}\int_{\mathcal{D}}(|\tilde{\eta}_\varepsilon|^2 - a(x))^2 + \frac{1}{\varepsilon^2}\int_{\mathcal{D}} a(x)\big||\tilde{\eta}_\varepsilon|^2 - a(x)\big| \\ &\le C|\ln\varepsilon| + \frac{C}{\varepsilon^2}\left(\int_{\mathcal{D}}(|\tilde{\eta}_\varepsilon|^2 - a(x))^2\right)^{1/2} \\ &\le C\varepsilon^{-1}|\ln\varepsilon|^{1/2}.\end{aligned}$$

Hence we have

$$|k_\varepsilon| \le C\varepsilon^{-1}|\ln\varepsilon|^{1/2}. \tag{3.49}$$

Step 4 : Proof of (3.44). We rewrite equation (3.48) as

$$-\Delta\tilde{\eta}_\varepsilon = \frac{1}{\varepsilon^2}(a_\varepsilon(x) - |\tilde{\eta}_\varepsilon|^2)\tilde{\eta}_\varepsilon \quad \text{in } \mathbb{R}^2, \tag{3.50}$$

with

$$a_\varepsilon(x) = a_0 + k_\varepsilon\varepsilon^2 - |x|^2. \tag{3.51}$$

Since $\tilde{\eta}_\varepsilon \ge 0$ and $\|\tilde{\eta}_\varepsilon\|_{L^2(\mathbb{R}^2)} = 1$, we necessarily have $\tilde{\eta}_\varepsilon > 0$ in $\mathbb{R}^2$ by the maximum principe. Setting for ε small enough,

$$\vartheta_\varepsilon(x) = \frac{\sqrt{a_0}}{\sqrt{a_0 + k_\varepsilon\varepsilon^2}}\,\tilde{\eta}_\varepsilon\Big(\frac{\sqrt{a_0 + k_\varepsilon\varepsilon^2}\,x}{\sqrt{a_0}}\Big), \tag{3.52}$$

a straightforward computation shows that

$$\begin{cases} -\tilde{\varepsilon}^2\Delta\vartheta_\varepsilon = (a(x) - |\vartheta_\varepsilon|^2)\vartheta_\varepsilon & \text{in } \mathbb{R}^2, \\ \vartheta_\varepsilon > 0 & \text{in } \mathbb{R}^2 \end{cases}$$

with $\tilde{\varepsilon} = \frac{a_0\varepsilon}{a_0 + k_\varepsilon\varepsilon^2}$. For ε sufficiently small we have $\tilde{\varepsilon} < \frac{a_0}{2}$ and by Theorem 3.2, it leads to

$$\vartheta_\varepsilon \equiv \eta_{\tilde{\varepsilon}} \tag{3.53}$$

Combining this identity with (3.52) we obtain (3.44).

Step 5 : Proof of (3.43). From (3.53) we infer that

$$E_{\tilde{\varepsilon}}(\vartheta_\varepsilon) = I(\tilde{\varepsilon})$$

where $I(\cdot)$ is defined by (3.37). On the other hand, we easily see from (3.52) that

$$E_{\tilde{\varepsilon}}(\vartheta_\varepsilon) = \frac{a_0}{a_0 + k_\varepsilon\varepsilon^2}\,\tilde{E}_\varepsilon(\tilde{\eta}_\varepsilon)$$

with $\tilde{E}_\varepsilon$ defined by

$$\tilde{E}_\varepsilon(u) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2 + \frac{1}{4\varepsilon^2}\int_{\mathbb{R}^2} (a_\varepsilon(x) - |u|^2)^2 - (a_\varepsilon^-(x))^2 \tag{3.54}$$

and $a_\varepsilon(x)$ given by (3.51). Therefore

$$\tilde{E}_\varepsilon(\tilde{\eta}_\varepsilon) = \frac{a_0 + k_\varepsilon \varepsilon^2}{a_0} I(\tilde{\varepsilon}). \tag{3.55}$$

But since $\|\tilde{\eta}_\varepsilon\|_{L^2(\mathbb{R}^2)} = 1$, we have

$$\begin{aligned}\tilde{E}_\varepsilon(\tilde{\eta}_\varepsilon) &= E_\varepsilon(\tilde{\eta}_\varepsilon) - \frac{k_\varepsilon}{2}\int_{\mathbb{R}^2} |\tilde{\eta}_\varepsilon|^2 + \frac{1}{4\varepsilon^2}\int_{\mathbb{R}^2} (a_\varepsilon^+(x))^2 - (a^+(x))^2 \\ &= E_\varepsilon(\tilde{\eta}_\varepsilon) - \frac{k_\varepsilon}{2} + \frac{1}{4\varepsilon^2}\int_{\mathbb{R}^2} (a_\varepsilon^+(x))^2 - (a^+(x))^2 \qquad (3.56) \\ &\geq I(\varepsilon) - \frac{k_\varepsilon}{2} + \frac{1}{4\varepsilon^2}\int_{\mathbb{R}^2} (a_\varepsilon^+(x))^2 - (a^+(x))^2. \qquad (3.57)\end{aligned}$$

Using the fact that $\int_{\mathbb{R}^2} a^+ = 1$, a simple computation leads to

$$-\frac{k_\varepsilon}{2} + \frac{1}{4\varepsilon^2}\int_{\mathbb{R}^2} (a_\varepsilon^+(x))^2 - (a^+(x))^2 = \frac{\pi a_0 k_\varepsilon^2 \varepsilon^2}{4} + \frac{\pi k_\varepsilon^3 \varepsilon^4}{12}\,. \tag{3.58}$$

Combining (3.55), (3.57) and (3.58), we are led to

$$\frac{\pi a_0 k_\varepsilon^2 \varepsilon^2}{4} \leq |I(\tilde{\varepsilon}) - I(\varepsilon)| + \frac{|k_\varepsilon|\varepsilon^2}{a_0} I(\tilde{\varepsilon}) + \frac{\pi |k_\varepsilon|^3 \varepsilon^4}{12}\,. \tag{3.59}$$

Then we estimate using (3.38), (3.49) and *3.1.a)* in Proposition 3.1,

$$\big|I(\tilde{\varepsilon}) - I(\varepsilon)\big| \leq C\varepsilon^{-1}|\ln\varepsilon||\tilde{\varepsilon} - \varepsilon| \leq C|k_\varepsilon|\varepsilon^2|\ln\varepsilon| \tag{3.60}$$

and

$$\frac{|k_\varepsilon|\varepsilon^2}{a_0} I(\tilde{\varepsilon}) \leq C|k_\varepsilon|\varepsilon^2|\ln\varepsilon|, \quad \frac{\pi|k_\varepsilon|^3\varepsilon^4}{12} \leq C|k_\varepsilon|\varepsilon^2|\ln\varepsilon|.$$

Inserting this estimates in (3.59), we deduce that $|k_\varepsilon| \leq C|\ln\varepsilon|$.

Step 6 : Uniqueness. Let $\hat{\eta}_\varepsilon$ be another solution of (3.6). As for $\tilde{\eta}_\varepsilon$, we may assume that $\hat{\eta}_\varepsilon$ is a real positive function. Let $\hat{k}_\varepsilon$ be the Lagrange multiplier associated to $\hat{\eta}_\varepsilon$, i.e., $\hat{\eta}_\varepsilon$ satisfies

$$-\Delta\hat{\eta}_\varepsilon = \frac{1}{\varepsilon^2}(a(x) - |\hat{\eta}_\varepsilon|^2)\hat{\eta}_\varepsilon + \hat{k}_\varepsilon\hat{\eta}_\varepsilon \quad \text{in } \mathbb{R}^2.$$

By Step 4, whenever ε is small enough, solution $\hat{\eta}_\varepsilon$ is characterized by

$$\hat{\eta}_\varepsilon(x) = \frac{\sqrt{a_0 + \hat{k}_\varepsilon\varepsilon^2}}{\sqrt{a_0}}\,\eta_{\hat{\varepsilon}}\Big(\frac{\sqrt{a_0}\,x}{\sqrt{a_0 + \hat{k}_\varepsilon\varepsilon^2}}\Big) \quad \text{with} \quad \hat{\varepsilon} = \frac{a_0\varepsilon}{a_0 + \hat{k}_\varepsilon\varepsilon^2}\,.$$

Hence it suffices to prove that $\hat{k}_\varepsilon = k_\varepsilon$. We proceed by contradiction. Assume for instance that $k_\varepsilon < \hat{k}_\varepsilon$. Then $\hat{\eta}_\varepsilon$ satisfies

$$-\Delta \hat{\eta}_\varepsilon \geq \frac{1}{\varepsilon^2}(a(x) - |\hat{\eta}_\varepsilon|^2)\hat{\eta}_\varepsilon + k_\varepsilon \hat{\eta}_\varepsilon \quad \text{in } \mathbb{R}^2. \tag{3.61}$$

We consider the function

$$\hat{\vartheta}_\varepsilon(x) = \frac{\sqrt{a_0}}{\sqrt{a_0 + k_\varepsilon \varepsilon^2}}\, \hat{\eta}_\varepsilon\left(\frac{\sqrt{a_0 + k_\varepsilon \varepsilon^2}\, x}{\sqrt{a_0}}\right), \tag{3.62}$$

which satisfies by (3.61),

$$\begin{cases} -\tilde{\varepsilon}^2 \Delta \hat{\vartheta}_\varepsilon \geq (a(x) - |\hat{\vartheta}_\varepsilon|^2)\hat{\vartheta}_\varepsilon & \text{in } \mathbb{R}^2, \\ \hat{\vartheta}_\varepsilon > 0 & \text{in } \mathbb{R}^2. \end{cases}$$

Therefore $\hat{\vartheta}_\varepsilon$ is a supersolution of (3.23) with $\tilde{\varepsilon}$ instead of ε. By Remark 3.3, we infer that $\hat{\vartheta}_\varepsilon \geq \eta_{\tilde{\varepsilon}}$ in $\mathbb{R}^2$. By (3.44) and (3.62), it leads to $\hat{\eta}_\varepsilon \geq \tilde{\eta}_\varepsilon$ in $\mathbb{R}^2$. Since $\|\hat{\eta}_\varepsilon\|_{L^2(\mathbb{R}^2)} = \|\tilde{\eta}_\varepsilon\|_{L^2(\mathbb{R}^2)} = 1$, we conclude that $\hat{\eta}_\varepsilon \equiv \tilde{\eta}_\varepsilon$ and hence $k_\varepsilon = \hat{k}_\varepsilon$, contradiction.

Step 7 : Proof of (3.45). By (3.43), (3.55), (3.60) and *3.1.a)* in Proposition 3.1, we have

$$\tilde{E}_\varepsilon(\tilde{\eta}_\varepsilon) = E_\varepsilon(\eta_\varepsilon) + \mathcal{O}(\varepsilon^2 |\ln \varepsilon|^2). \tag{3.63}$$

On the other hand, by (3.43), (3.56) and (3.58), we also have

$$\tilde{E}_\varepsilon(\tilde{\eta}_\varepsilon) = E_\varepsilon(\tilde{\eta}_\varepsilon) + \mathcal{O}(\varepsilon^2 |\ln \varepsilon|^2).$$

and (3.45) follows. ■

3.3 Minimizing F_ε under the mass constraint

Our aim in this section is to make a first description of minimizers u_ε of F_ε under the mass constraint. We prove the existence of u_ε and that $|u_\varepsilon|$ is concentrated in $\mathcal{D}$. We also present some tools that we will use in the sequel, in particular the splitting of energy (3.11).

3.3.1 Existence and first properties of minimizers

First, we seek minimizers u_ε of F_ε under the constraint $\|u_\varepsilon\|_{L^2(\mathbb{R}^2)} = 1$ and then study some first asymptotic properties. We want to perform the minimization in $\mathcal{H}$ and we shall see that F_ε is well defined on $\mathcal{H}$:

Lemma 3.3. *For any $u \in \mathcal{H}$, $\sigma > 0$ and $R > \sqrt{a_0}$, we have*

$$|R_\varepsilon(u)| \le \sigma \int_{\mathbb{R}^2} |\nabla u|^2 + \frac{\Omega^2 R^2}{8\sigma(R^2 - a_0)} \int_{\mathbb{R}^2} \left[(a(x) - |u|^2)^2 - (a^-(x))^2\right] + C_{R,\sigma}\,\Omega^2.$$

In particular, the functional F_ε is well defined on $\mathcal{H}$.

Proposition 3.2. *Assume that $\Omega < \varepsilon^{-1}$. Then there exists at least one map u_ε which minimizes F_ε in $\{u \in \mathcal{H}, \|u\|_{L^2(\mathbb{R}^2)} = 1\}$. Moreover, u_ε is smooth and there exists $\ell_\varepsilon \in \mathbb{R}$ such that u_ε satisfies*

$$-\Delta u_\varepsilon + 2i\Omega x^\perp \cdot \nabla u_\varepsilon = \frac{1}{\varepsilon^2}(a(x) - |u_\varepsilon|^2)u_\varepsilon + \ell_\varepsilon u_\varepsilon \quad \textit{in } \mathbb{R}^2. \tag{3.64}$$

We emphasize that we state the result for an angular velocity Ω strictly less than $1/\varepsilon$ but we also recall that we only consider the case of an angular velocity Ω at most of order $|\ln\varepsilon|$. In the sequel, we assume that

$$\Omega \le \omega_0 |\ln\varepsilon| \tag{3.65}$$

for some positive constant ω_0.

Before proving Lemma 3.3 and Proposition 3.2, we present some basic properties of any minimizer u_ε. We point out that the exponential decay of $|u_\varepsilon|$ outside the domain $\mathcal{D}$ (see *3.3.c)* below) shows that almost all the mass of u_ε is concentrated in $\mathcal{D}$.

Proposition 3.3. *For ε sufficiently small,*

3.3.a) $E_\varepsilon(u_\varepsilon) \le C_{\omega_0}|\ln\varepsilon|^2$,

3.3.b) $|\ell_\varepsilon| \le C_{\omega_0}\,\varepsilon^{-1}|\ln\varepsilon|$,

3.3.c) $|u_\varepsilon(x)| \le C_{\omega_0}\varepsilon^{1/3}|\ln\varepsilon|^{1/2}\exp\left(\dfrac{a_0 - |x|^2}{4\varepsilon^{2/3}}\right)$ *for $x \in \mathbb{R}^2\setminus\mathcal{D}$ with* $|x| \ge \sqrt{a_0 + 2\varepsilon^{1/3}}$,

3.3.d) $|u_\varepsilon(x)| \le \sqrt{a(x) + |\ell_\varepsilon|\varepsilon^2 + \varepsilon^2\Omega^2|x|^2}$ *for $x \in \mathcal{D}$ with* $\operatorname{dist}(x, \partial\mathcal{D}) \ge \varepsilon^{1/8}$,

3.3.e) $|u_\varepsilon| \le \sqrt{a_0} + C_{\omega_0}\varepsilon|\ln\varepsilon|$ *in* $\mathbb{R}^2$,

3.3.f) $\|\nabla u_\varepsilon\|_{L^\infty(K)} \le C_{\omega_0,K}\,\varepsilon^{-1}$ *for any compact set $K \subset \mathbb{R}^2$.*

Remark 3.5. As a direct consequence of *3.3.a)*, we have

$$\int_{\mathbb{R}^2\setminus\mathcal{D}} \left(|u_\varepsilon|^4 + 2a^-(x)|u_\varepsilon|^2\right) + \int_{\mathcal{D}} (|u_\varepsilon|^2 - a(x))^2 \le C_{\omega_0}\,\varepsilon^2|\ln\varepsilon|^2. \tag{3.66}$$

Proof of Lemma 3.3. Let $u \in \mathcal{H}$ and $\sigma \in (0,1)$. We have

$$4\sigma\,|R_\varepsilon(u)| \le 4\sigma^2 \int_{\mathbb{R}^2} |\nabla u|^2 + \Omega^2 \int_{\mathbb{R}^2} |x|^2|u|^2.$$

For $R > \sqrt{a_0}$, we have $|x|^2 \leq -\frac{R^2}{R^2-a_0}a(x)$ whenever $|x| \geq R$. Then we derive

$$4\sigma\,|R_\varepsilon(u)| \leq 4\sigma^2 \int_{\mathbb{R}^2} |\nabla u|^2 - \frac{\Omega^2 R^2}{2(R^2-a_0)} \int_{\mathbb{R}^2\setminus B_R} 2a(x)|u|^2 + \Omega^2 \int_{B_R} |x|^2|u|^2. \tag{3.67}$$

Now we notice that

$$\begin{aligned}\int_{B_R} |x|^2|u|^2 &= \frac{R^2}{2(R^2-a_0)} \int_{B_R} -2a(x)|u|^2 - \frac{a_0}{R^2-a_0}\int_{B_R} |x|^2|u|^2 + \frac{a_0 R^2}{R^2-a_0}\int_{B_R} |u|^2 \\ &\leq \frac{R^2}{2(R^2-a_0)} \int_{B_R} -2a(x)|u|^2 + \frac{R^2}{2(R^2-a_0)}\int_{B_R} |u|^4 + \frac{\pi R^4 a_0^2}{2(R^2-a_0)}.\end{aligned}$$

Inserting this estimate in (3.67), we obtain

$$|R_\varepsilon(u)| \leq \sigma \int_{\mathbb{R}^2} |\nabla u|^2 + \frac{\Omega^2 R^2}{8\sigma(R^2-a_0)} \int_{\mathbb{R}^2} \left[(a(x)-|u|^2)^2 - (a^-(x))^2\right] + \frac{\pi\Omega^2 R^4 a_0^2}{8\sigma(R^2-a_0)}$$

and the proof is complete. ∎

Proof of Proposition 3.2. Since $\Omega < \varepsilon^{-1}$, we can find $0 < \delta < 1$ such that $\Omega \leq \delta\varepsilon^{-1}$. Taking in Lemma 3.3

$$\sigma = \frac{\delta^2+1}{4} \quad \text{and} \quad R = \sqrt{\frac{2(1+\delta^2)a_0}{1-\delta^2}},$$

we infer that for any $u \in \mathcal{H}$,

$$\frac{1-\delta^2}{2}\, E_\varepsilon(u) - C_\delta\,\Omega^2 \leq F_\varepsilon(u) \leq 2\,E_\varepsilon(u) + C_\delta\,\Omega^2. \tag{3.68}$$

We easily check that E_ε is coercive in $\mathcal{H}$ (i.e., there exists a positive constant C such that $E_\varepsilon(u) \geq C(\|u\|^2_{\mathcal{H}} - 1)$) and by (3.68), F_ε is coercive, too. Let $(u_n)_{n\in\mathbb{N}} \subset \mathcal{H}$ be a minimizing sequence of F_ε in $\big\{u \in \mathcal{H},\ \|u\|_{L^2(\mathbb{R}^2)} = 1\big\}$. From the coerciveness of F_ε, we get that $(u_n)_{n\in\mathbb{N}}$ is bounded in $\mathcal{H}$ and therefore, there exists $u_\varepsilon \in \mathcal{H}$ such that up to a subsequence,

$$u_n \rightharpoonup u_\varepsilon \text{ weakly in } \mathcal{H} \ \text{ and } \ u_n \to u_\varepsilon \text{ in } L^4_{\text{loc}}(\mathbb{R}^2). \tag{3.69}$$

Arguing as in Step 1 in the proof of Theorem 3.3, we infer that $\|u_\varepsilon\|_{L^2(\mathbb{R}^2)} = 1$. Writing for $u \in \mathcal{H}$,

$$\begin{aligned}F_\varepsilon(u) =& \frac{1}{2}\int_{\mathbb{R}^2} \left|(\nabla - i\Omega x^\perp)u\right|^2 + \frac{1}{2\varepsilon^2}\int_{\{a^-(x)\geq\Omega^2\varepsilon^2|x|^2\}} \left[\frac{1}{2}\,|u|^4 + \big(a^-(x) - \varepsilon^2\Omega^2|x|^2\big)\,|u|^2\right] \\ &+ \frac{1}{4\varepsilon^2}\int_{\{a^-(x)\leq\Omega^2\varepsilon^2|x|^2\}} \left[(a(x)-|u|^2)^2 - (a^-(x))^2 - 2\Omega^2\varepsilon^2|x|^2\,|u|^2\right].\end{aligned}$$

we observe that the functional

$$u \in \mathcal{H} \mapsto \frac{1}{2}\int_{\mathbb{R}^2} \left|(\nabla - i\Omega x^\perp)u\right|^2 + \frac{1}{2\varepsilon^2}\int_{\{a^-(x)\geq\Omega^2\varepsilon^2|x|^2\}} \left[\frac{1}{2}\,|u|^4 + (a^-(x) - \varepsilon^2\Omega^2|x|^2)\,|u|^2\right]$$

is convex continuous on $\mathcal{H}$ for the strong topology. Then from (3.69), it follows that

$$F_\varepsilon(u_\varepsilon) \le \liminf_{n\to\infty} F_\varepsilon(u_n).$$

Hence u_ε is a minimizer of F_ε in $\{u \in \mathcal{H},\ \|u\|_{L^2(\mathbb{R}^2)} = 1\}$ and by the Lagrange multiplier rule, there exists $\ell_\varepsilon \in \mathbb{R}$ such that (3.64) holds. By standard elliptic regularity, we deduce that u_ε is smooth in $\mathbb{R}^2$. ■

Proof of Proposition 3.3. Proof of 3.3.a). Let $\hat{\eta}_\varepsilon$ be any real minimizer of E_ε under the constraint $\|\hat{\eta}_\varepsilon\|_{L^2(\mathbb{R}^2)} = 1$. Since $(i\hat{\eta}_\varepsilon, \nabla\hat{\eta}_\varepsilon) \equiv 0$, we derive from (3.47),

$$F_\varepsilon(u_\varepsilon) \le F_\varepsilon(\hat{\eta}_\varepsilon) = E_\varepsilon(\hat{\eta}_\varepsilon) \le C|\ln\varepsilon|. \tag{3.70}$$

Using Lemma 3.3 with $\sigma = 1/4$ and $R = \sqrt{2a_0}$, we infer that for ε small enough,

$$\frac{1}{2} E_\varepsilon(u_\varepsilon) - C\Omega^2 \le F_\varepsilon(u_\varepsilon). \tag{3.71}$$

Combining (3.70) and (3.71), we obtain *3.3.a)*.

Proof of 3.3.b). Multiplying equation (3.64) by u_ε, integrating by parts and using that $\int_{\mathbb{R}^2} |u_\varepsilon|^2 = 1$, we obtain

$$\ell_\varepsilon = \int_{\mathbb{R}^2} |\nabla u_\varepsilon|^2 - 2\Omega \int_{\mathbb{R}^2} x^\perp \cdot (iu_\varepsilon, \nabla u_\varepsilon) + \frac{1}{\varepsilon^2} \int_{\mathbb{R}^2} (|u_\varepsilon|^2 - a(x))|u_\varepsilon|^2. \tag{3.72}$$

From *3.3.a)* and Lemma 3.3, we derive

$$\left| \int_{\mathbb{R}^2} |\nabla u_\varepsilon|^2 - 2\Omega \int_{\mathbb{R}^2} x^\perp \cdot (iu_\varepsilon, \nabla u_\varepsilon) + \frac{1}{\varepsilon^2} \int_{\mathbb{R}^2\setminus\mathcal{D}} (|u_\varepsilon|^2 - a(x))|u_\varepsilon|^2 \right| \le C_{\omega_0} |\ln\varepsilon|^2 \tag{3.73}$$

and arguing as in the proof of (3.49), we obtain by (3.66),

$$\left| \frac{1}{\varepsilon^2} \int_{\mathcal{D}} (|u_\varepsilon|^2 - a(x))|u_\varepsilon|^2 \right| \le C_{\omega_0}\, \varepsilon^{-1} |\ln\varepsilon|. \tag{3.74}$$

Using (3.72), (3.73) and (3.74), we derive that $|\ell_\varepsilon| \le C_{\omega_0}\varepsilon^{-1}|\ln\varepsilon|$.

Proof of 3.3.c). We argue as in [2], Proposition 2.5. Setting $U_\varepsilon := |u_\varepsilon|^2$, we deduce from equation (3.64),

$$\frac{1}{2}\Delta U_\varepsilon = |\nabla u_\varepsilon|^2 - 2\Omega\, x^\perp \cdot (iu_\varepsilon, \nabla u_\varepsilon) - \frac{1}{\varepsilon^2}(a(x) - U_\varepsilon)U_\varepsilon - \ell_\varepsilon U_\varepsilon$$

and hence

$$-\Delta U_\varepsilon + \frac{2}{\varepsilon^2}\left(U_\varepsilon - (a(x) + \varepsilon^2|\ell_\varepsilon| + \varepsilon^2\Omega^2|x|^2)\right) U_\varepsilon \le 0 \quad \text{in } \mathbb{R}^2. \tag{3.75}$$

Let $T_\varepsilon = \{x \in \mathbb{R}^2 \setminus \mathcal{D},\, a^-(x) > 2(\varepsilon^2|\ell_\varepsilon| + \varepsilon^2\Omega^2|x|^2)\}$. From (3.75), we infer that

$$\Delta U_\varepsilon \geq \frac{1}{\varepsilon^2}\, a^-(x) U_\varepsilon \geq 0 \quad \text{in } T_\varepsilon \tag{3.76}$$

and thus U_ε is subharmonic in $T_\varepsilon \subset \mathbb{R}^2 \setminus \mathcal{D}$. Note that by (3.66),

$$\int_{\mathbb{R}^2\setminus\mathcal{D}} U_\varepsilon^2 \leq C_{\omega_0}\varepsilon^2 |\ln\varepsilon|^2. \tag{3.77}$$

Consider now

$$D_\varepsilon = \{x \in \mathbb{R}^2,\, \mathrm{dist}(x, \overline{\mathcal{D}}) > \varepsilon^{1/3}\}.$$

By *3.3.b)*, for ε small enough we have $\partial T_\varepsilon \subset \{x \in \mathbb{R}^2,\, |x| \leq a_0 + \frac{\varepsilon^{1/3}}{2}\}$. Then for ε small and any $x_0 \in D_\varepsilon$, we have $B(x_0, \frac{\varepsilon^{1/3}}{2}) \subset T_\varepsilon$. We infer from the subharmonicity of U_ε in T_ε and (3.77),

$$0 \leq U_\varepsilon(x_0) \leq \frac{4}{\pi\varepsilon^{2/3}} \int_{B(x_0, \frac{\varepsilon^{1/3}}{2})} U_\varepsilon \leq \frac{C}{\varepsilon^{1/3}} \left(\int_{B(x_0, \frac{\varepsilon^{1/3}}{2})} U_\varepsilon^2 \right)^{1/2} \leq C^\star_{\omega_0}\, \varepsilon^{2/3} |\ln\varepsilon| \quad \forall x_0 \in D_\varepsilon,$$

with a constant $C^\star_{\omega_0}$ independent of x_0. Hence we conclude that $U_\varepsilon \to 0$ locally uniformly in $\mathbb{R}^2 \setminus \overline{\mathcal{D}}$ as $\varepsilon \to 0$. It also follows that $u_\varepsilon \in L^\infty(\mathbb{R}^2)$ and $U_\varepsilon \in H^1(\mathbb{R}^2)$. By (3.76), U_ε is a subsolution of

$$\begin{cases} -\varepsilon^2 \Delta w + a^-(x) w = 0 & \text{in } D_\varepsilon, \\ w > 0 & \text{in } D_\varepsilon, \\ w = C^\star_{\omega_0}\, \varepsilon^{2/3} |\ln\varepsilon| & \text{on } \partial D_\varepsilon. \end{cases} \tag{3.78}$$

We check that for ε small enough,

$$v_{\text{out}}(x) = C^\star_{\omega_0} \varepsilon^{2/3} |\ln\varepsilon| \exp\left(\frac{a_0 + \varepsilon^{1/3} - |x|^2}{\varepsilon^{2/3}} \right)$$

is a supersolution of (3.78). Therefore

$$U_\varepsilon(x) = |u_\varepsilon(x)|^2 \leq v_{\text{out}}(x) \leq C^\star_{\omega_0} \varepsilon^{2/3} |\ln\varepsilon| \exp\left(\frac{a_0 - |x|^2}{2\varepsilon^{2/3}} \right) \quad \text{for } |x|^2 \geq a_0 + 2\varepsilon^{1/3}.$$

Proof of 3.3.d) and 3.3.e). We set $r_0 = \sqrt{a_0} - \varepsilon^{1/8}$ and

$$v_{\text{in}}(x) = \begin{cases} a(x) + |\ell_\varepsilon|\varepsilon^2 + \varepsilon^2\Omega^2\, |x|^2 & \text{if} \quad |x| \leq r_0 \\ -(1 - \varepsilon^2\Omega^2) r_0 (2|x| - r_0) + a_0 + |\ell_\varepsilon|\varepsilon^2 & \text{if} \quad r_0 \leq r \leq \sqrt{a_0} + \varepsilon^{1/3} \end{cases}$$

We easily verify that for ε sufficiently small, v_{in} satisfies

$$\begin{cases} -\varepsilon^2 \Delta v_{\text{in}} \geq 2\left(a(x) + |\ell_\varepsilon|\varepsilon^2 + \varepsilon^2\Omega^2|x|^2 - v_{\text{in}}\right) v_{\text{in}} & \text{in } B_{\sqrt{a_0}+\varepsilon^{1/3}}, \\ v_{\text{in}} > 0 & \text{in } B_{\sqrt{a_0}+\varepsilon^{1/3}}, \\ v_{\text{in}}(x) \geq C^\star_{\omega_0} \varepsilon^{2/3} |\ln\varepsilon| \geq U_\varepsilon(x) & \text{on } \partial B_{\sqrt{a_0}+\varepsilon^{1/3}} \end{cases} \tag{3.79}$$

and

$$v_{\rm in}(x) \geq a(x) + |\ell_\varepsilon|\varepsilon^2 + \varepsilon^2\Omega^2|x|^2 \quad \text{in } B_{\sqrt{a_0}+\varepsilon^{1/3}}.$$

Setting $V_\varepsilon = U_\varepsilon - v_{\rm in}$, we deduce from (3.75) and (3.79),

$$\begin{cases} -\varepsilon^2\Delta V_\varepsilon + b(x)V_\varepsilon \leq 0 & \text{in } B_{\sqrt{a_0}+\varepsilon^{1/3}}, \\ V_\varepsilon \leq 0 & \text{on } \partial B_{\sqrt{a_0}+\varepsilon^{1/3}}, \end{cases}$$

with

$$b(x) = 2\big(U_\varepsilon + v_{\rm in} - (a(x) + |\ell_\varepsilon|\varepsilon^2 + \varepsilon^2\Omega^2|x|^2)\big) \geq 0.$$

Hence $V_\varepsilon \leq 0$ which gives us *3.3.d)*. Then estimate *3.3.e)* directly follows from the construction of $v_{\rm in}$ and $v_{\rm out}$ and from *3.3.b)*.

Proof of 3.3.f). Without loss of generality, we may assume that $K = B_R$ with $R > 0$. Consider the rescaled function

$$\tilde{u}_\varepsilon(x) = u_\varepsilon(\varepsilon x), \, \forall x \in B_{3+\frac{R}{\varepsilon}}.$$

From (3.64), we obtain

$$-\Delta\tilde{u}_\varepsilon = (a(\varepsilon x) - |\tilde{u}_\varepsilon|^2)\tilde{u}_\varepsilon - 2i\Omega\varepsilon^2 x^\perp \cdot \nabla\tilde{u}_\varepsilon + \ell_\varepsilon\varepsilon^2\tilde{u}_\varepsilon \quad \text{in} \quad B_{3+\frac{R}{\varepsilon}}.$$

Take an arbitrary $x_0 \in B_{\frac{R}{\varepsilon}}$. It suffices to prove that exists a constant $C_R > 0$ independent of x_0 and ε such that

$$\|\nabla\tilde{u}_\varepsilon\|_{L^\infty(B(x_0,1))} \leq C_{\omega_0,R}. \tag{3.80}$$

Indeed, by *3.3.c)*, we know that $a(x)u_\varepsilon$ is bounded in $\mathbb{R}^2$. Using *3.3.a)*, *3.3.b)* and *3.3.e)*, we derive that

$$\begin{aligned} \|\Delta\tilde{u}_\varepsilon\|_{L^2(B(x_0,3))} &\leq C\big(\|(a(x) + \ell_\varepsilon\varepsilon^2 - |u_\varepsilon|^2)u_\varepsilon\|_{L^\infty(\mathbb{R}^2)} + \Omega\varepsilon^2\|x^\perp \cdot \nabla\tilde{u}_\varepsilon\|_{L^2(B(x_0,3))}\big) \\ &\leq C_{\omega_0}(1 + \Omega\varepsilon\|x^\perp \cdot \nabla u_\varepsilon\|_{L^2(B_{R+1})}) \\ &\leq C_{\omega_0,R}(1 + \Omega\varepsilon|\ln\varepsilon|) \\ &\leq C_{\omega_0,R}. \end{aligned}$$

Since $\|\tilde{u}_\varepsilon\|_{L^\infty(B(x_0,3))} \leq C_{\omega_0}$ by *3.3.e)*, it follows that $\|\tilde{u}_\varepsilon\|_{H^2(B(x_0,2))} \leq C_{\omega_0,R}$ by elliptic regularity. From Sobolev inequalities, we deduce that

$$\|\nabla\tilde{u}_\varepsilon\|_{L^4(B(x_0,2))} \leq C_{\omega_0,R}.$$

We repeat the above argument and it results

$$\|\Delta\tilde{u}_\varepsilon\|_{L^4(B(x_0,2))} \leq C_{\omega_0,R}(1 + \Omega\varepsilon^{3/2}\|\nabla\tilde{u}_\varepsilon\|_{L^4(B(x_0,2))}) \leq C_{\omega_0,R}.$$

It finally yields $\|\tilde{u}_\varepsilon\|_{W^{2,4}(B(x_0,1))} \leq C_{\omega_0,R}$ which implies (3.80). ■

3.3.2 Splitting the energy

In this section, we prove the splitting of the energy (3.11). The splitting technique has been introduced by L. Lassoued and P. Mironescu in [65]. The goal is to decouple the energy $F_\varepsilon(u)$ into two independent parts : the energy of the density profile η_ε and the reduced energy of the function u/η_ε (which plays the role, in some sense, of the phase of u). For $0 < \varepsilon < \frac{a_0}{2}$, we introduce the class

$$\mathcal{G}_\varepsilon = \left\{ v \in H^1_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{C}), \int_{\mathbb{R}^2} \eta_\varepsilon^2 |\nabla v|^2 + \eta_\varepsilon^4 (1 - |v|^2)^2 < +\infty \right\}.$$

We have the following result :

Lemma 3.4. *Let $u \in \mathcal{H}$ and $0 < \varepsilon < \frac{a_0}{2}$. Then $v = u/\eta_\varepsilon$ is well defined, belongs to $\mathcal{G}_\varepsilon$ and*

$$F_\varepsilon(u) = E_\varepsilon(\eta_\varepsilon) + \mathcal{F}_\varepsilon^{\eta_\varepsilon}(v). \tag{3.81}$$

Proof. Let $u \in \mathcal{H}$ and define $v = u/\eta_\varepsilon \in H^1_{\mathrm{loc}}(\mathbb{R}^2)$. We consider the sequence $(u_n)_{n\in\mathbb{N}} \subset \mathcal{H}$ defined by

$$u_n(x) = \zeta\left(n^{-1}|x|\right) u(x)$$

where ζ is the "cut-off" type function defined in (3.27). We easily check that $u_n \to u$ a.e. and $\nabla u_n \to \nabla u$ a.e. in $\mathbb{R}^2$. Setting $v_n = u_n/\eta_\varepsilon$, then we have $v_n \to v$ a.e. and $\nabla v_n \to \nabla v$ a.e. in $\mathbb{R}^2$. Since u_n has a compact support, we get that $v_n \in \mathcal{G}_\varepsilon$ for any $n \in \mathbb{N}$. We have

$$|\nabla u_n|^2 = |\nabla \eta_\varepsilon|^2 + \eta_\varepsilon^2 |\nabla v_n|^2 + (|v_n|^2 - 1)|\nabla \eta_\varepsilon|^2 + \eta_\varepsilon \nabla \eta_\varepsilon \cdot \nabla(|v_n|^2 - 1),$$

and therefore,

$$\begin{aligned} E_\varepsilon(u_n) = E_\varepsilon(\eta_\varepsilon) &+ \frac{1}{2}\int_{\mathbb{R}^2} \left(\eta_\varepsilon^2 |\nabla v_n|^2 + \frac{\eta_\varepsilon^4}{2\varepsilon^2}(|v_n|^2 - 1)^2\right) \\ &+ \frac{1}{2}\int_{\mathbb{R}^2} \left((|v_n|^2 - 1)|\nabla \eta_\varepsilon|^2 + \eta_\varepsilon \nabla \eta_\varepsilon \cdot \nabla(|v_n|^2 - 1) + \frac{1}{\varepsilon^2}\eta_\varepsilon^2(|v_n|^2 - 1)(\eta_\varepsilon^2 - a(x))\right). \end{aligned}$$

As in [65], the main idea is to multiply the equation (3.23) by $\eta_\varepsilon(|v_n|^2 - 1)$ and then to integrate by parts. It leads to

$$\int_{\mathbb{R}^2} \left\{ (|v_n|^2 - 1)|\nabla \eta_\varepsilon|^2 + \eta_\varepsilon \nabla \eta_\varepsilon \nabla |v_n|^2 + \frac{\eta_\varepsilon^2}{\varepsilon^2}(|v_n|^2 - 1)(\eta_\varepsilon^2 - a(x)) \right\} = 0$$

and we conclude that $E_\varepsilon(u_n) = E_\varepsilon(\eta_\varepsilon) + \mathcal{E}_\varepsilon^{\eta_\varepsilon}(v_n)$ for every $n \in \mathbb{N}$. Now we observe that

$$|u_n| \le |u| \quad \text{and} \quad |\nabla u_n| \le |\nabla u| + |u| \quad \text{a.e. in } \mathbb{R}^2, \tag{3.82}$$

and by dominated convergence, it results $E_\varepsilon(u_n) \to E_\varepsilon(u)$. Applying Fatou's lemma, we obtain

$$\mathcal{E}_\varepsilon^{\eta_\varepsilon}(v) \le \lim_{n\to+\infty} \mathcal{E}_\varepsilon^{\eta_\varepsilon}(v_n) = \lim_{n\to+\infty} E_\varepsilon(u_n) - E_\varepsilon(\eta_\varepsilon) = E_\varepsilon(u) - E_\varepsilon(\eta_\varepsilon) < +\infty,$$

and we conclude that $v \in \mathcal{G}_\varepsilon$. Since $\eta_\varepsilon^{-1}|u||\nabla\eta_\varepsilon| \le |\nabla u| + \eta_\varepsilon|\nabla v|$, we infer from (3.82) that

$$\eta_\varepsilon^2|\nabla v_n|^2 \le C(|\nabla u|^2 + |u|^2 + \eta_\varepsilon^2|\nabla v|^2)$$

and

$$\eta_\varepsilon^4(|v_n|^2 - 1)^2 \le 2(|u|^4 + \eta_\varepsilon^4).$$

By dominated convergence, we finally get that

$$\mathcal{E}_\varepsilon^{\eta_\varepsilon}(v) = \lim_{n\to+\infty} \mathcal{E}_\varepsilon^{\eta_\varepsilon}(v_n) = \lim_{n\to+\infty} E_\varepsilon(u_n) - E_\varepsilon(\eta_\varepsilon) = E_\varepsilon(u) - E_\varepsilon(\eta_\varepsilon).$$

The rest of the proof is trivial since $x^\perp \cdot (iu, \nabla u) = \eta_\varepsilon^2 x^\perp \cdot (iv, \nabla v)$ a.e. in $\mathbb{R}^2$. ■

Remark 3.6. By the splitting of the energy, one can deduce the uniqueness of positive minimizers η_ε of E_ε.

We now want to translate some of the properties of u_ε to the map $u_\varepsilon/\eta_\varepsilon$. To this aim, we define the subclass $\tilde{\mathcal{G}}_\varepsilon \subset \mathcal{G}_\varepsilon$ by

$$\tilde{\mathcal{G}}_\varepsilon = \big\{v \in \mathcal{G}_\varepsilon,\ \eta_\varepsilon v \in \mathcal{H} \text{ and } \|\eta_\varepsilon v\|_{L^2(\mathbb{R}^2)} = 1\big\}.$$

The result below directly follows from Proposition 3.1, Proposition 3.2 and Proposition 3.3.

Proposition 3.4. *For small $\varepsilon > 0$, let u_ε be a minimizer of F_ε in $\big\{u \in \mathcal{H},\ \|u\|_{L^2(\mathbb{R}^2)} = 1\big\}$. Then $v_\varepsilon = u_\varepsilon/\eta_\varepsilon$ minimizes $\mathcal{F}_\varepsilon^{\eta_\varepsilon}$ in $\tilde{\mathcal{G}}_\varepsilon$. Moreover, we have*

3.4.a) $\mathcal{E}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon) \le C_{\omega_0}|\ln\varepsilon|^2$,

3.4.b) $|v_\varepsilon(x)| \le 1 + C_{\omega_0}\,\varepsilon^{1/3}$ *for* $x \in \mathcal{D}$ *with* $\operatorname{dist}(x, \partial\mathcal{D}) \ge \varepsilon^{1/8}$,

3.4.c) $\|\nabla v_\varepsilon\|_{L^\infty(K)} \le C_{\omega_0,K}\,\varepsilon^{-1}$ *for any compact subset* $K \subset \mathcal{D}$.

3.3.3 Splitting the domain

The main goal in this section is to show that we can excise the region of $\mathbb{R}^2$ where the density $|u_\varepsilon|$ is very small (which corresponds roughly speaking to the exterior of $\mathcal{D}$) without modifying the relevant part in the energy.

Proposition 3.5. *For small ε and $\nu \in (1,2)$, we set*

$$\mathcal{D}_\varepsilon^\nu = \big\{x \in \mathbb{R}^2,\ a(x) > \nu|\ln\varepsilon|^{-3/2}\big\}. \tag{3.83}$$

We have

$$\mathcal{F}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon, \mathcal{D}_\varepsilon^\nu) \le C_{\omega_0}|\ln\varepsilon|^{-1}.$$

Proof. Since u_ε minimizes F_ε on $\{u \in \mathcal{H},\ \|u\|_{L^2(\mathbb{R}^2)} = 1\}$, we have for ε sufficiently small $F_\varepsilon(u_\varepsilon) \le F_\varepsilon(\tilde\eta_\varepsilon)$ (we recall that $\tilde\eta_\varepsilon$ is defined as the unique real positive solution of (3.6)). As before, $R_\varepsilon(\tilde\eta_\varepsilon) = 0$ since $\tilde\eta_\varepsilon$ is real valued. Then we get that $F_\varepsilon(u_\varepsilon) \le E_\varepsilon(\tilde\eta_\varepsilon)$ and by Lemma 3.4 it leads to

$$\mathcal{F}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon) \le E_\varepsilon(\tilde\eta_\varepsilon) - E_\varepsilon(\eta_\varepsilon).$$

Using (3.45), we deduce that

$$\mathcal{F}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon) \le C\varepsilon^2|\ln\varepsilon|^2. \tag{3.84}$$

We set $\mathcal{N}_\varepsilon^\nu = \mathbb{R}^2 \setminus \mathcal{D}_\varepsilon^\nu$. From the previous inequality, it suffices to prove that

$$\mathcal{F}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon, \mathcal{N}_\varepsilon^\nu) \ge -C_{\omega_0}|\ln\varepsilon|^{-1} \tag{3.85}$$

with $C_{\omega_0} > 0$ independent of ε and ν. Arguing as in the proof of Lemma 3.3 with $\sigma = 1/4$ and $R = 2\sqrt{a_0}$, we infer from (3.66),

$$\begin{aligned}
|\mathcal{R}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon, \mathcal{N}_\varepsilon^\nu)| \le& \frac{1}{4}\int_{\mathcal{N}_\varepsilon^\nu} \eta_\varepsilon^2|\nabla v_\varepsilon|^2 + \Omega^2 \int_{\mathcal{N}_\varepsilon^\nu} |x|^2|u_\varepsilon|^2 \\
\le& \frac{1}{4}\int_{\mathcal{N}_\varepsilon^\nu} \eta_\varepsilon^2|\nabla v_\varepsilon|^2 + \frac{2\Omega^2}{3}\int_{\mathbb{R}^2\setminus B_{2\sqrt{a_0}}} 2a^-(x)|u_\varepsilon|^2 + 4a_0\Omega^2 \int_{B_{2\sqrt{a_0}}\setminus\mathcal{D}_\varepsilon^\nu} |u_\varepsilon|^2 \\
\le& \frac{1}{4}\int_{\mathcal{N}_\varepsilon^\nu} \eta_\varepsilon^2|\nabla v_\varepsilon|^2 + 4a_0\Omega^2 \int_{B_{2\sqrt{a_0}}\setminus\mathcal{D}_\varepsilon^\nu} |u_\varepsilon|^2 + C_{\omega_0}\varepsilon^2|\ln\varepsilon|^4.
\end{aligned}$$

By (3.66), we may also estimate

$$\begin{aligned}
\int_{B_{2\sqrt{a_0}}\setminus\mathcal{D}_\varepsilon^\nu} |u_\varepsilon|^2 =& \int_{B_{2\sqrt{a_0}}\setminus B_{\sqrt{a_0}}} |u_\varepsilon|^2 + \int_{B_{\sqrt{a_0}}\setminus\mathcal{D}_\varepsilon^\nu} (|u_\varepsilon|^2 - a(x)) + \int_{B_{\sqrt{a_0}}\setminus\mathcal{D}_\varepsilon^\nu} a(x) \\
\le& C\Big(\int_{B_{2\sqrt{a_0}}\setminus B_{\sqrt{a_0}}} |u_\varepsilon|^4\Big)^{1/2} + C\Big(\int_{B_{\sqrt{a_0}}\setminus\mathcal{D}_\varepsilon^\nu} (|u_\varepsilon|^2 - a(x))^2\Big)^{1/2} + C|\ln\varepsilon|^{-3} \\
\le& C_{\omega_0}(|\ln\varepsilon|^{-3} + \varepsilon|\ln\varepsilon|).
\end{aligned}$$

Then it follows that

$$|\mathcal{R}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon, \mathcal{N}_\varepsilon^\nu)| \le \frac{1}{2}\,\mathcal{E}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon, \mathcal{N}_\varepsilon^\nu) + C_{\omega_0}|\ln\varepsilon|^{-1} \tag{3.86}$$

which leads to (3.85). ∎

For some technical reasons, it will be easier to deal with a^+ instead of η_ε. We now prove that the energy estimates inside $\mathcal{D}_\varepsilon^\nu$ remain unchanged if one replaces η_ε^2 by a^+ in the energies.

Proposition 3.6. *We have*

$$\mathcal{E}_\varepsilon^a(v_\varepsilon, \mathcal{D}_\varepsilon^\nu) \le C_{\omega_0}|\ln\varepsilon|^2 \quad \textit{and} \quad \mathcal{F}_\varepsilon^a(v_\varepsilon, \mathcal{D}_\varepsilon^\nu) \le C_{\omega_0}|\ln\varepsilon|^{-1}.$$

Proof. From *3.1.c)* in Proposition 3.1, we infer that

$$\left\| \frac{a-\eta_\varepsilon^2}{\eta_\varepsilon^2} \right\|_{L^\infty(\mathcal{D}_\varepsilon^\nu)} \leq C\varepsilon^{1/3} \quad \text{and} \quad \left\| \frac{a^2-\eta_\varepsilon^4}{\eta_\varepsilon^4} \right\|_{L^\infty(\mathcal{D}_\varepsilon^\nu)} \leq C\varepsilon^{1/3}$$

and then *3.4.a)* in Proposition 3.4 yields

$$|\mathcal{E}_\varepsilon^a(v_\varepsilon, \mathcal{D}_\varepsilon^\nu) - \mathcal{E}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon, \mathcal{D}_\varepsilon^\nu)| \leq C\varepsilon^{1/3}\, \mathcal{E}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon, \mathcal{D}_\varepsilon^\nu) \leq C_{\omega_0}\varepsilon^{1/3}|\ln\varepsilon|^2. \tag{3.87}$$

Using *3.3.a)* and *3.3.e)* in Proposition 3.3, we derive

$$\begin{aligned} |\mathcal{R}_\varepsilon^a(v_\varepsilon, \mathcal{D}_\varepsilon^\nu) - \mathcal{R}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon, \mathcal{D}_\varepsilon^\nu)| &\leq \Omega \int_{\mathcal{D}_\varepsilon^\nu} \frac{a-\eta_\varepsilon^2}{\eta_\varepsilon^2}\, |u_\varepsilon|\, |\nabla u_\varepsilon| \\ &\leq C\varepsilon^{1/3}\Omega(E_\varepsilon(u_\varepsilon, \mathcal{D}_\varepsilon^\nu))^{1/2} \leq C_{\omega_0}\varepsilon^{1/3}|\ln\varepsilon|^2. \end{aligned}$$

Therefore, it follows that

$$|\mathcal{F}_\varepsilon^a(v_\varepsilon, \mathcal{D}_\varepsilon^\nu) - \mathcal{F}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon, \mathcal{D}_\varepsilon^\nu)| \leq C_{\omega_0}\varepsilon^{1/3}|\ln\varepsilon|^2. \tag{3.88}$$

Then the conclusion comes immediately from *3.4.a)* in Proposition 3.4 and Proposition 3.5. ∎

3.4 Energy and degree estimates

In this section we find some a priori estimates of the energy and of the number of vortices. The main ingredients are the construction of vortex balls and an asymptotic expansion of the rotational energy in terms of these balls. From this formula, we show the non existence of vortices for velocities strictly less than Ω_1. For larger angular speeds, we give a first result about the location and number of the vortices inside $\mathcal{D}$. We also prove a fundamental energy estimate (Proposition 3.11) which is the starting point for our analysis in the next section.

3.4.1 Construction of the vortex balls

We present here a first vortex structure. It relies on the construction of vortex balls by a method due to E. Sandier [73] and E. Sandier and S. Serfaty [74].

Proposition 3.7. *Assume that* (3.65) *holds. Then there exists a positive constant* Λ_0 *such that for small* ε*, there exist* $\nu_\varepsilon \in (1,2)$ *and a finite collection of disjoint balls* $\{B_i\}_{i\in I_\varepsilon} := \{B(p_i, r_i)\}_{i\in I_\varepsilon}$ *satisfying the conditions :*

(i) $B_i \subset\subset \mathcal{D}_\varepsilon := \mathcal{D}_\varepsilon^{\nu_\varepsilon}$ *for every* $i \in I_\varepsilon$ *(where* $\mathcal{D}_\varepsilon^{\nu_\varepsilon}$ *is defined by* (3.83)*),*

(ii) $\{x \in \mathcal{D}_\varepsilon,\, |v_\varepsilon(x)| < 1 - |\ln\varepsilon|^{-5}\} \subset \cup_{i\in I_\varepsilon} B_i$,

(iii) $\displaystyle\sum_{i\in I_\varepsilon} r_i \le |\ln\varepsilon|^{-10}$,

(iv) $\displaystyle\int_{B_i} a(x)\left(\frac{1}{2}|\nabla v_\varepsilon|^2 - \Omega x^\perp \cdot (iv_\varepsilon, \nabla v_\varepsilon)\right) \ge \pi a(p_i)|d_i|\big(|\ln\varepsilon| - \Lambda_0 \ln|\ln\varepsilon|\big)$,

where $d_i = \deg\left(\dfrac{v_\varepsilon}{|v_\varepsilon|}, \partial B_i\right)$ *for every* $i \in I_\varepsilon$.

Proof. Using the method of E. Sandier [73] and E. Sandier and S. Serfaty [74], we prove as in [2] (by the estimates in Proposition 3.6 with $\nu = 1$) the existence of a finite collection of disjoint balls $\{B_i\}_{i\in I_\varepsilon}$ such that the conditions *(ii)* and *(iii)* are fulfilled for $\mathcal{D}_\varepsilon^1 = \{x \in \mathbb{R}^2,\, a(x) > |\ln\varepsilon|^{-3/2}\}$ and we have

$$\int_{B_i} \frac{a(x)}{2}|(\nabla - i\Omega x^\perp)v_\varepsilon|^2 \ge \pi a(p_i)|d_i|\big(|\ln\varepsilon| - \Lambda_0 \ln|\ln\varepsilon|\big),\ \forall i \in I_\varepsilon.$$

Therefore, by *(iii)*, we can find $\nu_\varepsilon \in (1,2)$ such that

$$\partial\mathcal{D}_\varepsilon^{\nu_\varepsilon} \cap \cup_{i\in I_\varepsilon} B_i = \emptyset.$$

By cancelling the balls B_i that are not in $\mathcal{D}_\varepsilon^{\nu_\varepsilon}$, it remains a finite collection of balls which satisfies *(i)*, *(ii)* and *(iii)* for $\mathcal{D}_\varepsilon^{\nu_\varepsilon}$. Notice now that *(iv)* takes place since

$$\Omega^2 \int_{B_i} \frac{a}{2}|x|^2|v_\varepsilon|^2 \le \Omega^2 \int_{B_i} |x|^2|u_\varepsilon|^2 \le C\Omega^2 r_i^2 = o(|\ln\varepsilon|^{-10})$$

and this term can be absorbed by $\Lambda_0 \ln|\ln\varepsilon|$ (up to a different constant $\Lambda_0 + 1$). ■

3.4.2 Expansion of the rotation energy

We are now in a position to compute an asymptotic expansion of the rotation energy according to the center of each vortex ball B_i and the associated degree d_i. We have :

Proposition 3.8. *For small* ε,

$$\mathcal{R}_\varepsilon^a\big(v_\varepsilon, \mathcal{D}_\varepsilon \setminus \cup_{i\in I_\varepsilon} B_i\big) = \frac{\pi\Omega}{2}\sum_{i\in I_\varepsilon}(a^2(p_i) - \nu_\varepsilon^2|\ln\varepsilon|^{-3})\, d_i \;+\; o(|\ln\varepsilon|^{-5})$$

Proof. By Proposition 3.7, $\mathcal{D}_\varepsilon \setminus \cup_{i\in I_\varepsilon} B_i \subset \mathcal{D}_\varepsilon \setminus \{|v_\varepsilon| < 1/2\}$ if $0 < \varepsilon < \varepsilon_2$. For $x \in \mathcal{D}_\varepsilon$ such that $|v_\varepsilon(x)| \ge 1/2$, we set

$$w_\varepsilon(x) = \frac{v_\varepsilon(x)}{|v_\varepsilon(x)|}.$$

Since $(iv_\varepsilon, \nabla v_\varepsilon) = |v_\varepsilon|^2 (iw_\varepsilon, \nabla w_\varepsilon)$ in $\mathcal{D}_\varepsilon \setminus \{|v_\varepsilon| < 1/2\}$, we have

$$\begin{aligned}\mathcal{R}^a_\varepsilon\left(v_\varepsilon, \mathcal{D}_\varepsilon \setminus \cup_{i\in I_\varepsilon} B_i\right) =& \,\Omega \int_{\mathcal{D}_\varepsilon \setminus \cup_{i\in I_\varepsilon} B_i} a(x) x^\perp \cdot (iw_\varepsilon, \nabla w_\varepsilon) \\ &+ \Omega \int_{\mathcal{D}_\varepsilon \setminus \cup_{i\in I_\varepsilon} B_i} a(x)(|v_\varepsilon|^2 - 1) x^\perp \cdot (iw_\varepsilon, \nabla w_\varepsilon). \end{aligned} \tag{3.89}$$

Then we estimate using Proposition 3.6,

$$\begin{aligned}\left| \int_{\mathcal{D}_\varepsilon \setminus \cup_{i\in I_\varepsilon} B_i} a(x)(|v_\varepsilon|^2 - 1)\, x^\perp \cdot (iw_\varepsilon, \nabla w_\varepsilon) \right| &\le C\varepsilon \left(\mathcal{E}^a_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon)\right)^{1/2} \|\nabla w_\varepsilon\|_{L^2(\mathcal{D}_\varepsilon \setminus \{|v_\varepsilon|<1/2\})} \\ &\le C\varepsilon |\ln\varepsilon| \|\nabla w_\varepsilon\|_{L^2(\mathcal{D}_\varepsilon \setminus \{|v_\varepsilon|<1/2\})}\,. \end{aligned} \tag{3.90}$$

In $\mathcal{D}_\varepsilon \setminus \{|v_\varepsilon| < 1/2\}$, we have $|\nabla w_\varepsilon| \le 2(|\nabla v_\varepsilon| + |\nabla |v_\varepsilon||) \le 4|\nabla v_\varepsilon|$. We deduce that

$$\int_{\mathcal{D}_\varepsilon \setminus \{|v_\varepsilon|<1/2\}} |\nabla w_\varepsilon|^2 \le 16 \int_{\mathcal{D}_\varepsilon} |\nabla v_\varepsilon|^2 \le 16 |\ln\varepsilon|^{3/2} \int_{\mathcal{D}_\varepsilon} a(x) |\nabla v_\varepsilon|^2 \le C |\ln\varepsilon|^{7/2} \tag{3.91}$$

and hence we obtain combining (3.89), (3.90) and (3.91),

$$\mathcal{R}^a_\varepsilon\left(v_\varepsilon, \mathcal{D}_\varepsilon \setminus \cup_{i\in I_\varepsilon} B_i\right) = \Omega \int_{\mathcal{D}_\varepsilon \setminus \cup_{i\in I_\varepsilon} B_i} a(x)\, x^\perp \cdot (iw_\varepsilon, \nabla w_\varepsilon) + \mathcal{O}(\varepsilon |\ln\varepsilon|^4). \tag{3.92}$$

We now define the function $\mathcal{P}_\varepsilon : \mathcal{D}_\varepsilon \to \mathbb{R}$ by

$$\mathcal{P}_\varepsilon(x) = \frac{a^2(x) - \nu_\varepsilon^2 |\ln\varepsilon|^{-3}}{4}.$$

The function $\mathcal{P}_\varepsilon$ satisfies

$$\begin{cases} \nabla \mathcal{P}_\varepsilon(x) = -a(x)\, x & \text{for } x \in \mathcal{D}_\varepsilon, \\ \mathcal{P}_\varepsilon(x) = 0 & \text{for } x \in \partial\mathcal{D}_\varepsilon. \end{cases}$$

Since $(iw_\varepsilon, \nabla w_\varepsilon) = w_\varepsilon \wedge \nabla w_\varepsilon$, we derive that

$$\begin{aligned}\int_{\mathcal{D}_\varepsilon \setminus \cup_{i\in I_\varepsilon} B_i} a(x)\, x^\perp \cdot (iw_\varepsilon, \nabla w_\varepsilon) &= - \int_{\mathcal{D}_\varepsilon \setminus \cup_{i\in I_\varepsilon} B_i} \nabla^\perp \mathcal{P}_\varepsilon(x) \cdot (w_\varepsilon \wedge \nabla w_\varepsilon) \\ &= \sum_{i\in I_\varepsilon} \int_{\partial B_i} \mathcal{P}_\varepsilon(x) \left(w_\varepsilon \wedge \frac{\partial w_\varepsilon}{\partial \tau} \right) \end{aligned}$$

where τ denotes the counterclockwise oriented unit tangent vector to ∂B_i. The smoothness of v_ε implies the existence of $\alpha_\varepsilon \in (\frac{1}{2}, \frac{2}{3})$ such that $\mathcal{U} = \left\{x \in \mathbb{R}^2\,,\, |v_\varepsilon| < \alpha_\varepsilon\right\}$ is a smooth open set. Then we set for $i \in I_\varepsilon$,

$$\mathcal{U}_i = B_i \cap \mathcal{U}$$

(note that $\mathcal{U}_i \subset\subset B_i$ for ε small enough by Proposition 3.7). For each $i \in I_\varepsilon$, we have by (3.91),

$$\begin{aligned}\left|\int_{\partial B_i} \mathcal{P}_\varepsilon(x)\left(w_\varepsilon \wedge \frac{\partial w_\varepsilon}{\partial \tau}\right) - \int_{\partial \mathcal{U}_i} \mathcal{P}_\varepsilon(x)\left(w_\varepsilon \wedge \frac{\partial w_\varepsilon}{\partial \tau}\right)\right| &= \left|\int_{B_i\setminus\mathcal{U}_i} \nabla^\perp \mathcal{P}_\varepsilon(x)\cdot(w_\varepsilon \wedge \nabla w_\varepsilon)\right| \\ &\leq C\, r_i\, \|\nabla w_\varepsilon\|_{L^2(\mathcal{D}_\varepsilon\setminus\{|v_\varepsilon|<1/2\})} \\ &\leq C\, r_i\, |\ln\varepsilon|^{7/4}\end{aligned}$$

and since $|v_\varepsilon| < 2/3$ in $\mathcal{U}_i$, it results from *3.4.b)* in Proposition 3.4 and Proposition 3.6,

$$\begin{aligned}\left|\int_{\partial \mathcal{U}_i} (\mathcal{P}_\varepsilon(x) - \mathcal{P}_\varepsilon(p_i))\left(w_\varepsilon \wedge \frac{\partial w_\varepsilon}{\partial \tau}\right)\right| &\leq C\left|\int_{\partial \mathcal{U}_i} (\mathcal{P}_\varepsilon(x) - \mathcal{P}_\varepsilon(p_i))\left(v_\varepsilon \wedge \frac{\partial v_\varepsilon}{\partial \tau}\right)\right| \\ &\leq C\left|\int_{\mathcal{U}_i} a(x)\, x^\perp \cdot (iv_\varepsilon, \nabla v_\varepsilon)\right| \\ &\quad + C\left|\int_{\mathcal{U}_i} (\mathcal{P}_\varepsilon(x) - \mathcal{P}_\varepsilon(p_i))\,\det(\nabla v_\varepsilon)\right| \\ &\leq C\,(r_i\, \|\sqrt{a}\nabla v_\varepsilon\|_{L^2(\mathcal{D}_\varepsilon)} + r_i\, |\ln\varepsilon|^{3/2}\, \|\sqrt{a}\nabla v_\varepsilon\|^2_{L^2(\mathcal{U}_i)}) \\ &\leq C\, r_i\, |\ln\varepsilon|^{7/2}\,.\end{aligned}$$

Therefore we conclude by *(iii)* in Proposition 3.7 that

$$\begin{aligned}\mathcal{R}^a_\varepsilon\,(v_\varepsilon, \mathcal{D}_\varepsilon \setminus \cup_{i\in I_\varepsilon} B_i) &= \Omega \sum_{i\in I_\varepsilon} \mathcal{P}_\varepsilon(p_i) \int_{\partial \mathcal{U}_i} w_\varepsilon \wedge \frac{\partial w_\varepsilon}{\partial \tau} \;+\; o(|\ln\varepsilon|^{-5}) \\ &= 2\pi\Omega \sum_{i\in I_\varepsilon} \mathcal{P}_\varepsilon(p_i)\, d_i \;+\; o(|\ln\varepsilon|^{-5})\end{aligned}$$

and the proof is complete. ■

3.4.3 Asymptotic behavior for subcritical velocities

We are now in a position to prove point *(i)* in Theorem 3.1 for small angular velocities. In terms of the map v_ε, the result takes the following form (using the notations in Proposition 3.7) :

Proposition 3.9. *Assume that*

$$\Omega \leq \omega_0 |\ln\varepsilon| \quad \textit{with} \quad \omega_0 < \frac{2}{a_0}\,. \tag{3.93}$$

Then for ε small enough, we have that

$$\sum_{i\in I_\varepsilon} |d_i| = 0 \tag{3.94}$$

and

$$|v_\varepsilon| \to 1 \quad in \ L^\infty_{\rm loc}(\mathcal{D}) \ \ as \ \varepsilon \to 0. \tag{3.95}$$

Moreover,

$$\mathcal{F}^{\eta_\varepsilon}_\varepsilon(v_\varepsilon) = o(1) \quad and \quad \mathcal{E}^{\eta_\varepsilon}_\varepsilon(v_\varepsilon) = o(1). \tag{3.96}$$

Proof. Combining Proposition 3.6 and Proposition 3.7, we get that

$$\begin{aligned}\mathcal{O}(|\ln\varepsilon|^{-1}) \geq \mathcal{F}^a_\varepsilon(v_\varepsilon,\mathcal{D}_\varepsilon) \geq \frac{1}{2}\int_{\mathcal{D}_\varepsilon\setminus\cup_{i\in I_\varepsilon}B_i} a(x)|\nabla v_\varepsilon|^2 + \frac{1}{4\varepsilon^2}\int_{\mathcal{D}_\varepsilon} a^2(x)(1-|v_\varepsilon|^2)^2 \\ + \pi\sum_{i\in I_\varepsilon} a(p_i)|d_i|\left(|\ln\varepsilon| - \Lambda_0\ln|\ln\varepsilon|\right) - \mathcal{R}^a_\varepsilon\left(v_\varepsilon,\mathcal{D}_\varepsilon\setminus\cup_{i\in I_\varepsilon}B_i\right).\end{aligned} \tag{3.97}$$

Since $a^2(p_i) - \nu_\varepsilon^2|\ln\varepsilon|^{-3} \leq a_0\, a(p_i)$, we infer from Proposition 3.8 that

$$\mathcal{R}^a_\varepsilon\left(v_\varepsilon,\mathcal{D}_\varepsilon\setminus\cup_{i\in I_\varepsilon}B_i\right) \leq \frac{\pi a_0\omega_0}{2}\sum_{i\in I_\varepsilon,\, d_i\geq 0} a(p_i)|d_i||\ln\varepsilon| + o(|\ln\varepsilon|^{-5}) \tag{3.98}$$

Since $\omega_0 < 2/a_0$, we infer from (3.97) and (3.98) that for ε small enough,

$$\frac{1}{2}\int_{\mathcal{D}_\varepsilon\setminus\cup_{i\in I_\varepsilon}B_i} a(x)|\nabla v_\varepsilon|^2 + \int_{\mathcal{D}_\varepsilon}\frac{a^2}{4\varepsilon^2}(1-|v_\varepsilon|^2)^2 + C\sum_{i\in I_\varepsilon} a(p_i)|d_i||\ln\varepsilon| \leq \mathcal{O}(|\ln\varepsilon|^{-1}), \tag{3.99}$$

for a positive constant C independent of ε. Since $a(p_i) \geq |\ln\varepsilon|^{-3/2}$ in $\mathcal{D}_\varepsilon$, we derive that $\sum_{i\in I_\varepsilon}|d_i| \leq \mathcal{O}(|\ln\varepsilon|^{-1/2})$ and therefore (3.94) holds for ε sufficiently small. Coming back to (3.99), (3.94) implies

$$\frac{1}{\varepsilon^2}\int_{\mathcal{D}_\varepsilon} a^2(x)(1-|v_\varepsilon|^2)^2 \leq o(1). \tag{3.100}$$

Then the proof of (3.95) follows as in [19] using the estimate *3.4.c)* in Proposition 3.4 on $|\nabla v_\varepsilon|$.

Since $\sum_{i\in I_\varepsilon}|d_i| = 0$, we derive from Proposition 3.8 that $\mathcal{R}^a_\varepsilon(v_\varepsilon,\mathcal{D}_\varepsilon\setminus\cup_{i\in I_\varepsilon}B_i) = o(1)$. On the other hand, *3.1.c)* in Proposition 3.1, *3.3.a)* in Proposition 3.3 and *(iii)* in Proposition 3.7 yield

$$\left|\mathcal{R}^a_\varepsilon\left(v_\varepsilon,\cup_{i\in I_\varepsilon}B_i\right)\right| \leq \Omega\sum_{i\in I_\varepsilon}\int_{B_i} a(x)\eta_\varepsilon^{-2}\left|(iu_\varepsilon,\nabla u_\varepsilon)\right| \leq C\Omega\|\nabla u_\varepsilon\|_{L^2(B_i)}\sum_{i\in I_\varepsilon} r_i = o(1) \tag{3.101}$$

and we conclude that $\mathcal{R}^a_\varepsilon(v_\varepsilon,\mathcal{D}_\varepsilon) = o(1)$. Since $\mathcal{F}^a_\varepsilon(v_\varepsilon,\mathcal{D}_\varepsilon) \leq o(1)$, we deduce that

$$\mathcal{E}^a_\varepsilon(v_\varepsilon,\mathcal{D}_\varepsilon) = o(1) \tag{3.102}$$

and hence we have $\mathcal{F}^a_\varepsilon(v_\varepsilon,\mathcal{D}_\varepsilon) = o(1)$. By (3.87) and (3.88), it leads to $\mathcal{E}^{\eta_\varepsilon}_\varepsilon(v_\varepsilon,\mathcal{D}_\varepsilon) = o(1)$ and $\mathcal{F}^{\eta_\varepsilon}_\varepsilon(v_\varepsilon,\mathcal{D}_\varepsilon) = o(1)$. Using (3.84) and (3.85), then we get

$$o(1) \leq \mathcal{F}^{\eta_\varepsilon}_\varepsilon(v_\varepsilon,\mathcal{N}^{\nu_\varepsilon}_\varepsilon) \leq -\mathcal{F}^{\eta_\varepsilon}_\varepsilon(v_\varepsilon,\mathcal{D}_\varepsilon) = o(1) \tag{3.103}$$

and therefore $\mathcal{F}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon) = o(1)$. By (3.86), we have

$$\mathcal{F}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon, \mathcal{N}_\varepsilon^{\nu_\varepsilon}) = \mathcal{E}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon, \mathcal{N}_\varepsilon^{\nu_\varepsilon}) + \mathcal{R}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon, \mathcal{N}_\varepsilon^{\nu_\varepsilon}) \geq \frac{1}{2}\mathcal{E}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon, \mathcal{N}_\varepsilon^{\nu_\varepsilon}) + o(1)$$

and we conclude from (3.103) that $\mathcal{E}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon, \mathcal{N}_\varepsilon^{\nu_\varepsilon}) = o(1)$. ■

Remark 3.7. Assuming that (3.93) holds, it follows from (3.102) and Proposition 3.1 that for any sequence $\varepsilon_n \to 0$ we can extract a subsequence (still denoted by ε_n) such that $u_{\varepsilon_n} \to \sqrt{a^+}e^{i\alpha}$ in $H^1_{\mathrm{loc}}(\mathcal{D})$ for some constant $\alpha \in \mathbb{R}$. By Proposition 3.1, Proposition 3.3 and Proposition of 3.9, we also have $|u_\varepsilon| \to \sqrt{a^+}$ as $\varepsilon \to 0$ in $L^\infty_{\mathrm{loc}}(\mathbb{R}^2 \setminus \partial\mathcal{D})$.

3.4.4 Degree estimates near the critical velocity

In this section, we are going to prove that the number of vortex balls with nonzero degree present in a slightly smaller domain than $\mathcal{D}_\varepsilon$, is bounded. To this aim, we need to distinguish different types of vortex balls. We divide I_ε into three pieces : $I_\varepsilon = I_0 \cup I_* \cup I_-$ where

$$\begin{aligned} I_0 =& \big\{ i \in I_\varepsilon \,,\, d_i \geq 0 \text{ and } |p_i| < |\ln\varepsilon|^{-1/6} \big\}, \\ I_* =& \big\{ i \in I_\varepsilon \,,\, d_i \geq 0 \text{ and } |p_i| \geq |\ln\varepsilon|^{-1/6} \big\}, \\ I_- =& \big\{ i \in I_\varepsilon \,,\, d_i < 0 \big\}. \end{aligned}$$

Then the result can be stated as follows.

Proposition 3.10. *Assume that*

$$\Omega \leq \frac{2}{a_0}\left(|\ln\varepsilon| + \omega_1 \ln|\ln\varepsilon|\right), \tag{3.104}$$

for some constant $\omega_1 \in \mathbb{R}$. Then

$$N_0 := \sum_{i\in I_0} |d_i| \leq C_{\omega_1} \tag{3.105}$$

and setting $\mathcal{B}_\varepsilon = \big\{ x \in \mathbb{R}^2,\, |x| \leq \sqrt{a_0 - |\ln\varepsilon|^{-1/2}}\ \big\}$, we have for ε sufficiently small,

$$\sum_{i\in I_*\cup I_-,\, p_i\in\mathcal{B}_\varepsilon} |d_i| = 0. \tag{3.106}$$

Proof. From Proposition 3.8, we derive that for ε small enough,

$$\begin{aligned} \mathcal{R}^a_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon \setminus \cup_{i\in I_\varepsilon} B_i) \leq& \frac{\pi a_0 \Omega}{2}\sum_{i\in I_0} a(p_i)|d_i| + \frac{\pi(a_0 - |\ln\varepsilon|^{-1/3})\Omega}{2}\sum_{i\in I_*} a(p_i)|d_i| + o(|\ln\varepsilon|^{-5}) \\ \leq& \pi \sum_{i\in I_0\cup I_*} a(p_i)|d_i||\ln\varepsilon| + \pi\omega_1 \sum_{i\in I_0} a(p_i)|d_i|\ln|\ln\varepsilon| \\ &- \frac{\pi}{2a_0}\sum_{i\in I_*} a(p_i)|d_i||\ln\varepsilon|^{2/3} + o(|\ln\varepsilon|^{-5}) \end{aligned} \tag{3.107}$$

(here we used that

$$(a_0 - |\ln\varepsilon|^{-1/3})\Omega \leq 2|\ln\varepsilon| - \frac{2}{a_0}|\ln\varepsilon|^{2/3} + 2|\omega_1|\ln|\ln\varepsilon| \leq 2|\ln\varepsilon| - \frac{1}{a_0}|\ln\varepsilon|^{2/3}$$

for $i \in I_*$ and ε small). Combining (3.107) and (3.97), we infer that for ε small enough,

$$\begin{aligned}\int_{\mathcal{D}_\varepsilon\setminus\cup_{i\in I_\varepsilon}B_i} a(x)|\nabla v_\varepsilon|^2 + \sum_{i\in I_*} a(p_i)|d_i||\ln\varepsilon|^{2/3} + \sum_{i\in I_-} a(p_i)|d_i||\ln\varepsilon| \leq \\ \leq C_0\big(\Lambda_0+\omega_1\big)\sum_{i\in I_0} a(p_i)|d_i|\ln|\ln\varepsilon| + \mathcal{O}(|\ln\varepsilon|^{-1}) \\ \leq C_0\big(\Lambda_0+\omega_1\big)a_0N_0\ln|\ln\varepsilon| + \mathcal{O}(|\ln\varepsilon|^{-1}) \qquad (3.108)\end{aligned}$$

for some positive constant C_0 independent of ε. We set

$$\tilde{I}_* = \left\{i \in I_*,\ |p_i| \leq \sqrt{a_0 - |\ln\varepsilon|^{-1/2}}\right\}, \quad N_* = \sum_{i\in\tilde{I}_*}|d_i|,$$

and

$$\tilde{I}_- = \left\{i \in I_-,\ |p_i| \leq \sqrt{a_0 - |\ln\varepsilon|^{-1/2}}\right\}, \quad N_- = \sum_{i\in\tilde{I}_-}|d_i|.$$

Since $a(p_i) \geq |\ln\varepsilon|^{-1/2}$ for any $i \in \tilde{I}_* \cup \tilde{I}_-$, we obtain from (3.108),

$$\int_{\mathcal{D}_\varepsilon\setminus\cup_{i\in I_\varepsilon}B_i} a(x)|\nabla v_\varepsilon|^2 + N_*|\ln\varepsilon|^{1/6} + N_-|\ln\varepsilon|^{1/2} \leq C_0|\Lambda_0+\omega_1|a_0N_0\ln|\ln\varepsilon| + \mathcal{O}(|\ln\varepsilon|^{-1}) \qquad (3.109)$$

which implies in particular,

$$\max\{N_*, N_-\} \leq \frac{N_0}{2} \qquad (3.110)$$

for ε sufficiently small. We now show that N_0 is uniformly bounded in ε. Consider the sets

$$\mathcal{I}_\varepsilon = \Big[\,|\ln\varepsilon|^{-1/6}, \frac{\sqrt{a_0}}{2}\,\Big] \quad \text{and} \quad \mathcal{J}_\varepsilon = \big\{r \in \mathcal{I}_\varepsilon\ :\ \partial B_r \cap (\cup_{i\in I_\varepsilon}\overline{B}_i) = \emptyset\big\}\,.$$

Notice that $\mathcal{J}_\varepsilon$ is a finite union of intervals verifying $|\mathcal{I}_\varepsilon \setminus \mathcal{J}_\varepsilon| \leq |\ln\varepsilon|^{-10}$. For $r \in \mathcal{J}_\varepsilon$ and ε small, we have $|v_\varepsilon| \geq \frac{1}{2}$ on ∂B_r and therefore, we can define

$$D(r) = \deg\left(\frac{v_\varepsilon}{|v_\varepsilon|}, \partial B_r(0)\right).$$

By (3.110), we obtain that for small ε,

$$|D(r)| = \Big|\sum_{|p_i|<r} d_i\Big| \geq N_0 - N_- \geq \frac{N_0}{2} \quad \text{for any } r \in \mathcal{J}_\varepsilon.$$

We have

$$\int_{B_{\frac{\sqrt{a_0}}{2}}\setminus\cup_{i\in I_\varepsilon}B_i} a|\nabla v_\varepsilon|^2 \geq \int_{\mathcal{J}_\varepsilon} a(r)\Big(\int_0^{2\pi}|\nabla v_\varepsilon|^2 r\,d\theta\Big)\,dr$$
$$\geq \frac{1}{4}\int_{\mathcal{J}_\varepsilon}\frac{a(r)}{r}\Big(\int_0^{2\pi}|v_\varepsilon\wedge\frac{\partial v_\varepsilon}{\partial\tau}|^2 r^2\,d\theta\Big)\,dr.$$

Set $w_\varepsilon = \frac{v_\varepsilon}{|v_\varepsilon|}$ in $B_{\frac{\sqrt{a_0}}{2}}\setminus\cup_{i\in I_\varepsilon}B_i$. Since

$$|v_\varepsilon\wedge\frac{\partial v_\varepsilon}{\partial\tau}| = |v_\varepsilon|^2|w_\varepsilon\wedge\frac{\partial w_\varepsilon}{\partial\tau}| \geq \frac{1}{4}|w_\varepsilon\wedge\frac{\partial w_\varepsilon}{\partial\tau}|,$$

it follows that

$$\int_{B_{\frac{\sqrt{a_0}}{2}}\setminus\cup_{i\in I_\varepsilon}B_i} a|\nabla v_\varepsilon|^2 \geq C\int_{\mathcal{J}_\varepsilon}\frac{a(r)}{r}\Big(\int_0^{2\pi}|w_\varepsilon\wedge\frac{\partial w_\varepsilon}{\partial\tau}|^2 r^2\,d\theta\Big)\,dr$$
$$\geq C\int_{\mathcal{J}_\varepsilon}\frac{1}{r}\Big(\int_0^{2\pi}w_\varepsilon\wedge\frac{\partial w_\varepsilon}{\partial\tau}r\,d\theta\Big)^2\,dr$$
$$\geq C\int_{\mathcal{J}_\varepsilon}\frac{D(r)^2}{r}\,dr \geq CN_0^2\int_{\mathcal{J}_\varepsilon}\frac{dr}{r}.$$

Notice now that

$$\Big|\int_{\mathcal{I}_\varepsilon}\frac{dr}{r}-\int_{\mathcal{J}_\varepsilon}\frac{dr}{r}\Big| \leq |\ln\varepsilon|^{1/6}|\mathcal{I}_\varepsilon\setminus\mathcal{J}_\varepsilon| = o(1)$$

and since $\displaystyle\int_{\mathcal{I}_\varepsilon}\frac{dr}{r} = C\ln|\ln\varepsilon| + \mathcal{O}(1)$, we finally get that

$$\int_{B_{\frac{\sqrt{a_0}}{2}}\setminus\cup_{i\in I_\varepsilon}B_i}\frac{a}{2}|\nabla v_\varepsilon|^2 \geq C_1\ln|\ln\varepsilon|N_0^2.$$

for some positive constant C_1 independent of ε. From (3.109), we derive that

$$\big(C_1N_0^2 - C_0|\Lambda_0+\omega_1|a_0N_0\big)\ln|\ln\varepsilon| \leq \mathcal{O}(|\ln\varepsilon|^{-1})$$

which implies that for ε small enough, $C_1N_0^2 - C_0|\Lambda_0+\omega_1|a_0N_0 \leq 1$ and hence N_0 is necessarily bounded in ε. Then it follows by (3.109) that

$$N_* \leq \mathcal{O}(\frac{\ln|\ln\varepsilon|}{|\ln\varepsilon|^{1/6}}) \quad\text{and}\quad N_- \leq \mathcal{O}(\frac{\ln|\ln\varepsilon|}{|\ln\varepsilon|^{1/2}}).$$

Therefore, $N_- = N_* = 0$ for ε sufficiently small. ■

3.4.5 Energy estimates near the critical velocity

We give here some fundamental energy estimates. These estimates follow from Proposition 3.10 and will allow us to construct a fine vortex structure in the next section.

Proposition 3.11. *Assume that* (3.104) *holds. Then there exist two positive constants* $\mathcal{M}_1$ *and* $\mathcal{M}_2$ *(which only depend on* ω_1*) such that*

$$\mathcal{E}_\varepsilon^a(v_\varepsilon, \mathcal{D}_\varepsilon) \le \mathcal{M}_1 |\ln\varepsilon| \quad \textit{and} \quad \mathcal{E}_\varepsilon^a(v_\varepsilon, \mathcal{A}_\varepsilon) \le \mathcal{M}_2 \ln|\ln\varepsilon|,$$

where $\mathcal{A}_\varepsilon = \mathcal{D}_\varepsilon \setminus B_{2|\ln\varepsilon|^{-1/6}}$.

Proof. From Proposition 3.8 and (3.106), we infer that for ε small,

$$\begin{aligned}\mathcal{R}_\varepsilon^a(v_\varepsilon, \mathcal{D}_\varepsilon \setminus \cup_{i\in I_\varepsilon} B_i) &\le \frac{\pi a_0 \Omega}{2} \sum_{i\in I_0} a(p_i)|d_i| + \frac{\pi\Omega}{2} |\ln\varepsilon|^{-1/2} \sum_{i\in I_*\setminus \tilde{I}_*} a(p_i)|d_i| + o(|\ln\varepsilon|^{-5}) \\ &\le \pi \sum_{i\in I_0} a(p_i)|d_i| \big(|\ln\varepsilon| + \omega_1 \ln|\ln\varepsilon|\big) + \frac{2\pi}{a_0} \sum_{i\in I_*} a(p_i)|d_i||\ln\varepsilon|^{1/2} \\ &\qquad + o(|\ln\varepsilon|^{-5})\end{aligned}$$

Injecting this estimate in (3.97), we derive that

$$\sum_{i\in I_*} a(p_i)|d_i||\ln\varepsilon| \le C_{\omega_1} N_0 \ln|\ln\varepsilon|$$

and from (3.105), we deduce that $\sum_{i\in I_*} a(p_i)|d_i||\ln\varepsilon|^{1/2} = o(1)$. Hence

$$\mathcal{R}_\varepsilon^a(v_\varepsilon, \mathcal{D}_\varepsilon \setminus \cup_{i\in I_\varepsilon} B_i) \le \pi \sum_{i\in I_0} a(p_i)|d_i| \big(|\ln\varepsilon| + \omega_1 \ln|\ln\varepsilon|\big) + o(1). \tag{3.111}$$

By (3.101) we have $\mathcal{R}_\varepsilon^a(v_\varepsilon, \mathcal{D}_\varepsilon) = \mathcal{R}_\varepsilon^a(v_\varepsilon, \mathcal{D}_\varepsilon \setminus \cup_{i\in I_\varepsilon} B_i) + o(1)$ and since

$$\mathcal{O}(|\ln\varepsilon|^{-1}) \ge \mathcal{F}_\varepsilon^a(v_\varepsilon, \mathcal{D}_\varepsilon) = \mathcal{E}_\varepsilon^a(v_\varepsilon, \mathcal{D}_\varepsilon) - \mathcal{R}_\varepsilon^a(v_\varepsilon, \mathcal{D}_\varepsilon),$$

it follows by (3.111) and (3.105),

$$\begin{aligned}\mathcal{E}_\varepsilon^a(v_\varepsilon, \mathcal{D}_\varepsilon) &\le \pi \sum_{i\in I_0} a(p_i)|d_i| \big(|\ln\varepsilon| + \omega_1 \ln|\ln\varepsilon|\big) + o(1) \\ &\le C_{\omega_1} N_0 |\ln\varepsilon| \le C_{\omega_1} |\ln\varepsilon|.\end{aligned} \tag{3.112}$$

As in (3.101), we have that $\mathcal{R}_\varepsilon^a(v_\varepsilon, \cup_{i\in I_0} B_i) = o(1)$ and we infer from Proposition 3.7 that

$$\begin{aligned}\sum_{i\in I_0} \frac{1}{2} \int_{B_i} a(x)|\nabla v_\varepsilon|^2 &= \sum_{i\in I_0} \frac{1}{2} \int_{B_i} a(x)|\nabla v_\varepsilon|^2 - \mathcal{R}_\varepsilon^a(v_\varepsilon, \cup_{i\in I_0} B_i) + o(1) \\ &\ge \pi \sum_{i\in I_0} a(p_i)|d_i| \left(|\ln\varepsilon| - \Lambda_0 \ln|\ln\varepsilon|\right) + o(1).\end{aligned}$$

Matching this inequality with (3.112), we finally obtain

$$\begin{aligned}\mathcal{E}^a_\varepsilon(v_\varepsilon, \mathcal{A}_\varepsilon) \leq \mathcal{E}^a_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon \setminus \cup_{i\in I_0} B_i) &\leq \pi(\omega_1 + \Lambda_0) \sum_{i\in I_0} a(p_i)|d_i| \ln|\ln\varepsilon| + o(1) \\ &\leq C_{\omega_1} N_0 \ln|\ln\varepsilon| \leq C_{\omega_1} \ln|\ln\varepsilon|\end{aligned}$$

and the proof is complete. ■

3.5 Fine structure of vortices

The main goal of this section is to define a fine structure of vortices away from the boundary of $\mathcal{D}$. The analysis here follows the ideas in [20] and [21]. The main difficulty in our situation is due to the presence in the energy of the weight function $a(x)$ which vanishes on $\partial\mathcal{D}$ and it does not allow us to construct the structure up to the boundary. From now, we assume that (3.104) holds, i.e., $\Omega \leq \frac{2}{a_0}(|\ln\varepsilon| + \omega_1 \ln|\ln\varepsilon|)$ for some constant $\omega_1 \in \mathbb{R}$. We will prove the following result :

Theorem 3.4. 1) *For any $R \in (\frac{\sqrt{a_0}}{2}, \sqrt{a_0})$ there exists $\varepsilon_R > 0$ such that for any $\varepsilon < \varepsilon_R$,*

$$|v_\varepsilon| \geq \frac{1}{2} \quad \text{in } B_R \setminus B_{\frac{\sqrt{a_0}}{2}}.$$

2) *There exist some constants $N \in \mathbb{N}$, $\lambda_0 > 0$ and $\varepsilon_0 > 0$ (which only depend on ω_1) such that for any $\varepsilon < \varepsilon_0$, there exists a finite collection of points $\{x_j^\varepsilon\}_{j\in J_\varepsilon} \subset B_{\frac{\sqrt{a_0}}{4}}$ such that* $\mathrm{Card}(J_\varepsilon) \leq N$ *and*

$$|v_\varepsilon| \geq \frac{1}{2} \quad \text{in } \overline{B}_{\frac{\sqrt{a_0}}{2}} \setminus \big(\cup_{j\in J_\varepsilon} B(x_j^\varepsilon, \lambda_0\varepsilon)\big).$$

Remark 3.8. The statement of Theorem 3.4 also holds if the radius $\frac{\sqrt{a_0}}{2}$ is replaced by an arbitrary $r \in (0, R)$ but then the constants in Theorem 3.4 depend on r. For sake of simplicity, we prefered to fix $r = \frac{\sqrt{a_0}}{2}$.

3.5.1 Some local estimates

We start with a fundamental lemma. It strongly relies on Pohozaev's identity and it will play a similar role as Theorem III.2 in [20]. In our situation, we only derive local estimates as in [7, 21, 82]. Some of the arguments used in the proof are taken from [7, 21]. In the sequel, R denotes some arbitrary radius in $[\frac{\sqrt{a_0}}{2}, \sqrt{a_0})$ and we will write $R' = \frac{R+\sqrt{a_0}}{2}$.

Lemma 3.5. *For any $2/3 < \alpha < 1$, there exists a positive constant $C_{R,\alpha}$ such that*

$$\frac{1}{\varepsilon^2}\int_{B(x_0,\varepsilon^\alpha)} (1 - |v_\varepsilon|^2)^2 \leq C_{R,\alpha} \quad \text{for any } x_0 \in B_R.$$

Proof. Step 1. We claim that

$$E_\varepsilon(u_\varepsilon, B_{R'}) \leq C_R|\ln\varepsilon|. \tag{3.113}$$

Indeed, since $u_\varepsilon = \eta_\varepsilon v_\varepsilon$, we get that

$$|\nabla u_\varepsilon|^2 \leq (\sqrt{a_0}|\nabla v_\varepsilon| + |v_\varepsilon||\nabla\eta_\varepsilon|)^2 \leq C_R(|\nabla v_\varepsilon|^2 + |\nabla\eta_\varepsilon|^2) \quad \text{in } B_{R'}$$

(here we use *3.4.b)* in Proposition 3.4). Then it results

$$\int_{B_{R'}} |\nabla u_\varepsilon|^2 \leq C_R \left(\min_{y\in B_{R'}} a(y)\right)^{-1} \int_{B_{R'}} a(x)|\nabla v_\varepsilon|^2 + C_R \int_{B_{R'}} |\nabla\eta_\varepsilon|^2 \leq C_R|\ln\varepsilon|,$$

by *3.1.a)* in Proposition 3.1 and Proposition 3.11. On the other hand, we also have by the same propositions,

$$\begin{aligned}\frac{1}{\varepsilon^2}\int_{B_{R'}} (a(x) - |u_\varepsilon|^2)^2 \leq & \frac{C}{\varepsilon^2}\int_{B_{R'}} \left[(a(x) - \eta_\varepsilon^2)^2 + \eta_\varepsilon^4(1 - |v_\varepsilon|^2)^2\right] \\ \leq & \frac{C}{\varepsilon^2}\int_{B_{R'}} (a(x) - \eta_\varepsilon^2)^2 + \frac{C_R}{\varepsilon^2}\int_{B_{R'}} a^4(x)(1 - |v_\varepsilon|^2)^2 \leq C_R|\ln\varepsilon|\end{aligned}$$

and therefore (3.113) follows.

Step 2. We are going to show that one can find a constant $C_{R,\alpha} > 0$, independent of ε, such that for any $x_0 \in B_R$, there is some $r_0 \in (\varepsilon^\alpha, \varepsilon^{\alpha/2+1/3})$ satisfying

$$E_\varepsilon\left(u_\varepsilon, \partial B(x_0, r_0)\right) \leq \frac{C_{R,\alpha}}{r_0}.$$

We proceed by contradiction. Assume that for any $M > 0$, there is $x_M \in B_R$ such that

$$E_\varepsilon\left(u_\varepsilon, \partial B(x_M, r)\right) \geq \frac{M}{r}, \quad \forall r \in (\varepsilon^\alpha, \varepsilon^{\alpha/2+1/3}). \tag{3.114}$$

Without loss of generality we may assume that $B(x_M, \varepsilon^{\alpha/2+1/3}) \subset B_{R'}$ since ε is small. Integrating (3.114) in $r \in (\varepsilon^\alpha, \varepsilon^{\alpha/2+1/3})$, we derive that

$$E_\varepsilon\left(u_\varepsilon, B_{R'}\right) \geq M\int_{\varepsilon^\alpha}^{\varepsilon^{\alpha/2+1/3}} \frac{dr}{r} = M(\alpha/2 - 1/3)|\ln\varepsilon|$$

which contradicts Step 1 for M large enough.

Step 3. Fix $x_0 \in B_R$ and let $r_0 \in (\varepsilon^\alpha, \varepsilon^{\alpha/2+1/3})$ be given by Step 2. As in Step 2, we may assume that $B(x_0, r_0) \subset B_{R'}$. By Proposition 3.2, we have

$$-\Delta u_\varepsilon = \frac{1}{\varepsilon^2}(a(x_0) - |u_\varepsilon|^2)u_\varepsilon + \frac{1}{\varepsilon^2}(a(x) - a(x_0))u_\varepsilon - 2i\Omega x^\perp\cdot\nabla u_\varepsilon + \ell_\varepsilon u_\varepsilon \quad \text{in } B(x_0, r_0). \tag{3.115}$$

As in the proof of the Pohozaev identity, we multiply (3.115) by $(x-x_0)\cdot\nabla u_\varepsilon$ and we integrate by parts in $B(x_0,r_0)$. We have

$$\int_{B(x_0,r_0)} -\Delta u_\varepsilon\cdot[(x-x_0)\cdot\nabla u_\varepsilon] = \frac{r_0}{2}\int_{\partial B(x_0,r_0)}|\nabla u_\varepsilon|^2 - r_0\int_{\partial B(x_0,r_0)}\left|\frac{\partial u_\varepsilon}{\partial\nu}\right|^2 \tag{3.116}$$

and

$$\begin{aligned}\frac{1}{\varepsilon^2}\int_{B(x_0,r_0)}&(a(x_0)-|u_\varepsilon|^2)u_\varepsilon\cdot[(x-x_0)\cdot\nabla u_\varepsilon] = \\ &= \frac{1}{2\varepsilon^2}\int_{B(x_0,r_0)}(a(x_0)-|u_\varepsilon|^2)^2 - \frac{r_0}{4\varepsilon^2}\int_{\partial B(x_0,r_0)}(a(x_0)-|u_\varepsilon|^2)^2\end{aligned} \tag{3.117}$$

(where ν is the outer normal vector to $\partial B(x_0,r_0)$). From (3.115), (3.116) and (3.117) we derive that

$$\begin{aligned}\frac{1}{\varepsilon^2}\int_{B(x_0,r_0)}(a(x_0)-|u_\varepsilon|^2)^2 \le C\big(r_0&\int_{\partial B(x_0,r_0)}|\nabla u_\varepsilon|^2 + r_0\int_{\partial B(x_0,r_0)}\varepsilon^{-2}(a(x_0)-|u_\varepsilon|^2)^2 \\ &+ r_0\varepsilon^{-2}\int_{B(x_0,r_0)}|a(x)-a(x_0)||u_\varepsilon||\nabla u_\varepsilon| \\ &+ \Omega r_0\int_{B(x_0,r_0)}|\nabla u_\varepsilon|^2 + |\ell_\varepsilon| r_0\int_{B(x_0,r_0)}|u_\varepsilon||\nabla u_\varepsilon|\big).\end{aligned}$$

Then we estimate each integral term in the right hand side of the previous inequality. According to (3.113) and to *3.3.a)*, *3.3.b)*, *3.3.e)* in Proposition 3.3, we have

$$\begin{aligned}\varepsilon^{-2}\int_{\partial B(x_0,r_0)}(a(x_0)-|u_\varepsilon|^2)^2 &\le C\varepsilon^{-2}\int_{\partial B(x_0,r_0)}\left[(a(x_0)-a(x))^2+(a(x)-|u_\varepsilon|^2)^2\right] \\ &\le C\varepsilon^{-2}\int_{\partial B(x_0,r_0)}(a(x)-|u_\varepsilon|^2)^2 + C_R\varepsilon^{\frac{3}{2}\alpha-1}\end{aligned}$$

and

$$\Omega r_0\int_{B(x_0,r_0)}|\nabla u_\varepsilon|^2 \le \Omega r_0 E_\varepsilon(u_\varepsilon,B_R) \le C_R\,\varepsilon^{\alpha/2+1/3}|\ln\varepsilon|^2$$

and

$$\begin{aligned}r_0\varepsilon^{-2}\int_{B(x_0,r_0)}|a(x)-a(x_0)||u_\varepsilon||\nabla u_\varepsilon| &\le C_R\,r_0^2\,\varepsilon^{-2}\int_{B(x_0,r_0)}|\nabla u_\varepsilon| \\ &\le C_R\,r_0^3\,\varepsilon^{-2}[E_\varepsilon(u_\varepsilon,B_R)]^{1/2} \le C_R\,\varepsilon^{\frac{3}{2}\alpha-1}|\ln\varepsilon|^{1/2}\end{aligned}$$

and

$$|\ell_\varepsilon| r_0\int_{B(x_0,r_0)}|u_\varepsilon||\nabla u_\varepsilon| \le C_R|\ell_\varepsilon|\,r_0^2\,[E_\varepsilon(u_\varepsilon,B_R)]^{1/2} \le C_R\,\varepsilon^{\alpha-\frac{1}{3}}|\ln\varepsilon|^{3/2}$$

(here we used that $|a(x)-a(x_0)|\le C_R\,r_0$ for any $x,x_0\in B_{R'}$), so that we finally get that

$$\frac{1}{\varepsilon^2}\int_{B(x_0,r_0)}(a(x_0)-|u_\varepsilon|^2)^2 \le C_{R,\alpha}\big(1+r_0E_\varepsilon\left(u_\varepsilon,\partial B(x_0,r_0)\right)\big)$$

for some constant $C_{R,\alpha}$ independent of ε. By Step 2, we conclude that

$$\frac{1}{\varepsilon^2}\int_{B(x_0,\varepsilon^\alpha)}(a(x_0)-|u_\varepsilon|^2)^2 \le C_{R,\alpha}. \tag{3.118}$$

Using *3.1.e)* in Proposition 3.1, we may write

$$\begin{aligned}\frac{1}{\varepsilon^2}\int_{B(x_0,\varepsilon^\alpha)}(1-|v_\varepsilon|^2)^2 &\le \frac{C_R}{\varepsilon^2}\int_{B(x_0,\varepsilon^\alpha)}(\eta_\varepsilon^2-|u_\varepsilon|^2)^2\\ &\le \frac{C_R}{\varepsilon^2}\int_{B(x_0,\varepsilon^\alpha)}(a(x)-|u_\varepsilon|^2)^2+o(1)\\ &\le \frac{C_R}{\varepsilon^2}\int_{B(x_0,\varepsilon^\alpha)}(a(x_0)-|u_\varepsilon|^2)^2+o(1)\le C_{R,\alpha}\end{aligned}$$

and we conclude with (3.118). ■

The next result will allow us to define the notion of a bad disc as in [20].

Proposition 3.12. *There exist positive constants λ_R and μ_R such that if*

$$\frac{1}{\varepsilon^2}\int_{B_{R'}\cap B(x_0,2l)}(1-|v_\varepsilon|^2)^2 \le \mu_R \quad \textit{with } x_0\in B_R\,,\ \frac{l}{\varepsilon}\ge\lambda_R \ \textit{ and } \ l\le 1,$$

then $|v_\varepsilon|\ge 1/2$ in $B_{R'}\cap B(x_0,l)$.

Proof. By *3.4.c)* in Proposition 3.4, there exists a constant $C_R>0$ independent of ε such that

$$|\nabla v_\varepsilon|\le\frac{C_R}{\varepsilon}\quad\text{in } B_{R'}.$$

Then the result follows as in [20], Theorem III.3. ■

Definition 3.1. For $x\in B_R$, we say that $B(x,\lambda_R\varepsilon)$ is a **bad disc** if

$$\frac{1}{\varepsilon^2}\int_{B_{R'}\cap B(x,2\lambda_R\varepsilon)}(1-|v_\varepsilon|^2)^2\ge\mu_R.$$

Now we can give a local version of Theorem 3.4. We will see that Lemma 3.5 plays a crucial role in the proof.

Proposition 3.13. *Let $2/3<\alpha<1$. There exist positive constants $N_{R,\alpha}$ and $\varepsilon_{R,\alpha}$ such that for every $\varepsilon<\varepsilon_{R,\alpha}$ and $x_0\in B_R$ one can find $x_1,\dots,x_{N_\varepsilon}\in B(x_0,\varepsilon^\alpha)$ with $N_\varepsilon\le N_{R,\alpha}$ verifying*

$$|v_\varepsilon|\ge\frac{1}{2}\quad\textit{in}\quad B(x_0,\varepsilon^\alpha)\setminus\left(\cup_{k=1}^{N_\varepsilon}B(x_k,\lambda_R\varepsilon)\right).$$

Proof. First, choosing ε small enough, we may assume that $B(x_0,\varepsilon^\alpha) \subset B_{R'}$. Consider now a family of discs $\big\{B(x_i,\lambda_R\varepsilon)\big\}_{i\in\mathcal{F}}$ such that

$$\begin{cases} x_i \in B(x_0,\varepsilon^\alpha), \\ B\big(x_i,\lambda_R\varepsilon/4\big)\cap B\big(x_j,\lambda_R\varepsilon/4\big)=\emptyset \quad \text{for } i\neq j, \\ B(x_0,\varepsilon^\alpha)\subset \bigcup_{i\in\mathcal{F}} B(x_i,\lambda_R\varepsilon). \end{cases} \tag{3.119}$$

We denote by $\mathcal{F}'$ the set of indices $i\in\mathcal{F}$ such that $B(x_i,\lambda_R\varepsilon)$ is a bad disc. We derive from Proposition 3.12 that for ε small enough,

$$\mu_R \operatorname{Card}(\mathcal{F}') \leq \sum_{i\in\mathcal{F}} \frac{1}{\varepsilon^2}\int_{B_{R'}\cap B(x_i,2\lambda_R\varepsilon)} (1-|v_\varepsilon|^2)^2 \leq \frac{C}{\varepsilon^2}\int_{B(x_0,\varepsilon^{\alpha'})}(1-|v_\varepsilon|^2)^2$$

where C is some absolute constant and $\alpha' = 1/2(\alpha+2/3)$. The conclusion now follows by Lemma 3.5. ■

Remark 3.9. By proof of Proposition 3.13, any cover $\big\{B(x_i,\lambda_R\varepsilon)\big\}_{i\in\mathcal{F}}$ of $B(x_0,\varepsilon^\alpha)$ satisfying (3.119) contains at most $N_{R,\alpha}$ bad discs.

We will need the following lemma to prove that vortices of degree zero do not occur :

Lemma 3.6. *Let $D>0$, $0<\beta<1$ and $\gamma>1$ be given constants such that $\gamma\beta<1$. Let $0<\rho<\varepsilon^\beta$ be such that $\rho^\gamma>\lambda_R\varepsilon$. We assume that for $x_0\in B_R$,*

(i) $\displaystyle\int_{\partial B(x_0,\rho)} |\nabla v_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(1-|v_\varepsilon|^2)^2 < \frac{D}{\rho}$,

(ii) $|v_\varepsilon| \geq \dfrac{1}{2}$ *on* $\partial B(x_0,\rho)$,

(iii) $\deg\left(\dfrac{v_\varepsilon}{|v_\varepsilon|},\partial B(x_0,\rho)\right)=0$.

Then we have

$$|v_\varepsilon| \geq \frac{1}{2} \quad \text{in } B(x_0,\rho^\gamma).$$

Proof of Lemma 3.6. Step 1. We are going to construct a comparison function to obtain the following estimate :

$$\int_{B(x_0,\rho)} |\nabla v_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(1-|v_\varepsilon|^2)^2 \leq C_{\beta,R}. \tag{3.120}$$

Since the degree of v_ε restricted to $\partial B(x_0,\rho)$ is zero, we may write on $\partial B(x_0,\rho)$

$$v_\varepsilon = |v_\varepsilon| e^{i\phi_\varepsilon}$$

where ϕ_ε is a smooth map from $\partial B(x_0,\rho)$ into $\mathbb{R}$. Then we define $\hat{v}_\varepsilon : \mathbb{R}^2 \to \mathbb{C}$ by

$$\begin{cases} \hat{v}_\varepsilon = \chi_\varepsilon e^{i\psi_\varepsilon} & \text{in } B(x_0,\rho) \\ \hat{v}_\varepsilon = v_\varepsilon & \text{in } \mathbb{R}^2 \setminus B(x_0,\rho) \end{cases}$$

where ψ_ε is the solution of

$$\begin{cases} \Delta\psi_\varepsilon = 0 & \text{in } B(x_0,\rho) \\ \psi_\varepsilon = \phi_\varepsilon & \text{on } \partial B(x_0,\rho), \end{cases}$$

and χ_ε has the form, written in polar coordinates centered at x_0,

$$\chi_\varepsilon(r,\theta) = (|v_\varepsilon(\rho e^{i\theta})| - 1)\xi(r) + 1$$

and ξ is a smooth function taking values in $[0,1]$ with small support near ρ with $\xi(\rho) = 1$ (note that by *3.4.b)* in Proposition 3.4, $0 \le \chi_\varepsilon \le 1 + C\varepsilon^{1/3}$). Arguing as in [19], proof of Theorem 2, we may prove

$$\int_{B(x_0,\rho)} |\nabla\psi_\varepsilon|^2 \le C\rho \int_{\partial B(x_0,\rho)} \left|\frac{\partial\phi_\varepsilon}{\partial\tau}\right|^2 \le C\rho \int_{\partial B(x_0,\rho)} |\nabla v_\varepsilon|^2 \tag{3.121}$$

and

$$\int_{B(x_0,\rho)} |\nabla\chi_\varepsilon|^2 + \frac{1}{\varepsilon^2}(1-\chi_\varepsilon^2)^2 \le C\rho \int_{\partial B(x_0,\rho)} |\nabla v_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(1-|v_\varepsilon|^2)^2 + \mathcal{O}(\rho). \tag{3.122}$$

From (3.121), (3.122) and assumption *(i)*, we infer that

$$\int_{B(x_0,\rho)} |\nabla\hat{v}_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(1-|\hat{v}_\varepsilon|^2)^2 \le C\,. \tag{3.123}$$

We set $\tilde{v}_\varepsilon = m_\varepsilon^{-1}\,\hat{v}_\varepsilon$ with $m_\varepsilon = \|\eta_\varepsilon\hat{v}_\varepsilon\|_{L^2(\mathbb{R}^2)}$. Clearly we have $\tilde{v}_\varepsilon \in \tilde{\mathcal{G}}_\varepsilon$ and hence, by Proposition 3.4,

$$\mathcal{F}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon) \le \mathcal{F}_\varepsilon^{\eta_\varepsilon}(\tilde{v}_\varepsilon). \tag{3.124}$$

We claim that

$$\mathcal{F}_\varepsilon^{\eta_\varepsilon}(\tilde{v}_\varepsilon) \le \mathcal{F}_\varepsilon^{\eta_\varepsilon}(\hat{v}_\varepsilon) + C\rho|\ln\varepsilon|^2. \tag{3.125}$$

Indeed, using (3.123), $\|\eta_\varepsilon v_\varepsilon\|_{L^2(\mathbb{R}^2)} = 1$, $\hat{v}_\varepsilon = v_\varepsilon$ in $\mathbb{R}^2 \setminus B(x_0,\rho)$ and *3.4.a)* in Proposition 3.4, we estimate

$$\begin{aligned} m_\varepsilon^2 &= 1 + \int_{B(x_0,\rho)} \eta_\varepsilon^2(|\hat{v}_\varepsilon|^2 - 1) + \int_{B(x_0,\rho)} \eta_\varepsilon^2(1-|v_\varepsilon|^2) \\ &= 1 + \mathcal{O}(\rho\,\varepsilon|\ln\varepsilon|) \end{aligned} \tag{3.126}$$

From *3.4.a)* in Proposition 3.4, (3.123) and (3.126) we derive

$$\int_{\mathbb{R}^2} \eta_\varepsilon^2|\nabla\tilde{v}_\varepsilon|^2 = m_\varepsilon^{-2}\int_{\mathbb{R}^2} \eta_\varepsilon^2|\nabla\hat{v}_\varepsilon|^2 = \int_{\mathbb{R}^2} \eta_\varepsilon^2|\nabla\hat{v}_\varepsilon|^2 + \mathcal{O}(\rho\varepsilon|\ln\varepsilon|^3), \tag{3.127}$$

by *3.3.a)* in Proposition 3.3, Lemma 3.3, (3.123) and (3.126),

$$\mathcal{R}_\varepsilon^{\eta_\varepsilon}(\tilde v_\varepsilon) = m_\varepsilon^{-2}\mathcal{R}_\varepsilon^{\eta_\varepsilon}(\hat v_\varepsilon) = \mathcal{R}_\varepsilon^{\eta_\varepsilon}(\hat v_\varepsilon) - (1-m_\varepsilon^{-2})R_\varepsilon(\eta_\varepsilon \hat v_\varepsilon) = \mathcal{R}_\varepsilon^{\eta_\varepsilon}(\hat v_\varepsilon) + \mathcal{O}(\rho\varepsilon|\ln\varepsilon|^3), \quad (3.128)$$

and using also *3.3.e)* in Proposition 3.3, (3.66) and *3.4.a)* in Proposition 3.4,

$$\begin{aligned}\frac{1}{\varepsilon^2}\int_{\mathbb{R}^2}\eta_\varepsilon^4(1-|\tilde v_\varepsilon|^2)^2 &= \frac{1}{\varepsilon^2}\int_{\mathbb{R}^2}\eta_\varepsilon^4(1-|\hat v_\varepsilon|^2)^2 + \frac{2(1-m_\varepsilon^{-2})}{\varepsilon^2}\int_{\mathbb{R}^2}\eta_\varepsilon^2(1-|\hat v_\varepsilon|^2)|\eta_\varepsilon\hat v_\varepsilon|^2 \\ &\quad + \frac{(1-m_\varepsilon^{-2})^2}{\varepsilon^2}\int_{\mathbb{R}^2}|\eta_\varepsilon\hat v_\varepsilon|^4 \\ &\le \frac{1}{\varepsilon^2}\int_{\mathbb{R}^2}\eta_\varepsilon^4(1-|\hat v_\varepsilon|^2)^2 \\ &\quad + C\rho|\ln\varepsilon|\left(\frac{1}{\varepsilon^2}\int_{\mathbb{R}^2\setminus B(x_0,\rho)}\eta_\varepsilon^4(1-|v_\varepsilon|^2)^2\right)^{1/2}\left(\int_{\mathbb{R}^2\setminus B(x_0,\rho)}|u_\varepsilon|^4\right)^{1/2} \\ &\quad + C\rho^2|\ln\varepsilon|^2 \\ &\le \frac{1}{\varepsilon^2}\int_{\mathbb{R}^2}\eta_\varepsilon^4(1-|\hat v_\varepsilon|^2)^2 + C\rho|\ln\varepsilon|^2. \end{aligned} \quad (3.129)$$

We conclude from (3.127), (3.128) and (3.129) that (3.125) holds.

Since $\hat v_\varepsilon = v_\varepsilon$ in $\mathbb{R}^2\setminus B(x_0,\rho)$, we get from (3.124) and (3.125) that

$$\mathcal{F}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon, B(x_0,\rho)) \le \mathcal{F}_\varepsilon^{\eta_\varepsilon}(\hat v_\varepsilon, B(x_0,\rho)) + C\rho|\ln\varepsilon|^2.$$

By (3.123) we have $\mathcal{E}_\varepsilon^{\eta_\varepsilon}(\hat v_\varepsilon, B(x_0,\rho)) \le C$ and therefore,

$$\left|\mathcal{R}_\varepsilon^{\eta_\varepsilon}(\hat v_\varepsilon, B(x_0,\rho))\right| \le C_R\Omega\int_{B(x_0,\rho)}|\nabla\hat v_\varepsilon| \le C_R\Omega\rho\|\nabla\hat v_\varepsilon\|_{L^2(B(x_0,\rho))} = \mathcal{O}(\rho|\ln\varepsilon|).$$

Hence, $\mathcal{F}_\varepsilon^{\eta_\varepsilon}(\hat v_\varepsilon, B(x_0,\rho)) \le C$ and we conclude that

$$\mathcal{F}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon, B(x_0,\rho)) \le C_\beta.$$

As for $\hat v_\varepsilon$, using Proposition 3.11, we easily check that $|\mathcal{R}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon, B(x_0,\rho))| = \mathcal{O}(\rho|\ln\varepsilon|^{3/2})$ and we finally get that $\mathcal{E}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon, B(x_0,\rho)) \le C_\beta$ which clearly implies (3.120) by *3.1.c)* in Proposition 3.1.

Step 2. We deduce from (3.120) that

$$\int_{2\rho^\gamma}^{\rho}\left(\int_{\partial B(x_0,s)}|\nabla v_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(1-|v_\varepsilon|^2)^2\right)ds \le C_{\beta,R}.$$

Since $\int_{2\rho^\gamma}^{\rho}\frac{ds}{s|\ln s|^{1/2}} \ge C_\gamma|\ln\varepsilon|^{1/2}$, we derive that for small ε there exists $s_0\in[2\rho^\gamma,\rho]$ such that

$$\int_{\partial B(x_0,s_0)}|\nabla v_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(1-|v_\varepsilon|^2)^2 \le \frac{C_{\beta,R}}{s_0|\ln s_0|^{1/2}}\,.$$

Repeating the arguments used to prove (3.120), we find that

$$\int_{B(x_0,s_0)} |\nabla v_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(1-|v_\varepsilon|^2)^2 \le \frac{C_{\beta,R}}{|\ln s_0|^{1/2}} .$$

In particular, we have

$$\frac{1}{\varepsilon^2}\int_{B(x_0,2\rho^\gamma)} (1-|v_\varepsilon|^2)^2 = o(1)$$

and the conclusion follows by Proposition 3.12. ■

We now establish an estimate of the contribution in the energy of any vortex :

Proposition 3.14. *Let $x_0 \in B_R$ and $\frac{2}{3} < \alpha < 1$. Assume that $|v_\varepsilon(x_0)| < 1/2$. Then there exists a positive constant $C_{R,\alpha}$ (which only depends on R, α and ω_1) such that*

$$\int_{B(x_0,\varepsilon^\alpha)} |\nabla v_\varepsilon|^2 \ge C_{R,\alpha}|\ln\varepsilon|.$$

Proof. Let $N_{R,\alpha}$ and $x_1,\dots,x_{N_\varepsilon} \in B(x_0,\varepsilon^\alpha)$ be as in Proposition 3.13. Set

$$\delta_\alpha = \frac{\alpha^{1/2}-\alpha}{3(N_{R,\alpha}+1)}$$

and for $k = 0,\dots,3N_{R,\alpha}+2$ we consider

$$\alpha_k = \alpha^{1/2} - k\delta_\alpha\,,\ \mathcal{I}_k = \left[\varepsilon^{\alpha_k},\varepsilon^{\alpha_{k+1}}\right] \text{ and } \mathcal{C}_k = B(x_0,\varepsilon^{\alpha_{k+1}}) \setminus B(x_0,\varepsilon^{\alpha_k}).$$

Then there is some $k_0 \in \{1,\dots,3N_{R,\alpha}+1\}$ such that

$$\mathcal{C}_{k_0} \cap \left(\cup_{j=1}^{N_\varepsilon} B(x_j,\lambda_R\varepsilon)\right) = \emptyset. \tag{3.130}$$

Indeed, since $N_\varepsilon \le N_{R,\alpha}$ and $2\lambda_R\varepsilon < |\mathcal{I}_k|$ for small ε, the union of N_ε intervals of length $2\lambda_R\varepsilon$

$$\cup_{j=1}^{N_\varepsilon}(|x_i-x_0|-\lambda_R\varepsilon, |x_i-x_0|+\lambda_R\varepsilon)$$

cannot intersect all the intervals $\mathcal{I}_k$ of disjoint interior, for $1 \le k \le 3N_{R,\alpha}+1$. From (3.130) we deduce that

$$|v_\varepsilon(x)| \ge \frac{1}{2} \quad \forall x \in \mathcal{C}_{k_0}.$$

Therefore, for every $\rho \in \mathcal{I}_{k_0}$,

$$d_{k_0} = \deg\left(\frac{v_\varepsilon}{|v_\varepsilon|}, \partial B(x_0,\rho)\right)$$

is well defined and does not depend on ρ.

We claim that

$$d_{k_0} \neq 0. \tag{3.131}$$

By contradiction, we suppose that $d_{k_0} = 0$. From Proposition 3.11, it results that

$$\int_{B_{R'}} |\nabla v_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(1-|v_\varepsilon|^2)^2 \le C_R|\ln\varepsilon|.$$

Using the same argument as in Step 2 of the proof of Lemma 3.5, there is a constant $C_{R,\alpha}$ such that

$$\int_{\partial B(x_0,\rho_0)} |\nabla v_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(1-|v_\varepsilon|^2)^2 \le \frac{C_{R,\alpha}}{\rho_0} \quad \text{for some } \rho_0 \in \mathcal{I}_{k_0}.$$

According to Lemma 3.6 (where $\beta = \alpha_{k_0+1}$ and $\gamma = \frac{\alpha_{k_0-1}}{\alpha_{k_0}}$), we should have $|v_\varepsilon(x_0)| \ge 1/2$ which is a contradiction.

By (3.131), we obtain for every $\rho \in \mathcal{I}_{k_0}$,

$$1 \le |d_{k_0}| = \frac{1}{2\pi}\left|\int_{\partial B(x_0,\rho)} \frac{1}{|v_\varepsilon|^2}\big(v_\varepsilon \wedge \frac{\partial v_\varepsilon}{\partial\tau}\big)\right| \le C\int_{\partial B(x_0,\rho)} |\nabla v_\varepsilon|$$

(where we use that $|v_\varepsilon| \ge \frac{1}{2}$ in $\mathcal{C}_{k_0}$). Cauchy-Schwarz inequality yields

$$\int_{\partial B(x_0,\rho)} |\nabla v_\varepsilon|^2 \ge \frac{C}{\rho} \quad \forall \rho \in \mathcal{I}_{k_0}$$

and the conclusion follows integrating on $\mathcal{I}_{k_0}$. ∎

3.5.2 Proof of Theorem 3.4

The part 1) in Theorem 3.4 follows directly from Lemma 3.7 below.

Lemma 3.7. *There exists a constant $\varepsilon_R > 0$ such that for any $0 < \varepsilon < \varepsilon_R$,*

$$|v_\varepsilon| \ge \frac{1}{2} \quad \textit{in } B_R \setminus B_{\frac{\sqrt{a_0}}{5}}.$$

Proof. First, we fix some $\alpha \in (2/3, 1)$. We proceed by contradiction. Suppose that there is some $x_0 \in B_R \setminus B_{\frac{\sqrt{a_0}}{5}}$ such that $|v_\varepsilon(x_0)| < 1/2$. Then for any ε sufficiently small, we have $B(x_0, \varepsilon^\alpha) \subset \mathcal{A}_\varepsilon$ ($\mathcal{A}_\varepsilon$ is defined in Proposition 3.11) and therefore, by Proposition 3.11, we get that

$$\int_{B(x_0,\varepsilon^\alpha)} |\nabla v_\varepsilon|^2 \le C_R\,\mathcal{E}_\varepsilon^\alpha(v_\varepsilon, \mathcal{A}_\varepsilon) \le C_R \ln|\ln\varepsilon|$$

which contradicts Proposition 3.14 for ε small enough. ∎

Proof of 2) in Theorem 3.4. We fix some $2/3 < \alpha < 1$. As in the proof of Proposition 3.13, we consider a finite family of points $\{x_j\}_{j\in\mathcal{J}}$ satisfying

$$x_j \in B_{\frac{\sqrt{a_0}}{2}}$$

$$B\big(x_i, \lambda_0\varepsilon/4\big) \cap B\big(x_j, \lambda_0\varepsilon/4\big) = \emptyset \quad \text{for } i \ne j,$$

$$B_{\frac{\sqrt{a_0}}{2}} \subset \bigcup_{j\in\mathcal{J}} B\left(x_j, \lambda_0\varepsilon\right),$$

where $\lambda_0 := \lambda_{\frac{\sqrt{a_0}}{2}}$ (defined in Proposition 3.12 with $R = \frac{\sqrt{a_0}}{2}$) and we denote by J_ε the set of indices $j \in \mathcal{J}$ such that $B(x_j, \lambda_0\varepsilon)$ contains at least one point y_j verifying

$$|v_\varepsilon(y_j)| < \frac{1}{2}\,. \tag{3.132}$$

Applying Lemma 3.7 (with $R = \frac{3\sqrt{a_0}}{4}$), we infer that there exists ε_0 such that for any $0 < \varepsilon < \varepsilon_0$,

$$B(x_j, \lambda_0\varepsilon) \subset B_{\frac{\sqrt{a_0}}{4}} \quad \text{for any } j \in J_\varepsilon. \tag{3.133}$$

Then it remains to prove that $\mathrm{Card}(J_\varepsilon)$ is bounded independently of ε. Using (3.133) and Proposition 3.14 (with $R = \frac{\sqrt{a_0}}{2}$), we derive that for every $j \in J_\varepsilon$,

$$\int_{B(y_j, \varepsilon^\alpha)} |\nabla v_\varepsilon|^2 \geq C_\alpha |\ln \varepsilon| \tag{3.134}$$

for some positive constant C_α which only depends on α (where y_j is any point satisfying (3.132) in the ball $B(x_j, \lambda_0\varepsilon)$). We set for ε small enough

$$W = \bigcup_{j \in J_\varepsilon} B(x_j, 2\varepsilon^\alpha) \subset B_{\frac{\sqrt{a_0}}{3}}\,.$$

We claim that there is a positive integer M_α independent of ε such that for any $y \in W$, the point y belongs to at most M_α balls in the collection $\{B(x_j, 2\varepsilon^\alpha)\}_{j \in J_\varepsilon}$. Indeed, consider for $y \in W$ the subset K_y of J_ε defined by

$$K_y = \big\{ j \in J_\varepsilon,\ y \in B(x_j, 2\varepsilon^\alpha) \big\}.$$

We have for every $j \in K_y$,

$$x_j \in B(y, \varepsilon^{\alpha'}) \subset B_{\frac{\sqrt{a_0}}{2}} \tag{3.135}$$

with $\alpha' = 1/2(2/3 + \alpha)$. Obviously, the family $\big\{B(x_j, \lambda_0\varepsilon)\big\}_{j \in K_y}$ can be completed into a cover of $B(y, \varepsilon^{\alpha'})$ satisfying (3.119) (with $R = \frac{\sqrt{a_0}}{2}$) and by Remark 3.9, this cover contains at most M_α bad discs for a constant M_α independent of ε. On the other hand, $B(x_j, \lambda_0\varepsilon)$ is a bad disc for any $j \in J_\varepsilon$ by Proposition 3.12. Hence

$$\mathrm{Card}(K_y) \leq M_\alpha.$$

From (3.134), we infer that

$$\int_{B_{\frac{\sqrt{a_0}}{2}}} |\nabla v_\varepsilon|^2 \geq \int_W |\nabla v_\varepsilon|^2 \geq \frac{1}{M_\alpha} \sum_{j \in J} \int_{B(x_j, 2\varepsilon^\alpha)} |\nabla v_\varepsilon|^2 \geq C_\alpha \mathrm{Card}(J_\varepsilon) |\ln \varepsilon|. \tag{3.136}$$

Moreover, we know from Proposition 3.11,

$$\int_{B_{\frac{\sqrt{a_0}}{2}}} |\nabla v_\varepsilon|^2 \leq C \int_{B_{\frac{\sqrt{a_0}}{2}}} a(x) |\nabla v_\varepsilon|^2 \leq C |\ln \varepsilon| \tag{3.137}$$

for some constant C independent of ε. We deduce that $\mathrm{Card}(J_\varepsilon)$ is bounded independently of ε matching (3.136) with (3.137). ■

3.5.3 Modifying the bad discs

In this section, we refine the vortex structure given by *2)* in Theorem 3.4. We obtain the following result as in [78] combining Theorem 3.4 with an adaptation of Theorem V.1 in [7] (the method comes from a preliminary version of [20]).

Proposition 3.15. *Let $0<\beta<\mu<1$ be given constants such that $\overline{\mu}:=\mu^{N+1}>\beta$ and let $\{x_j^\varepsilon\}_{j\in J_\varepsilon}$ be the collection of points given by 2) in Theorem 3.4. There exists $0<\varepsilon_1<\varepsilon_0$ such that for any $\varepsilon<\varepsilon_1$, we can find $\tilde{J}_\varepsilon\subset J_\varepsilon$ and $\rho>0$ verifying*

(i) $\lambda_0\varepsilon\le\varepsilon^\mu\le\rho\le\varepsilon^{\overline{\mu}}<\varepsilon^\beta$,

(ii) $|v_\varepsilon|\ge\dfrac{1}{2}$ *in* $\overline{B}_{\frac{\sqrt{a_0}}{2}}\setminus\cup_{j\in\tilde{J}_\varepsilon}B(x_j^\varepsilon,\rho)$,

(iii) $|v_\varepsilon|\ge 1-\dfrac{2}{|\ln\varepsilon|^2}$ *on* $\partial B(x_j^\varepsilon,\rho)$ *for every* $j\in\tilde{J}_\varepsilon$,

(iv) $\displaystyle\int_{\partial B(x_j^\varepsilon,\rho)}|\nabla v_\varepsilon|^2+\frac{1}{2\varepsilon^2}(1-|v_\varepsilon|^2)^2\le\frac{C(\beta,\mu)}{\rho}$ *for every* $j\in\tilde{J}_\varepsilon$,

(v) $|x_i^\varepsilon-x_j^\varepsilon|\ge 8\rho$ *for every* $i,j\in\tilde{J}_\varepsilon$ *with* $i\ne j$.

Moreover, for each $j\in\tilde{J}_\varepsilon$, we have

$$D_j:=\deg\left(\frac{v_\varepsilon}{|v_\varepsilon|},\partial B(x_j^\varepsilon,\rho)\right)\ne 0. \tag{3.138}$$

Proof. By Theorem 3.4, we have for ε small enough,

$$\cup_{j\in J_\varepsilon}B(x_j^\varepsilon,\lambda_0\varepsilon)\subset B_{\frac{\sqrt{a_0}}{3}}.$$

From *(iii)* in Proposition 3.7, there exists a radius $r_\varepsilon\in(\frac{\sqrt{a_0}}{3},\frac{\sqrt{a_0}}{2}]$ such that

$$\bar{B}_i\cap\partial B_{r_\varepsilon}=\emptyset\quad\text{for every } i\in I_\varepsilon \tag{3.139}$$

where $\{B_i\}_{i\in I_\varepsilon}$ denotes the collection of vortex balls constructed in Proposition 3.7. Hence we have

$$|v_\varepsilon|\ge 1-|\ln\varepsilon|^{-5}\quad\text{on } \partial B_{r_\varepsilon}.$$

The existence of a subset $\tilde{J}_\varepsilon\subset J_\varepsilon$ satisfying *(i)-(v)* can now be proved identically as Proposition 3.2 in [78]. It remains to prove (3.138). From the proof of Theorem 3.4, we know (by construction) that each disc $B(x_k^\varepsilon,\lambda_0\varepsilon)$, $k\in J_\varepsilon$, contains at least one point y_k such that $|v_\varepsilon(y_k)|<\frac{1}{2}$. Therefore each disc $B(x_j^\varepsilon,\rho)$, $j\in\tilde{J}_\varepsilon$, contains at least one of the y_k's with $|x_j^\varepsilon-y_k|<\lambda_0\varepsilon$. Assume now that $D_j=0$. By Lemma 3.6 with $\gamma=\mu^{-1/2}$, it would lead to $|v_\varepsilon|\ge\frac{1}{2}$ in $B(x_j^\varepsilon,\rho^\gamma)$ and then $|v_\varepsilon(y_k)|\ge\frac{1}{2}$ for ε small enough, contradiction. ∎

Remark 3.10. We emphasize that each ball $B(x_j^\varepsilon,\rho)$ carries at least one zero of v_ε since $D_j\ne 0$ for any $j\in\tilde{J}_\varepsilon$.

The previous result also gives us a control on the degrees D_j :

Lemma 3.8. *For every $j \in \tilde{J}_\varepsilon$, we have*

$$|D_j| \le C$$

for a constant C independent of ε.

Proof. We have

$$|D_j| = \frac{1}{2\pi}\left|\int_{\partial B(x_j^\varepsilon,\rho)} \frac{1}{|v_\varepsilon|^2}\big(v_\varepsilon \wedge \frac{\partial v_\varepsilon}{\partial \tau}\big)\right| \le C\|\nabla v_\varepsilon\|_{L^2(\partial B(x_j^\varepsilon,\rho))}\sqrt{\rho} \le C$$

by *(iv)* in Proposition 3.15. ∎

3.6 Lower energy estimates

In this section, we obtain various lower energy estimates for v_ε in terms of the vortex structure defined in Section 3.5.3, Proposition 3.15. We start by proving a lower bound on the kinetic energy away from the vortices which brings out the interaction between vortices. The method we use is based on the techniques developped in [7], [20] and [78]. As in the previous section, the main difficulty is due to the degenerate behavior near the boundary of $\mathcal{D}$ of the weight function $a(x)$. To avoid this problem, we shall establish your estimates in B_R^Λ for an arbitrary radius $R \in [\sqrt{a_0}/2, \sqrt{a_0}\,)$. To emphasize the possible dependence on R in the "error term", we will denote by $\mathcal{O}_R(1)$ (respectively $o_R(1)$) any quantity which remains uniformly bounded in ε for fixed R (respectively any quantity which tends to 0 as $\varepsilon \to 0$ for fixed R). In the rest of the chapter, we consider that ε is sufficiently small and we write $\tilde{J}_\varepsilon = \{1, \dots, n\}$. By Theorem 3.4, we may also assume

$$\cup_{j=1}^n B(x_j^\varepsilon, \rho) \subset B_{\frac{\sqrt{a_0}}{3}}. \tag{3.140}$$

3.6.1 A lower estimate away from the vortices

Proposition 3.16. *Setting $\Theta_\rho = B_R \setminus \cup_{j=1}^n B(x_j^\varepsilon, \rho)$, we have*

$$\frac{1}{2}\int_{\Theta_\rho} a(x)|\nabla v_\varepsilon|^2 \ge \pi\sum_{j=1}^n D_j^2\, a(x_j^\varepsilon)|\ln\rho| + W_{R,\varepsilon}\big((x_1^\varepsilon, D_1), \dots, (x_n^\varepsilon, D_n)\big) + \mathcal{O}_R(1) \tag{3.141}$$

where

$$W_{R,\varepsilon}\big((x_1^\varepsilon, D_1), \dots, (x_n^\varepsilon, D_n)\big) = -\pi\sum_{i\ne j} D_i D_j\, a(x_j^\varepsilon)\ln|x_i^\varepsilon - x_j^\varepsilon| - \pi\sum_{j=1}^n D_j \Psi_{R,\varepsilon}(x_j^\varepsilon)$$

and $\Psi_{R,\varepsilon}$ is defined by (3.146). Moreover, if $\frac{\rho}{|x_i^\varepsilon - x_j^\varepsilon|} \to 0$ as $\varepsilon \to 0$ for any $i \ne j$ then the term $\mathcal{O}_R(1)$ is in fact $o_R(1)$.

Remark 3.11. We point out that the dependence on R in the interaction term $W_{R,\varepsilon}$ only appears in the function $\Psi_{R,\varepsilon}$. Moreover, for $\Psi_{R,\varepsilon}$ to be well defined, $1/a(x)$ has to be bounded inside B_R^Λ (see (3.146)) so that we can not pass to the limit $R \to \sqrt{a_0}$ in (3.141) without an *a priori* deterioration of the error term.

Proof. We consider the solution Φ_ρ of the linear problem

$$\begin{cases} \operatorname{div}\big(\frac{1}{a}\nabla\Phi_\rho\big) = 0 & \text{in } \Theta_\rho, \\ \Phi_\rho = 0 & \text{on } \partial B_R, \\ \Phi_\rho = \text{const.} & \text{on } \partial B(x_j^\varepsilon,\rho), \\ \displaystyle\int_{\partial B(x_j^\varepsilon,\rho)} \frac{1}{a}\frac{\partial \Phi_\rho}{\partial \nu} = 2\pi D_j & j = 1,\dots,n, \end{cases}$$

and $\Phi_{R,\varepsilon}$ the solution of

$$\begin{cases} \operatorname{div}\big(\frac{1}{a}\nabla\Phi_{R,\varepsilon}\big) = 2\pi \displaystyle\sum_{j=1}^n D_j\,\delta_{x_j^\varepsilon} & \text{in } B_R \\ \Phi_{R,\varepsilon} = 0 & \text{on } \partial B_R \end{cases} \tag{3.142}$$

For $x \in \Theta_\rho$, we set $w_\varepsilon(x) = \frac{v_\varepsilon(x)}{|v_\varepsilon(x)|}$ and

$$S = \Big(-w_\varepsilon \wedge \frac{\partial w_\varepsilon}{\partial x_2} + \frac{1}{a}\frac{\partial \Phi_\rho}{\partial x_1}\,,\; w_\varepsilon \wedge \frac{\partial w_\varepsilon}{\partial x_1} + \frac{1}{a}\frac{\partial \Phi_\rho}{\partial x_2}\Big).$$

We easily check that

$$\operatorname{div} S = 0 \quad \text{in } \Theta_\rho \quad \text{and} \quad \int_{\partial B_{R_\varepsilon}} S\cdot\nu = \int_{\partial B(x_j^\varepsilon,\rho)} S\cdot\nu = 0.$$

By Lemma I.1 in [20], there exists $H \in C^1(\overline{\Theta}_\rho)$ such that $S = \nabla^\perp H$ and hence we can write the Hodge-de Rham type decomposition

$$w_\varepsilon \wedge \nabla w_\varepsilon = \frac{1}{a}\,\nabla^\perp \Phi_\rho + \nabla H.$$

Consequently,

$$\begin{aligned} \int_{\Theta_\rho} a(x)|\nabla w_\varepsilon|^2 &= \int_{\Theta_\rho} \frac{1}{a(x)}\,|\nabla\Phi_\rho|^2 + 2\int_{\Theta_\rho} \nabla^\perp\Phi_\rho\cdot\nabla H + \int_{\Theta_\rho} a(x)|\nabla H|^2 \\ &\geq \int_{\Theta_\rho} \frac{1}{a(x)}\,|\nabla\Phi_\rho|^2 + 2\int_{\Theta_\rho} \nabla^\perp\Phi_\rho\cdot\nabla H. \end{aligned}$$

The last term is in fact equal to zero since it is the integral of a Jacobian and Φ_ρ is constant on $\partial\Theta_\rho$. Hence

$$\int_{\Theta_\rho} a(x)|\nabla w_\varepsilon|^2 \geq \int_{\Theta_\rho} \frac{1}{a(x)}\,|\nabla\Phi_\rho|^2.$$

Since $|\nabla v_\varepsilon|^2 \geq |v_\varepsilon|^2 |\nabla w_\varepsilon|^2$ in Θ_ρ, we derive that

$$\int_{\Theta_\rho} a(x)|\nabla v_\varepsilon|^2 \geq \int_{\Theta_\rho} \frac{1}{a(x)} |\nabla \Phi_\rho|^2 + T_1 + 2T_2$$

with

$$T_1 = \int_{\Theta_\rho} \left(|v_\varepsilon|^2 - 1\right) \frac{1}{a(x)} |\nabla \Phi_\rho|^2 \quad \text{and} \quad T_2 = \int_{\Theta_\rho} \left(|v_\varepsilon|^2 - 1\right) \nabla \Phi_\rho^\perp \cdot \nabla H.$$

Arguing as in [7] (see Step 4 in the proof of Theorem 6), it turns out that $T_1 = o_R(1)$ and $T_2 = o_R(1)$ and therefore

$$\int_{\Theta_\rho} a(x)|\nabla v_\varepsilon|^2 \geq \int_{\Theta_\rho} \frac{1}{a(x)} |\nabla \Phi_\rho|^2 + o_R(1). \tag{3.143}$$

On the other hand, we have

$$\int_{\Theta_\rho} \frac{1}{a(x)} |\nabla \Phi_\rho|^2 = \int_{\partial\Theta_\rho} \frac{1}{a(x)} \frac{\partial \Phi_\rho}{\partial \nu} \Phi_\rho = -2\pi \sum_{j=1}^n D_j \, \Phi_\rho(z_j)$$

for any point $z_j \in \partial B(x_j^\varepsilon, \rho)$. By Lemma 3.8, we may write this equality as

$$\int_{\Theta_\rho} \frac{1}{a(x)} |\nabla \Phi_\rho|^2 = -2\pi \sum_{j=1}^n D_j \, \Phi_{R,\varepsilon}(z_j) + \mathcal{O}(\|\Phi_{R,\varepsilon} - \Phi_\rho\|_{L^\infty(\Theta_\rho)}) \tag{3.144}$$

Using an adaptation of Lemma I.4 in [20] (see e.g. [15], Lemma 3.5), we derive that

$$\|\Phi_{R,\varepsilon} - \Phi_\rho\|_{L^\infty(\Theta_\rho)} \leq \sum_{j=1}^n \left(\sup_{\partial B(x_j^\varepsilon, \rho)} \Phi_{R,\varepsilon} - \inf_{\partial B(x_j^\varepsilon, \rho)} \Phi_{R,\varepsilon} \right). \tag{3.145}$$

Now we define for $x \in B_R$,

$$\Psi_{R,\varepsilon}(x) = \Phi_{R,\varepsilon}(x) - \sum_{j=1}^n D_j \, a(x_j^\varepsilon) \ln |x - x_j^\varepsilon|.$$

Since $\Phi_{R,\varepsilon}$ satisfies (3.142), we easily derive that $\Psi_{R,\varepsilon}$ verifies

$$\begin{cases} \operatorname{div}\left(\frac{1}{a}\nabla \Psi_{R,\varepsilon}\right) = -\sum_{j=1}^n D_j \, a(x_j^\varepsilon) \, \nabla \left(\frac{1}{a}\right) \cdot \nabla \left(\ln |x - x_j^\varepsilon|\right) & \text{in } B_R, \\ \Psi_{R,\varepsilon} = -\sum_{j=1}^n D_j \, a(x_j^\varepsilon) \ln |x - x_j^\varepsilon| & \text{on } \partial B_R. \end{cases} \tag{3.146}$$

By elliptic regularity, we have (recall that all the x_j^ε's remain close to the origin)

$$\|\Psi_{R,\varepsilon}\|_{W^{2,p}(B_R)} \leq C_{R,p} \quad \text{for any } 1 \leq p < 2.$$

In particular, $\Psi_{R,\varepsilon}$ is uniformly bounded with respect to ε in $C^{0,1/2}(B_R)$ and hence

$$\sup_{\partial B(x_j^\varepsilon,\rho)} \Psi_{R,\varepsilon} - \inf_{\partial B(x_j^\varepsilon,\rho)} \Psi_{R,\varepsilon} \leq C_R\sqrt{\rho} = o_R(1).$$

Since $|x_j^\varepsilon - x_i^\varepsilon| \geq 8\rho$, we have by Lemma 3.8,

$$\sup_{\partial B(x_j^\varepsilon,\rho)} \left(\sum_{i=1}^n D_i\, a(x_i^\varepsilon) \ln |x - x_i^\varepsilon| \right) - \inf_{\partial B(x_j^\varepsilon,\rho)} \left(\sum_{i=1}^n D_i\, a(x_i^\varepsilon) \ln |x - x_i^\varepsilon| \right)$$
$$\leq \rho \sum_{i=1}^n a(x_i^\varepsilon) \sup_{\partial B(x_j^\varepsilon,\rho)} \frac{|D_i|}{|x - x_i^\varepsilon|} \leq \mathcal{O}(1),$$

(respectively $\leq o(1)$ if $\frac{\rho}{|x_i^\varepsilon - x_j^\varepsilon|} \to 0$ as $\varepsilon \to 0$ for any $i \neq j$). Coming back to (3.145), we deduce that

$$\|\Phi_{R,\varepsilon} - \Phi_\rho\|_{L^\infty(\Theta_\rho)} \leq \mathcal{O}_R(1)$$

(respectively $\leq o_R(1)$ if $\frac{\rho}{|x_i^\varepsilon - x_j^\varepsilon|} \to 0$ as $\varepsilon \to 0$ for any $i \neq j$). Inserting this estimate in (3.144), we get that

$$\begin{aligned}\int_{\Theta_\rho} \frac{1}{a(x)}\, |\nabla \Phi_\rho|^2 =& -2\pi \sum_{j=1}^n D_j\, \Phi_{R,\varepsilon}(z_j) + \mathcal{O}_R(1)\\ =& -2\pi \sum_{j=1}^n D_j\, \Psi_{R,\varepsilon}(z_j) - 2\pi \sum_{i\neq j} D_i D_j\, a(x_i^\varepsilon) \ln |z_j - x_i^\varepsilon|\\ &+ 2\pi \sum_{j=1}^n D_j^2\, a(x_j^\varepsilon) |\ln \rho| + \mathcal{O}_R(1)\end{aligned}$$

(respectively $+o_R(1)$ as $\varepsilon \to 0$). Since $\Psi_{R,\varepsilon}$ is uniformly bounded with respect to ε in $C^{0,1/2}(B_R)$, we have

$$|\Psi_{R,\varepsilon}(z_j) - \Psi_{R,\varepsilon}(x_j^\varepsilon)| \leq C_R\sqrt{\rho} = o_R(1).$$

By Lemma 3.8 and since $|x_j^\varepsilon - x_i^\varepsilon| \geq 8\rho$, we derive

$$\begin{aligned}\Big| \sum_{i\neq j} D_i D_j\, a(x_i^\varepsilon)(\ln |z_j - x_i^\varepsilon| - \ln |x_j^\varepsilon - x_i^\varepsilon|)\Big| &\leq \sum_{i\neq j} |D_i|\, |D_j| \ln \Big|1 + \frac{z_j - x_j^\varepsilon}{x_j^\varepsilon - x_i^\varepsilon}\Big|\\ &\leq \sum_{i\neq j} |D_i|\, |D_j|\, \frac{\rho}{|x_j^\varepsilon - x_i^\varepsilon|} \leq \mathcal{O}(1)\end{aligned}$$

(respectively $\leq o(1)$ as $\varepsilon \to 0$) and we conclude that

$$\begin{aligned}\int_{\Theta_\rho} \frac{1}{a(x)}\, |\nabla \Phi_\rho|^2 =& -2\pi \sum_{j=1}^n D_j\, \Psi_{R,\varepsilon}(x_j^\varepsilon) - 2\pi \sum_{i\neq j} D_i D_j\, a(x_i^\varepsilon) \ln |x_j^\varepsilon - x_i^\varepsilon|\\ &+ 2\pi \sum_{j=1}^n D_j^2\, a(x_j^\varepsilon) |\ln \rho| + \mathcal{O}_R(1)\end{aligned}$$

(respectively $+o_R(1)$ as $\varepsilon \to 0$). Combining this estimate with (3.143), we obtain the result. ■

Remark 3.12. It would be interesting to know if the estimates on $\Psi_{R,\varepsilon}$ and $\Phi_{R,\varepsilon}$ hold independently on R when ε is small.

3.6.2 Lower estimate for $\mathcal{E}^a_\varepsilon$

From Proposition 3.16 and Proposition 3.15, we derive the following lower bounds estimating the contribution of any vortex.

Lemma 3.9. *We have*

$$\mathcal{E}^a_\varepsilon(v_\varepsilon, B_R) \geq \pi \sum_{j=1}^{n} D_j^2\, a(x_j^\varepsilon)|\ln\rho| + \pi \sum_{j=1}^{n} |D_j|\, a(x_j^\varepsilon) \ln\frac{\rho}{\varepsilon} + W_{R,\varepsilon} + \mathcal{O}_R(1) \tag{3.147}$$

and

$$\mathcal{E}^a_\varepsilon(v_\varepsilon, B_R) \geq \pi \sum_{j=1}^{n} |D_j|\, a(x_j^\varepsilon) \ln\frac{\rho}{\varepsilon} + \mathcal{O}(1). \tag{3.148}$$

Proof. By Proposition 3.16, it is sufficient to show that

$$\mathcal{E}^a_\varepsilon(v_\varepsilon, B(x_j^\varepsilon,\rho)) \geq \pi|D_j|\, a(x_j^\varepsilon)\ln\frac{\rho}{\varepsilon} + \mathcal{O}(1)$$

which is equivalent to prove

$$\frac{1}{2}\int_{B(x_j^\varepsilon,\rho)} |\nabla v_\varepsilon|^2 + \frac{a(x_j^\varepsilon)}{2\varepsilon^2}(1-|v_\varepsilon|^2)^2 \geq \pi|D_j|\, \ln\frac{\rho}{\varepsilon} + \mathcal{O}(1) \tag{3.149}$$

(we used that $|a(x)-a(x_j^\varepsilon)| \leq C\rho$ for $x \in B(x_j^\varepsilon,\rho)$ and $\mathcal{E}^a_\varepsilon(v_\varepsilon, B_R) \leq C|\ln\varepsilon|$). We consider the change of variable $\tilde{x} = \dfrac{x - x_j^\varepsilon}{\rho}$ and we set

$$\tilde{v}(\tilde{x}) = v_\varepsilon(x) \quad \text{and} \quad \tilde{\varepsilon} = \frac{\varepsilon}{\rho\sqrt{a(x_j^\varepsilon)}}\,.$$

From *(iii)* in Proposition 3.15 we have $\tilde{v} \geq 1 - \frac{2}{|\ln\varepsilon|}$ on ∂B_1 and by *(iv)* in Proposition 3.15,

$$\int_{\partial B_1} \frac{|\nabla\tilde{v}|^2}{2} + \frac{1}{4\tilde{\varepsilon}^2}(1-|\tilde{v}|^2)^2 = \rho\int_{\partial B(x_j^\varepsilon,\rho)} \frac{|\nabla v_\varepsilon|^2}{2} + \frac{a(x_j^\varepsilon)}{4\varepsilon^2}(1-|v_\varepsilon|^2)^2 \leq C$$

and

$$\frac{1}{2}\int_{B_1} |\nabla\tilde{v}|^2 + \frac{1}{2\tilde{\varepsilon}^2}(1-|\tilde{v}|^2)^2 = \frac{1}{2}\int_{B(x_j^\varepsilon,\rho)} |\nabla v_\varepsilon|^2 + \frac{a(x_j^\varepsilon)}{2\varepsilon^2}(1-|v_\varepsilon|^2)^2.$$

As in the proof of Lemma VI.1 in [7], we infer that for ε small enough,

$$\frac{1}{2}\int_{B_1} |\nabla\tilde{v}|^2 + \frac{1}{2\tilde{\varepsilon}^2}(1-|\tilde{v}|^2)^2 \geq \pi|D_j|\,|\ln\tilde{\varepsilon}| + \mathcal{O}(1) = \pi|D_j|\, \ln\frac{\rho}{\varepsilon} + \mathcal{O}(1)$$

and hence (3.149) holds. ∎

3.6.3 Lower estimate for $\mathcal{F}_\varepsilon^a$

We are now able to give some lower expansions for $\mathcal{F}_\varepsilon^a(v_\varepsilon, \mathcal{D}_\varepsilon)$.

Lemma 3.10. *We have*

$$\mathcal{F}_\varepsilon^a(v_\varepsilon, \mathcal{D}_\varepsilon) \geq \pi \sum_{j=1}^n D_j^2\, a(x_j^\varepsilon) |\ln \rho| + \pi \sum_{j=1}^n |D_j|\, a(x_j^\varepsilon) \ln \frac{\rho}{\varepsilon} - \frac{\pi\Omega}{2} \sum_{j=1}^n a^2(x_j^\varepsilon)\, D_j + \\ + W_{R,\varepsilon} + \mathcal{O}_R(1) \qquad (3.150)$$

and

$$\mathcal{F}_\varepsilon^a(v_\varepsilon, \mathcal{D}_\varepsilon) \geq \pi \sum_{j=1}^n |D_j|\, a(x_j^\varepsilon) \ln \frac{\rho}{\varepsilon} - \frac{\pi\Omega}{2} \sum_{j=1}^n a^2(x_j^\varepsilon)\, D_j + \mathcal{O}(1). \qquad (3.151)$$

Proof. Consider the family of vortex ball $\{B_i\}_{i\in I_\varepsilon}$ given in Proposition 3.7. As in the proof of Proposition 3.15, we can find $r_\varepsilon \in [R, \frac{R+\sqrt{a_0}}{2}]$ such that (3.139) holds. We set

$$\hat{I}_* = \left\{ i \in I_*,\, p_i \notin \overline{B}_{r_\varepsilon} \right\} \quad \text{and} \quad \hat{I}_- = \left\{ i \in I_-,\, p_i \notin \overline{B}_{r_\varepsilon} \right\} \qquad (3.152)$$

where I_* and I_- are defined in Section 4.4. By construction, we have

$$\overline{B}_i \subset \mathcal{D}_\varepsilon \setminus \overline{B}_{r_\varepsilon} \quad \text{for any } i \in \hat{I}_* \cup \hat{I}_-.$$

Setting $\Xi_\varepsilon = \mathcal{D}_\varepsilon \setminus \left(\bigcup_{i\in \hat{I}_*\cup\hat{I}_-} B_i \cup \bigcup_{j=1}^n B(x_j^\varepsilon, \rho) \right)$, we derive from Proposition 3.7, *1)* in Theorem 3.4 and Proposition 3.15, that for ε small enough,

$$|v_\varepsilon| \geq \frac{1}{2} \quad \text{in } \Xi_\varepsilon.$$

Arguing exactly as in the proof of Proposition 3.8 we obtain that

$$\mathcal{R}_\varepsilon^a(v_\varepsilon, \Xi_\varepsilon) = \frac{\pi\Omega}{2} \sum_{j=1}^n a^2(x_j^\varepsilon)\, D_j + \frac{\pi\Omega}{2} \sum_{i\in\hat{I}_*\cup\hat{I}_-} \left(a^2(p_i) - \nu_\varepsilon^2 |\ln \varepsilon|^{-3} \right) d_i + o_R(1). \qquad (3.153)$$

Now we remark that

$$\left| \mathcal{R}_\varepsilon^a(v_\varepsilon, B(x_j^\varepsilon, \rho)) \right| \leq C\Omega\rho \|\nabla v_\varepsilon\|_{L^2(\mathcal{D}_\varepsilon)} = o(\varepsilon^\beta |\ln \varepsilon|^{3/2})$$

(here we use Proposition 3.11, *3.1.c)* in Proposition 3.1 and *3.4.b)* in Proposition 3.4) and using Proposition 3.7, we deduce that

$$\begin{aligned} \mathcal{F}_\varepsilon^a(v_\varepsilon, \mathcal{D}_\varepsilon) &\geq \mathcal{E}_\varepsilon^a(v_\varepsilon, B_{r_\varepsilon}) - \mathcal{R}_\varepsilon^a(v_\varepsilon, \Xi_\varepsilon) + \sum_{i\in\hat{I}_*\cup\hat{I}_-} \mathcal{F}_\varepsilon^a(v_\varepsilon, B_i) + o_R(1) \\ &\geq \mathcal{E}_\varepsilon^a(v_\varepsilon, B_{r_\varepsilon}) - \frac{\pi\Omega}{2} \sum_{j=1}^n a^2(x_j^\varepsilon)\, D_j + \pi \sum_{i\in\hat{I}_*\cup\hat{I}_-} a(p_i)|d_i| \left(|\ln \varepsilon| - \Lambda_0 \ln |\ln \varepsilon| \right) \\ &\quad - \frac{\pi\Omega}{2} \sum_{i\in\hat{I}_*\cup\hat{I}_-} \left(a^2(p_i) - \nu_\varepsilon^2 |\ln \varepsilon|^{-3} \right) d_i + o_R(1). \end{aligned} \qquad (3.154)$$

Since $p_i \notin \overline{B}_{r_\varepsilon}$ for $i \in \hat{I}_* \cup \hat{I}_-$, we have $a(p_i) \ll a_0$ and we infer that for ε small enough,

$$\pi \sum_{i \in \hat{I}_* \cup \hat{I}_-} a(p_i)|d_i|\big(|\ln\varepsilon| - \Lambda_0 \ln|\ln\varepsilon|\big) - \frac{\pi\Omega}{2} \sum_{i \in \hat{I}_* \cup \hat{I}_-} \big(a^2(p_i) - \nu_\varepsilon^2 |\ln\varepsilon|^{-3}\big) d_i \geq 0$$

which leads to (since $r_\varepsilon \geq R$)

$$\mathcal{F}_\varepsilon^a(v_\varepsilon, \mathcal{D}_\varepsilon) \geq \mathcal{E}_\varepsilon^a(v_\varepsilon, B_R) - \frac{\pi\Omega}{2} \sum_{j=1}^{n} a^2(x_j^\varepsilon)\, D_j + o_R(1). \tag{3.155}$$

Combining (3.155) and (3.147) we obtain (3.150). In the same way, (3.155) with $R = \frac{\sqrt{a_0}}{2}$ and (3.148) yield (3.151). ■

3.7 Proof of Theorem 3.1

In this section, we are going to prove Theorem 3.1 in terms of the map v_ε. We write

$$\Omega = \frac{2}{a_0}(|\ln\varepsilon| + \omega(\varepsilon)\ln|\ln\varepsilon|) \tag{3.156}$$

so that assumption (3.104) can be reformulated as $\omega(\varepsilon) \leq \omega_1$.

3.7.1 Vortices have degree one

Lemma 3.11. *We have*

$$D_j = +1 \quad \text{for } j = 1, \dots, n,$$

for ε sufficiently small.

Proof. By (3.140) we may use the estimates in Section 6 with $R = \frac{\sqrt{a_0}}{2}$. Combining Proposition 3.6 and Lemma 3.10, we get that

$$\pi \sum_{j=1}^{n} |D_j|\, a(x_j^\varepsilon) \ln\frac{\rho}{\varepsilon} - \frac{\pi a_0 \Omega}{2} \sum_{j=1}^{n} a(x_j^\varepsilon)\, D_j \leq \pi \sum_{j=1}^{n} |D_j|\, a(x_j^\varepsilon) \ln\frac{\rho}{\varepsilon} - \frac{\pi\Omega}{2} \sum_{j=1}^{n} a^2(x_j^\varepsilon)\, D_j \leq \mathcal{O}(1)$$

Using (3.156), we derive

$$\sum_{j=1}^{n} |D_j|\, a(x_j^\varepsilon) \ln\frac{\rho}{\varepsilon} \leq \sum_{D_j > 0} |D_j|\, a(x_j^\varepsilon) |\ln\varepsilon| + o(|\ln\varepsilon|)$$

Since $\rho \geq \varepsilon^\mu$, it yields (recall that $D_j \neq 0$)

$$(1-\mu) \sum_{D_j < 0} |D_j|\, a(x_j^\varepsilon) |\ln\varepsilon| \leq \mu \sum_{D_j > 0} |D_j|\, a(x_j^\varepsilon) |\ln\varepsilon| + o(|\ln\varepsilon|).$$

For ε small we have $a_0 \geq a(x_j^\varepsilon) \geq \frac{a_0}{2}$ and therefore (using Lemma 3.8)

$$\sum_{D_j < 0} |D_j| \leq \frac{2\mu}{1-\mu} \sum_{D_j > 0} |D_j| + o(1) \leq \frac{C\mu}{1-\mu} + o(1).$$

Choosing μ and then ε sufficiently small, we obtain $\sum_{D_j<0}|D_j| \equiv 0$, i.e.,

$$D_j > 0 \quad \text{for any } j = 1, \dots, n.$$

Since all the x_j^ε 's remain close to the origin, we have for ε small enough,

$$-\pi \sum_{i \neq j} D_i D_j \, a(x_j^\varepsilon) \ln |x_i^\varepsilon - x_j^\varepsilon| \geq \mathcal{O}(1)$$

and hence $W_{\frac{\sqrt{a_0}}{2},\varepsilon} \geq -\pi \sum_{j=1}^n D_j \Psi_{\frac{\sqrt{a_0}}{2},\varepsilon}(x_j^\varepsilon) = \mathcal{O}(1)$. We deduce from Lemma 3.6 and Lemma 3.10,

$$\pi \sum_{j=1}^n D_j^2 \, a(x_j^\varepsilon) |\ln \rho| + \pi \sum_{j=1}^n D_j \, a(x_j^\varepsilon) \ln \frac{\rho}{\varepsilon} - \frac{\pi\Omega}{2} \sum_{j=1}^n a^2(x_j^\varepsilon) \, D_j \leq \mathcal{O}(1).$$

As previously, we derive from (3.156),

$$\sum_{j=1}^n (D_j^2 - D_j) \, a(x_j^\varepsilon) |\ln \rho| \leq o(|\ln \varepsilon|).$$

Since $\rho \leq \varepsilon^{\overline{\mu}}$ and the x_j^ε 's are closed to 0,

$$\overline{\mu} \sum_{j=1}^n (D_j^2 - D_j) \leq o(1)$$

which leads to $D_j = +1$ for ε sufficiently small. ∎

We now derive an easy estimate for the energy.

Corollary 3.1. *We have*

$$\mathcal{F}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon) \geq \pi \sum_{j=1}^n a(x_j^\varepsilon) |\ln \varepsilon| - \frac{\pi\Omega}{2} \sum_{j=1}^n a^2(x_j^\varepsilon) + W_{R,\varepsilon} + \mathcal{O}_R(1).$$

Proof. This estimate follows directly from Lemma 3.10, Lemma 3.11, (3.85) and (3.88). ∎

3.7.2 The subcritical case

In this section, we extend Proposition 3.9 to higher rotational speeds which remain below Ω_1.

Proposition 3.17. *Assume that $\omega_1 < 0$. Then the conclusion of Proposition 3.9 holds.*

Proof. We fix $\frac{\sqrt{a_0}}{2} < R_0 < \sqrt{a_0}$. We get from Corollary 3.1 with $R = \frac{\sqrt{a_0}}{2}$ and (3.84) that

$$\pi \sum_{j=1}^n a(x_j^\varepsilon) |\ln \varepsilon| - \frac{\pi a_0 \Omega}{2} \sum_{j=1}^n a(x_j^\varepsilon) \leq \pi \sum_{j=1}^n a(x_j^\varepsilon) |\ln \varepsilon| - \frac{\pi\Omega}{2} \sum_{j=1}^n a^2(x_j^\varepsilon) \leq \mathcal{O}(1)$$

Using (3.156), we obtain that

$$-\frac{\omega_1 \, n a_0}{2} \ln |\ln \varepsilon| \leq -\omega_1 \sum_{j=1}^n a(x_j^\varepsilon) \ln |\ln \varepsilon| \leq \mathcal{O}(1)$$

and then $n \leq \frac{C}{|\omega_1| \ln|\ln\varepsilon|}$ which implies that $n = 0$ for ε small enough. Therefore we have $\tilde{J}_\varepsilon = \emptyset$ i.e.,

$$|v_\varepsilon| \geq \frac{1}{2} \quad \text{in } \overline{B}_{\frac{\sqrt{a_0}}{2}}.$$

By *1)* in Theorem 3.4, for $\varepsilon < \varepsilon_{R_0}$ we have

$$|v_\varepsilon| \geq \frac{1}{2} \quad \text{in } \overline{B}_{R_0}.$$

Using the notation (3.152), we infer from Proposition 3.6 and (3.154),

$$\pi \sum_{i \in \hat{I}_* \cup \hat{I}_-} a(p_i)|d_i|\big(|\ln\varepsilon| - \Lambda_0 \ln|\ln\varepsilon|\big) - \frac{\pi\Omega}{2} \sum_{i \in \hat{I}_* \cup \hat{I}_-} \big(a^2(p_i) - \nu_\varepsilon^2 |\ln\varepsilon|^{-3}\big) d_i \leq \mathcal{O}(|\ln\varepsilon|^{-1}).$$

Since $a(p_i) \ll a_0$ for $i \in \hat{I}_* \cup \hat{I}_-$, we infer that exists $c > 0$ independent of ε such that

$$c \sum_{i \in \hat{I}_* \cup \hat{I}_-} a(p_i)|d_i||\ln\varepsilon| \leq \pi \sum_{i \in \hat{I}_* \cup \hat{I}_-} a(p_i)|d_i|\big(|\ln\varepsilon| - \Lambda_0 \ln|\ln\varepsilon|\big) - \frac{\pi\Omega}{2} \sum_{i \in \hat{I}_* \cup \hat{I}_-} \big(a^2(p_i) - \nu_\varepsilon^2 |\ln\varepsilon|^{-3}\big) d_i$$

and since $a(x) \geq |\ln\varepsilon|^{-3/2}$ in $\mathcal{D}_\varepsilon$, we finally obtain

$$\sum_{i \in \hat{I}_* \cup \hat{I}_-} |d_i| \leq \mathcal{O}(|\ln\varepsilon|^{-1/2}).$$

Hence $\sum_{i \in \hat{I}_* \cup \hat{I}_-} |d_i| = 0$ for ε sufficiently small and we conclude from (3.153),

$$\mathcal{R}_\varepsilon^a(v_\varepsilon, \mathcal{D}_\varepsilon \setminus \cup_{i \in \hat{I}_* \cup \hat{I}_-} B_i) = o(1).$$

Then the rest of the proof follows as in the proof of Proposition 3.9. ■

3.7.3 The supercritical case

From now, we assume that

$$\omega(\varepsilon) \geq \delta > 0$$

for some constant δ independent of ε. We are going to prove that vortices appear in this regime. We will use explicit test functions constructed in Section 3.8. We start with :

Lemma 3.12. *v_ε has at least one vortex (i.e., $n \geq 1$) for any ε sufficiently small.*

Proof. By Theorem 3.5 in Section 3.8 (with $d = 1$), there exists for ε small enough, $\tilde{u}_\varepsilon \in \mathcal{H}$ such that $\|\tilde{u}_\varepsilon\|_{L^2(\mathbb{R}^2)} = 1$ and

$$F_\varepsilon(\tilde{u}_\varepsilon) \leq E_\varepsilon(\eta_\varepsilon) - \pi a_0 \omega(\varepsilon) \ln|\ln\varepsilon| + \mathcal{O}(1).$$

By the minimizing property of u_ε and Lemma 3.4, we have $E_\varepsilon(\eta_\varepsilon) + \mathcal{F}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon) = F_\varepsilon(u_\varepsilon) \leq F_\varepsilon(\tilde{u}_\varepsilon)$ and then we deduce that

$$\mathcal{F}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon) \leq -\pi a_0 \omega(\varepsilon) \ln|\ln\varepsilon| + \mathcal{O}(1).$$

From here, it turns out by Corollary 3.1 with $R = \frac{\sqrt{a_0}}{2}$ (recall that $W_{\frac{\sqrt{a_0}}{2},\varepsilon} \geq \mathcal{O}(1)$),

$$\begin{aligned}-\pi a_0 \omega(\varepsilon) \ln|\ln\varepsilon| + \mathcal{O}(1) \geq \mathcal{F}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon) &\geq \pi \sum_{j=1}^n a(x_j^\varepsilon)|\ln\varepsilon| - \frac{\pi\Omega}{2}\sum_{j=1}^n a^2(x_j^\varepsilon)\\ &\geq \pi \sum_{j=1}^n a(x_j^\varepsilon)\left(-\omega(\varepsilon)\ln|\ln\varepsilon| + \frac{\Omega|x_j^\varepsilon|^2}{2}\right)\\ &\geq -\pi a_0\omega(\varepsilon) n \ln|\ln\varepsilon|.\end{aligned}$$

Hence $n \geq 1 + o(1)$ and the conclusion follows. ■

We shall use this first development of energy :

Proposition 3.18. *We have*

$$\mathcal{F}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon) = -\pi a_0 \omega(\varepsilon) n \ln|\ln\varepsilon| + \frac{\pi a_0}{2}(n^2 - n)\ln|\ln\varepsilon| + \mathcal{O}(1)$$

Proof. In the case $n = 1$, we have already proved the result in the proof of the previous lemma. Then we may assume that $n \geq 2$. Since $\|\Psi_{\frac{\sqrt{a_0}}{2},\varepsilon}\|_\infty = \mathcal{O}(1)$, we get from Corollary 3.1 with $R = \frac{\sqrt{a_0}}{2}$,

$$\begin{aligned}\mathcal{F}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon) &\geq \pi\sum_{j=1}^n a(x_j^\varepsilon)\Bigg(|\ln\varepsilon| - \sum_{\substack{i=1\\ i\neq j}}^n \ln|x_i^\varepsilon - x_j^\varepsilon| - \frac{\Omega}{2}\,a(x_j^\varepsilon)\Bigg) + \mathcal{O}(1)\\ &\geq \pi\sum_{j=1}^n a(x_j^\varepsilon)\Bigg(-\omega(\varepsilon)\ln|\ln\varepsilon| - \sum_{\substack{i=1\\ i\neq j}}^n \ln|x_i^\varepsilon - x_j^\varepsilon| + \frac{\Omega}{2}\,|x_j^\varepsilon|^2\Bigg) + \mathcal{O}(1). \qquad (3.157)\end{aligned}$$

By Proposition 3.5, $\mathcal{F}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon) \leq o(1)$ and therefore

$$-\sum_{i\neq j}\ln|x_i^\varepsilon - x_j^\varepsilon| + \frac{\Omega}{2}\sum_{j=1}^n |x_j^\varepsilon|^2 \leq C\ln|\ln\varepsilon|$$

On the other hand $-\sum_{i\neq j}\ln|x_i^\varepsilon - x_j^\varepsilon| \geq \mathcal{O}(1)$ so that $|x_j^\varepsilon|^2 \leq C\big(\ln|\ln\varepsilon|\big)|\ln\varepsilon|^{-1}$ and hence

$$\pi\sum_{j=1}^n a(x_j^\varepsilon)\Bigg(-\omega(\varepsilon)\ln|\ln\varepsilon| - \sum_{\substack{i=1\\ i\neq j}}^n \ln|x_i^\varepsilon - x_j^\varepsilon| + \frac{\Omega}{2}\,|x_j^\varepsilon|^2\Bigg) = \qquad (3.158)$$

$$= -\pi a_0\omega(\varepsilon) n\ln|\ln\varepsilon| - \pi a_0\sum_{i\neq j}\ln|x_i^\varepsilon - x_j^\varepsilon| + \frac{\pi a_0\Omega}{2}\sum_{j=1}^n |x_j^\varepsilon|^2 + o(1)$$

Setting $r = \max_j |x_j^\varepsilon|$, we remark that

$$-\sum_{i\neq j} \ln|x_i^\varepsilon - x_j^\varepsilon| + \frac{\Omega}{2}\sum_{j=1}^n |x_j^\varepsilon|^2 \geq -(n^2-n)\ln 2r + \frac{\Omega r^2}{2} \geq \frac{n^2-n}{2}\ln|\ln\varepsilon| + \mathcal{O}(1). \quad (3.159)$$

Combining this estimate with (3.157) and (3.158), we finally obtain

$$\mathcal{F}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon) \geq -\pi a_0 \omega(\varepsilon) n \ln|\ln\varepsilon| + \frac{\pi a_0}{2}(n^2-n)\ln|\ln\varepsilon| + \mathcal{O}(1). \quad (3.160)$$

By Theorem 3.5 in Section 8, there exists $\tilde{u}_\varepsilon \in \mathcal{H}$ such that $\|\tilde{u}_\varepsilon\|_{L^2(\mathbb{R}^2)} = 1$ and

$$F_\varepsilon(\tilde{u}_\varepsilon) \leq E_\varepsilon(\eta_\varepsilon) - \pi a_0\omega(\varepsilon) n \ln|\ln\varepsilon| + \frac{\pi a_0}{2}(n^2-n)\ln|\ln\varepsilon| + \mathcal{O}(1).$$

Since $E_\varepsilon(\eta_\varepsilon) + \mathcal{F}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon) = F_\varepsilon(u_\varepsilon) \leq F_\varepsilon(\tilde{u}_\varepsilon)$, we obtain the reverse inequality in (3.160) and the proof is complete. ∎

Now we are in position to derive the critical rotational velocities for which v_ε has exactly d vortices.

Proposition 3.19. *Assume that $(d-1)+\delta \leq \omega(\varepsilon) \leq d-\delta$ for some integer $d \geq 1$ and $0 < \delta \ll 1$. Then, for ε sufficiently small, v_ε has exactly d vortices of degree one, i.e., $n = d$.*

Proof. We start with proving that $n \geq d$. The case $d = 1$ is given by Lemma 3.12. Now we assume that $d \geq 2$. By Proposition 3.18 and using the test functions in Theorem 3.5 as in the proof of Proposition 3.18, we infer that

$$-\pi a_0\omega(\varepsilon) n\ln|\ln\varepsilon| + \frac{\pi a_0}{2}(n^2-n)\ln|\ln\varepsilon| \leq -\pi a_0 \omega(\varepsilon) d\ln|\ln\varepsilon| + \frac{\pi a_0}{2}(d^2-d)\ln|\ln\varepsilon| + \mathcal{O}(1).$$

Hence we have

$$-\omega(\varepsilon)n + \frac{n^2-n}{2} \leq -\omega(\varepsilon)d + \frac{d^2-d}{2} + o(1)$$

and it yields

$$\omega(\varepsilon)(d-n) \leq \frac{(d-n)(d+n-1)}{2} + o(1). \quad (3.161)$$

If assume that $n \leq d-1$, it would lead to

$$(d-1)+\delta \leq \frac{d+n-1}{2} + o(1) \leq d-1+o(1)$$

which is impossible for ε small enough.

Assume now that $n \geq d+1$. As previously we infer that (3.161) holds and therefore

$$d-\delta \geq \frac{d+n-1}{2} + o(1) \geq d + o(1)$$

which is also impossible for ε small. ∎

3.7.4 Vortex location and final expansion of the energy

In this section, we assume that $(d-1)+\delta \leq \omega(\varepsilon) \leq d-\delta$ for some integer $d \geq 1$ and $0<\delta \ll 1$. By Proposition 3.19, we may assume that v_ε has exactly d vortices. We obtain here a precise information on their location.

Lemma 3.13. *Under the assumptions above, we have*

$$|x_j^\varepsilon| \leq \frac{C}{|\ln\varepsilon|^{1/2}} \quad \text{for } j=1,\dots,d$$

and for $d \geq 2$,

$$|x_i^\varepsilon - x_j^\varepsilon| \geq \frac{C}{|\ln\varepsilon|^{1/2}} \quad \text{for } i \neq j.$$

Proof. Combining Proposition 3.18, (3.157) and (3.158), we get that

$$-\pi a_0 \sum_{i\neq j} \ln|x_i^\varepsilon - x_j^\varepsilon| + \frac{\pi a_0 \Omega}{2} \sum_{j=1}^d |x_j^\varepsilon|^2 \leq \frac{\pi a_0}{2}(d^2-d)\ln|\ln\varepsilon| + \mathcal{O}(1)$$

Hence

$$\sum_{j=1}^d \left(-\sum_{i\neq j} \ln\left(\sqrt{|\ln\varepsilon|}\,|x_i^\varepsilon - x_j^\varepsilon|\right) + \frac{\Omega|x_j^\varepsilon|^2}{2} \right) \leq \mathcal{O}(1)$$

and the conclusion follows. ■

Since $\frac{\rho}{|x_i^\varepsilon - x_j^\varepsilon|} = o(1)$ and $D_j = 1$, we may now improve the lower estimates obtained in Lemma 3.9 :

Lemma 3.14. *We have*

$$\mathcal{E}_\varepsilon^a(v_\varepsilon, B_R) \geq \pi a_0 \sum_{j=1}^d a(x_j^\varepsilon)|\ln\varepsilon| + W_{R,\varepsilon}(x_1^\varepsilon,\dots,x_d^\varepsilon) + \frac{\pi a_0 d}{2}\ln a_0 + a_0 d\gamma_0 + o_R(1)$$

where γ_0 is an absolute constant.

Proof. Since $\frac{\rho}{|x_i^\varepsilon - x_j^\varepsilon|} = o(1)$ and $D_j = 1$, we obtain by Proposition 3.16,

$$\frac{1}{2}\int_{\Theta_\rho} a(x)|\nabla v_\varepsilon|^2 \geq \pi \sum_{j=1}^d a(x_j^\varepsilon)|\ln\rho| + W_{R,\varepsilon}(x_1^\varepsilon,\dots,x_d^\varepsilon) + o_R(1) \tag{3.162}$$

and it remains to estimate $\mathcal{E}_\varepsilon^a(v_\varepsilon, B(x_j^\varepsilon,\rho))$ for $j=1,\dots,d$. We proceed as follows. Since $D_j = 1$, we may write on $\partial B(x_j^\varepsilon,\rho)$ in polar coordinates with center x_j^ε,

$$v_\varepsilon(x) = |v_\varepsilon(x)|\, e^{i(\theta+\psi_j(\theta))}, \quad \theta \in [0,2\pi]$$

where $\psi_j \in H^1([0,2\pi],\mathbb{R})$ and $\psi_j(0) = \psi_j(2\pi) = 0$. Then in each disc $B(x_j^\varepsilon, 2\rho)$, we consider the map $\hat{v}_\varepsilon$ defined by

$$\hat{v}_\varepsilon(x) = v_\varepsilon(x) \quad \text{if } x \in B(x_j^\varepsilon, \rho)$$

and if $x \in B(x_j^\varepsilon, 2\rho) \setminus B(x_j^\varepsilon, \rho)$,

$$\hat{v}_\varepsilon(x) = \big(\frac{r-\rho}{\rho} + \frac{2\rho - r}{\rho}|v_\varepsilon(x_j^\varepsilon + \rho\, e^{i\theta})|\big)\exp i\left(\theta + \psi_j(\theta)\frac{2\rho - r}{\rho} + \psi_j(0)\frac{\rho - r}{\rho}\right).$$

Exactly as in the proof of Proposition 5.2 in [78, 79], we prove that

$$\left|\mathcal{E}_\varepsilon^a(\hat{v}_\varepsilon, B(x_j^\varepsilon, 2\rho) \setminus B(x_j^\varepsilon, \rho)) - \pi a(x_j^\varepsilon)\ln 2\right| = o(1). \tag{3.163}$$

Since $|a(x) - a(x_j^\varepsilon)| = \mathcal{O}(\rho)$ on $B(x_j^\varepsilon, 2\rho)$, we may write

$$\mathcal{E}_\varepsilon^a(\hat{v}_\varepsilon, B(x_j^\varepsilon, 2\rho)) = \frac{a(x_j^\varepsilon)}{2}\int_{B(x_j^\varepsilon, 2\rho)} |\nabla \hat{v}_\varepsilon|^2 + \frac{a(x_j^\varepsilon)}{2\varepsilon^2}(1 - |\hat{v}_\varepsilon|^2)^2 + o(1). \tag{3.164}$$

Now we should recall a result in [20]. For $\tilde{\varepsilon} > 0$, consider

$$I(\tilde{\varepsilon}) = \operatorname*{Min}_{u \in \mathcal{C}} \frac{1}{2}\int_{B(0,1)} |\nabla u|^2 + \frac{1}{2\tilde{\varepsilon}^2}(1 - |u|^2)^2$$

where

$$\mathcal{C} = \left\{u \in H^1(B(0,1),\mathbb{C}),\ u(x) = \frac{x}{|x|} \text{ on } \partial B(0,1)\right\}.$$

Then we have

$$\lim_{\tilde{\varepsilon}\to 0}\big(I(\tilde{\varepsilon}) + \pi\ln\tilde{\varepsilon}\big) \equiv \gamma_0. \tag{3.165}$$

Since $\hat{v}_\varepsilon(x) = \frac{x - x_j^\varepsilon}{|x - x_j^\varepsilon|}\, e^{i\psi_j(0)}$ on $\partial B(x_j^\varepsilon, 2\rho)$, we obtain by scaling

$$\begin{aligned}\frac{1}{2}\int_{B(x_j^\varepsilon, 2\rho)} |\nabla \hat{v}_\varepsilon|^2 + \frac{a(x_j^\varepsilon)}{2\varepsilon^2}(1 - |\hat{v}_\varepsilon|^2)^2 &\ge I\left(\frac{\varepsilon}{2\rho\sqrt{a(x_j^\varepsilon)}}\right)\\ &= \pi\ln\frac{\rho}{\varepsilon} + \pi\ln 2 + \frac{\pi}{2}\ln a(x_j^\varepsilon) + \gamma_0 + o(1).\end{aligned}$$

With (3.163) and (3.164), we derive that for $j = 1,\dots,d$,

$$\begin{aligned}\mathcal{E}_\varepsilon^a(v_\varepsilon, B(x_j^\varepsilon, \rho)) &\ge \pi a(x_j^\varepsilon)\ln\frac{\rho}{\varepsilon} + \frac{\pi a(x_j^\varepsilon)}{2}\ln a(x_j^\varepsilon) + a(x_j^\varepsilon)\gamma_0 + o(1)\\ &\ge \pi a(x_j^\varepsilon)\ln\frac{\rho}{\varepsilon} + \frac{\pi a_0}{2}\ln a_0 + a_0\gamma_0 + o(1).\end{aligned}$$

Combining this estimate with (3.162), we get the result. ■

Proposition 3.20. *Setting $\tilde{x}_j^\varepsilon = \sqrt{\Omega}\, x_j^\varepsilon$ for $j = 1, \dots, d$, as $\varepsilon \to 0$ the $\tilde{x}_j^\varepsilon$'s tend to minimize the renormalized energy w given by*

$$w(b_1, \dots, b_d) = -\pi a_0 \sum_{i \neq j} \ln |b_i - b_j| + \frac{\pi a_0}{2} \sum_{j=1}^{d} |b_j|^2.$$

Moreover, we have

$$\mathcal{F}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon) = -\pi a_0 \omega(\varepsilon) d \ln|\ln\varepsilon| + \frac{\pi a_0}{2}(d^2 - d)\ln|\ln\varepsilon| + \operatorname*{Min}_{b \in \mathbb{R}^{2d}} w(b_1, \dots, b_d) + Q_d + o(1) \quad (3.166)$$

where $Q_d = \dfrac{\pi a_0}{2}(d^2 - d)\ln 2 + \pi a_0 d \ln a_0 - \dfrac{\pi a_0 d^2}{2} + a_0 d \gamma_0$.

Proof. Step 1. From Lemma 3.14 and (3.155), we infer that for any $[\sqrt{a_0}/2, \sqrt{a_0}\,)$,

$$\mathcal{F}_\varepsilon^a(v_\varepsilon, \mathcal{D}_\varepsilon) \geq \pi \sum_{j=1}^{n} a(x_j^\varepsilon)|\ln\varepsilon| - \frac{\pi\Omega}{2} \sum_{j=1}^{n} a^2(x_j^\varepsilon) + W_{R,\varepsilon} + \frac{\pi a_0 d}{2} \ln a_0 + a_0 d \gamma_0 + o_R(1).$$

By (3.85), it implies

$$\mathcal{F}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon) \geq \pi \sum_{j=1}^{n} a(x_j^\varepsilon)|\ln\varepsilon| - \frac{\pi\Omega}{2} \sum_{j=1}^{n} a^2(x_j^\varepsilon) + W_{R,\varepsilon} + \frac{\pi a_0 d}{2} \ln a_0 + a_0 d \gamma_0 + o_R(1).$$

Expanding Ω and writing $a(x_j^\varepsilon) = a_0 - |x_j^\varepsilon|^2$, we derive that

$$\mathcal{F}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon) \geq \pi \sum_{j=1}^{n} a(x_j^\varepsilon) \left(-\omega(\varepsilon) \ln|\ln\varepsilon| + \frac{\Omega |x_j^\varepsilon|^2}{2} \right) + W_{R,\varepsilon} + \frac{\pi a_0 d}{2} \ln a_0 + a_0 d \gamma_0 + o_R(1)$$

and by Lemma 3.13, it yields

$$\mathcal{F}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon) \geq -\pi a_0 \omega(\varepsilon) d \ln|\ln\varepsilon| + \frac{\pi a_0}{2} \sum_{j=1}^{n} \Omega |x_j^\varepsilon|^2 + W_{R,\varepsilon} + \frac{\pi a_0 d}{2} \ln a_0 + a_0 d \gamma_0 + o_R(1). \quad (3.167)$$

Step 2. By Lemma 3.13, we may write

$$W_{R,\varepsilon} = -\pi a_0 \sum_{i \neq j} \ln |x_i^\varepsilon - x_j^\varepsilon| - \pi \sum_{j=1}^{d} \Psi_{R,\varepsilon}(x_j^\varepsilon) + o(1). \quad (3.168)$$

By (3.146) and since $D_j = 1$ for any j, the function $\Psi_{R,\varepsilon}$ satisfies the equation

$$\begin{cases} \operatorname{div}\left(\dfrac{1}{a} \nabla \Psi_{R,\varepsilon}\right) = -\displaystyle\sum_{j=1}^{d} a(x_j^\varepsilon)\, \nabla \left(\frac{1}{a}\right) \cdot \nabla \left(\ln|x - x_j^\varepsilon|\right) & \text{in } B_R, \\ \Psi_{R,\varepsilon} = -\displaystyle\sum_{j=1}^{d} a(x_j^\varepsilon) \ln|x - x_j^\varepsilon| & \text{on } \partial B_R. \end{cases} \quad (3.169)$$

Now we remark that in B_R, we have

$$-\sum_{j=1}^{d} a(x_j^\varepsilon)\,\nabla\left(\frac{1}{a}\right)\cdot\nabla\left(\ln|x-x_j^\varepsilon|\right) = \frac{-2a_0 d}{a^2(x)} + 2\sum_{j=1}^{d}\left(\frac{|x_j^\varepsilon|^2}{a^2(x)} - \frac{a(x_j^\varepsilon)x_j^\varepsilon\cdot(x-x_j^\varepsilon)}{a^2(x)|x-x_j^\varepsilon|^2}\right)$$
$$= \frac{-2a_0 d}{a^2(x)} + f_\varepsilon(x).$$

Moreover, for any $p\in[1,2)$ fixed, we have by Lemma 3.13,

$$\|f_\varepsilon\|_{L^p(B_R)} = o_R(1). \tag{3.170}$$

We also have by Lemma 3.13,

$$\left\|da_0\ln R - \sum_{j=1}^{d} a(x_j^\varepsilon)\ln|x-x_j^\varepsilon|\,\right\|_{C^1(\partial B_R)} = o(1). \tag{3.171}$$

Let us now define Ψ_R^* to be the solution of the equation

$$\begin{cases} \operatorname{div}\left(\dfrac{1}{a}\nabla\Psi_R^*\right) = \dfrac{-2da_0}{a^2(x)} & \text{in } B_R,\\ \Psi_R^* = -da_0\ln R & \text{on } \partial B_R. \end{cases} \tag{3.172}$$

It follows by (3.170), (3.171) and classical results that

$$\|\Psi_{R,\varepsilon} - \Psi_R^*\|_{L^\infty(B_R)} = o_R(1). \tag{3.173}$$

We are going to compute explicitly the function Ψ_R^*. Since $a(x)$ is a radial function, it follows by uniqueness that Ψ_R^* is radial. Setting $\Psi_R^*(x) = g(|x|)$, we have to solve the equation

$$\left(\frac{g'}{a(r)}\right)' + \frac{g'}{ra(r)} = \frac{-2da_0}{a^2(r)} \quad \text{in } (0,R) \tag{3.174}$$

together with the conditions

$$g(R) = -da_0\ln R \quad\text{and}\quad g'(0)=0.$$

Multiplying (3.174) by r and integrating the equation, we obtain that

$$\frac{rg'(r)}{a(r)} = -2da_0\int_0^r \frac{sds}{a^2(s)}$$

and it yields

$$g(r) = -2da_0\int_0^r\left(\int_0^t \frac{a(t)s}{ta^2(s)}\,ds\right)dt + c$$

where c denotes the constant determined for g to satisfy the condition $g(R) = -da_0\ln R$. Therefore we found

$$g(r) = \frac{d(R^2-r^2)}{2} - da_0\ln(R)$$

and consequently

$$\Psi^*_R(x) = \frac{d(R^2 - |x|^2)}{2} - da_0 \ln(R).$$

Hence we may write

$$\lim_{\varepsilon\to 0}\left\{-\pi\sum_{j=1}^{d}\Psi^*_R(x_j^\varepsilon)\right\} = -\frac{\pi a_0 d^2}{2} + \frac{\pi a_0 d^2}{2}\ln a_0 + \mathcal{O}\big(|R-\sqrt{a_0}|\big).$$

By (3.168), it follows

$$\lim_{\varepsilon\to 0}\big\{W_{R,\varepsilon}(x_1^\varepsilon,\ldots,x_d^\varepsilon) + \pi a_0\sum_{i\neq j}\ln|x_i^\varepsilon - x_j^\varepsilon|\big\} = -\frac{\pi a_0 d^2}{2} + \frac{\pi a_0 d^2}{2}\ln a_0 + \\ + \mathcal{O}\big(|R-\sqrt{a_0}|\big). \tag{3.175}$$

Step 3. We derive from (3.167) and (3.175) that

$$\liminf_{\varepsilon\to 0}\big\{\mathcal{F}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon) + \pi a_0\omega(\varepsilon)d\ln|\ln\varepsilon| + \pi a_0\sum_{i\neq j}\ln|x_i^\varepsilon - x_j^\varepsilon| - \frac{\pi a_0}{2}\sum_{j=1}^{n}\Omega|x_j^\varepsilon|^2\big\} \geq \\ \geq -\frac{\pi a_0 d^2}{2} + \frac{\pi a_0 d^2}{2}\ln a_0 + \frac{\pi a_0 d}{2}\ln a_0 + a_0 d\gamma_0 + \mathcal{O}\big(|R-\sqrt{a_0}|\big).$$

Setting $\tilde{x}_j^\varepsilon = \sqrt{\Omega}\, x_j^\varepsilon$ for $j = 1,\ldots,d$, we deduce

$$\liminf_{\varepsilon\to 0}\big\{\mathcal{F}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon) + \pi a_0\omega(\varepsilon)d\ln|\ln\varepsilon| - \frac{\pi a_0}{2}(d^2-d)\ln|\ln\varepsilon| - w(\tilde{x}_1^\varepsilon,\ldots,\tilde{x}_d^\varepsilon)\big\} \geq \\ \geq Q_d + \mathcal{O}\big(|R-\sqrt{a_0}|\big)$$

Letting $R\to\sqrt{a_0}$, we finally conclude that

$$\liminf_{\varepsilon\to 0}\big\{\mathcal{F}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon) + \pi a_0\omega(\varepsilon)d\ln|\ln\varepsilon| - \frac{\pi a_0}{2}(d^2-d)\ln|\ln\varepsilon| - w(\tilde{x}_1^\varepsilon,\ldots,\tilde{x}_d^\varepsilon)\big\} \geq Q_d \tag{3.176}$$

and hence

$$\liminf_{\varepsilon\to 0}\big\{\mathcal{F}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon) + \pi a_0\omega(\varepsilon)d\ln|\ln\varepsilon| - \frac{\pi a_0}{2}(d^2-d)\ln|\ln\varepsilon|\big\} \geq \\ \geq \operatorname*{Min}_{b\in\mathbb{R}^{2d}} w(b_1,\ldots,b_d) + Q_d. \tag{3.177}$$

Step 4. End of the proof. Let $\hat{b} = (\hat{b}_1,\ldots,\hat{b}_d)\in\mathbb{R}^{2d}$ be a minimizing configuration for the renormalized energy w, i.e.,

$$w(\hat{b}_1,\ldots,\hat{b}_d) = \operatorname*{Min}_{b\in\mathbb{R}^{2d}} w(b_1,\ldots,b_d) \tag{3.178}$$

(and therefore $b_i \neq b_j$ for $i\neq j$). By Theorem 3.5 in Section 8, for any $\delta' > 0$, there exists $(\tilde{u}_\varepsilon)_{\varepsilon>0}\subset\mathcal{H}$ such that $\|\tilde{u}_\varepsilon\|_{L^2(\mathbb{R}^2)} = 1$ and

$$\limsup_{\varepsilon\to 0}\big\{F_\varepsilon(\tilde{u}_\varepsilon) - E_\varepsilon(\eta_\varepsilon) + \pi a_0\omega(\varepsilon)d\ln|\ln\varepsilon| - \frac{\pi a_0}{2}(d^2-d)\ln|\ln\varepsilon|\big\} \leq w(\hat{b}_1,\ldots,\hat{b}_d) + Q_d + \delta'$$

As in the proof of Proposition 3.18, $F_\varepsilon(u_\varepsilon) \le F_\varepsilon(\tilde{u}_\varepsilon)$ implies

$$\limsup_{\varepsilon\to 0} \left\{ \mathcal{F}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon) + \pi a_0 \omega(\varepsilon) d \ln|\ln\varepsilon| - \frac{\pi a_0}{2}(d^2-d)\ln|\ln\varepsilon| \right\} \le w(\hat{b}_1,\dots,\hat{b}_d) + Q_d + \delta'.$$

Letting $\delta' \to 0$, we infer from (3.178) that

$$\limsup_{\varepsilon\to 0} \left\{ \mathcal{F}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon) + \pi a_0 \omega(\varepsilon) d \ln|\ln\varepsilon| - \frac{\pi a_0}{2}(d^2-d)\ln|\ln\varepsilon| \right\} \le$$
$$\le \operatorname*{Min}_{b\in\mathbb{R}^{2d}} w(b_1,\dots,b_d) + Q_d. \qquad (3.179)$$

Matching (3.177) with (3.179), we conclude

$$\lim_{\varepsilon\to 0} \left\{ \mathcal{F}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon) + \pi a_0 \omega(\varepsilon) d \ln|\ln\varepsilon| - \frac{\pi a_0}{2}(d^2-d)\ln|\ln\varepsilon| \right\} = \operatorname*{Min}_{b\in\mathbb{R}^{2d}} w(b_1,\dots,b_d) + Q_d.$$

Coming back to (3.176), we are led to

$$\operatorname*{Min}_{b\in\mathbb{R}^{2d}} w(b_1,\dots,b_d) + Q_d - \limsup_{\varepsilon\to 0} w(\tilde{x}_1^\varepsilon,\dots,\tilde{x}_d^\varepsilon) \ge Q_d$$

and therefore $\lim_{\varepsilon\to 0} w(\tilde{x}_1^\varepsilon,\dots,\tilde{x}_d^\varepsilon) = \operatorname*{Min}_{b\in\mathbb{R}^{2d}} w(b_1,\dots,b_d)$ which ends the proof. ■

Remark 3.13. In the case $d = 1$, the expansion of the energy takes the simpler form

$$\mathcal{F}_\varepsilon^{\eta_\varepsilon}(v_\varepsilon) = -\pi a_0 \omega(\varepsilon) \ln|\ln\varepsilon| + Q_1 + o(1)$$

with $Q_1 = \pi a_0 \ln a_0 - \dfrac{\pi a_0}{2} + a_0\gamma_0$ and the renormalized energy w reduces to

$$w(b) = \frac{\pi a_0 |b|^2}{2}.$$

In particular, if x^ε denotes the single vortex of v_ε, we have $\sqrt{\Omega}\, x^\varepsilon \to 0$ as ε goes to 0.

3.8 Upper bound of the energy

In this section, we give the construction of the test functions used in the previous section. For any integer $d \ge 1$, we consider an arbitrary configuration of d distinct points $b = (b_1,\dots,b_d)$ in $\mathbb{R}^2$. We assume that $\Omega \le \frac{2}{a_0}\big(|\ln\varepsilon| + \omega_1 \ln|\ln\varepsilon|\big)$ for some constant $\omega_1 \in \mathbb{R}$. Using notation (3.156), we have

Theorem 3.5. *For any $\delta' > 0$, there exists $(\tilde{u}_\varepsilon)_{\varepsilon>0} \subset \mathcal{H}$ such that $\|\tilde{u}_\varepsilon\|_{L^2(\mathbb{R}^2)} = 1$ and*

$$\limsup_{\varepsilon\to 0} \left\{ F_\varepsilon(\tilde{u}_\varepsilon) - E_\varepsilon(\eta_\varepsilon) + \pi a_0 \omega(\varepsilon) d \ln|\ln\varepsilon| - \frac{\pi a_0}{2}(d^2-d)\ln|\ln\varepsilon| \right\} \le w(b_1,\dots,b_d) + Q_d + \delta'$$

where the constant Q_d is defined in Proposition 3.20.

3.8.1 First construction

Using a slight modification of a result of N. André and I. Shafrir (see [12], Lemma 2.6), we obtain the following.

Proposition 3.21. *For any $\delta' > 0$, there exists $(\hat{v}_\varepsilon)_{\varepsilon>0}$ such that $\eta_\varepsilon \hat{v}_\varepsilon \in \mathcal{H}$ and*

$$\limsup_{\varepsilon\to 0} \left\{\mathcal{F}_\varepsilon^{\eta_\varepsilon}(\hat{v}_\varepsilon) + \pi a_0 \omega(\varepsilon) d \ln|\ln\varepsilon| - \frac{\pi a_0}{2}(d^2-d)\ln|\ln\varepsilon|\right\} \le w(b_1,\dots,b_d) + Q_d + \delta'.$$

Proof. Step 1. Let $\sigma > 0$ and κ be two small parameters that we will choose later. We define in $\mathcal{D}$ the function a_σ by

$$a_\sigma(x) = \begin{cases} a(x) & \text{if } |x| \le \sqrt{a_0-\sigma}, \\ -2\sqrt{a_0-\sigma}\,|x| + 2a_0 - \sigma & \text{if } \sqrt{a_0-\sigma} \le |x| \le \sqrt{a_0}. \end{cases}$$

It turns out that $a_\sigma \in C^1(\overline{\mathcal{D}})$, $a_\sigma \ge a$ and $a_\sigma \ge C\sigma^2$ in $\overline{\mathcal{D}}$ for some positive constant C. We infer from the supersolution of (3.23) given by (3.31) that exists $\varepsilon_\sigma > 0$ such that for any $0 < \varepsilon < \varepsilon_\sigma$,

$$\eta_\varepsilon^2(x) \le a_\sigma(x) \quad \text{for } x \in \mathcal{D}. \tag{3.180}$$

Step 2. We consider $\Phi_\sigma : \mathcal{D} \to \mathbb{R}$ the solution of the equation

$$\begin{cases} \mathrm{div}(\frac{1}{a_\sigma}\nabla\Phi_\sigma) = 2\pi d\,\delta_0 & \text{in } \mathcal{D}, \\ \Phi_\sigma = 0 & \text{on } \partial\mathcal{D}. \end{cases} \tag{3.181}$$

By the results in Chap. I of [20], we may find a map $v_0^\sigma \in C^2(\overline{\mathcal{D}} \setminus \{0\}, S^1)$ satisfying

$$v_0^\sigma \wedge \nabla v_0^\sigma = \frac{1}{a_\sigma}\nabla^\perp \Phi_\sigma \quad \text{in } \mathcal{D}\setminus\{0\}. \tag{3.182}$$

For ε small, we set $\Theta_\kappa = \mathcal{D} \setminus B(0, \kappa^{-1}\Omega^{-1/2})$. By (3.181) and (3.182), we have

$$\begin{aligned} \frac{1}{2}\int_{\Theta_\kappa} a_\sigma |\nabla v_0^\sigma|^2 = \frac{1}{2}\int_{\Theta_\kappa} \frac{1}{a_\sigma}|\nabla\Phi_\sigma|^2 &= -\int_{\partial B(0,\kappa^{-1}\Omega^{-1/2})} \frac{1}{a}\frac{\partial \Phi_\sigma}{\partial\nu}\Phi_\sigma \\ &= -\int_{\partial B(0,\kappa^{-1}\Omega^{-1/2})} \frac{1}{a}\Big(\frac{\partial\Psi_\sigma}{\partial\nu} + \frac{a_0 d}{|x|}\Big)\big(\Psi_\sigma + a_0 d\ln|x|\big) \end{aligned} \tag{3.183}$$

where $\Psi_\sigma(x) = \Phi_\sigma(x) - a_0 d \ln|x|$. Notice that Ψ_σ is of class C^2 in $\overline{\mathcal{D}}$ since it satisfies the equation

$$\begin{cases} \mathrm{div}(\frac{1}{a_\sigma}\nabla\Psi_\sigma) = f_\sigma(x) & \text{in } \mathcal{D}, \\ \Psi_\sigma = -\frac{a_0 d}{2}\ln a_0 & \text{on } \partial\mathcal{D} \end{cases} \tag{3.184}$$

with

$$f_\sigma(x) = -a_0 d\,\nabla\Big(\frac{1}{a_\sigma(x)}\Big)\cdot\frac{x}{|x|^2} = \begin{cases} \dfrac{-2a_0 d}{a_\sigma^2(x)} & \text{if } |x| \leq \sqrt{a_0-\sigma}, \\ \dfrac{-2a_0 d}{a_\sigma^2(x)}\dfrac{\sqrt{a_0-\sigma}}{|x|} & \text{otherwise.} \end{cases}$$

Arguing as in Step 3 in the proof of Proposition 3.20, we infer that

$$\Psi_\sigma(x) = -\int_{|x|}^{\sqrt{a_0}} \frac{a_\sigma(t)}{t}\left(\int_0^t f_\sigma(s)sds\right)dt - \frac{a_0 d}{2}\ln a_0.$$

A straightforward computation gives for $|x| \leq \sqrt{a_0-\sigma}$,

$$\nabla\Psi_\sigma(x) = -dx \quad \text{and} \quad \Psi_\sigma(x) = \frac{d(a_0-|x|^2)}{2} - \frac{a_0 d}{2}\ln a_0 + \mathcal{O}(\sigma).$$

By (3.183), we conclude that choosing σ small enough,

$$\lim_{\varepsilon\to 0}\left\{\frac{1}{2}\int_{\Theta_\kappa} a_\sigma|\nabla v_0^\sigma|^2 - \pi a_0 d^2\ln(\kappa\Omega^{1/2})\right\} = -\frac{\pi a_0 d^2}{2} + \frac{\pi a_0 d^2}{2}\ln a_0 + \frac{\delta'}{2} \tag{3.185}$$

In $\mathbb{R}^2 \setminus B(0,\kappa^{-1}\Omega^{-1/2})$, we define

$$\hat{v}_\varepsilon(x) = \begin{cases} v_0^\sigma(x) & \text{if } x \in \Theta_\kappa, \\ v_0^\sigma(\sqrt{a_0}\,\dfrac{x}{|x|}) & \text{if } x \in \mathbb{R}^2\setminus\mathcal{D}. \end{cases}$$

Since $\hat{v}_\varepsilon$ does not depend on ε in $\mathbb{R}^2\setminus\mathcal{D}_\varepsilon$ (for ε small enough) and $|\hat{v}_\varepsilon| = 1$ in $\mathbb{R}^2\setminus\mathcal{D}_\varepsilon$, we derive from (3.32) and *3.1.b)* in Proposition 3.1,

$$\lim_{\varepsilon\to 0}\mathcal{E}_\varepsilon^{\eta_\varepsilon}(\hat{v}_\varepsilon, \mathbb{R}^2\setminus\mathcal{D}_\varepsilon) = 0 \tag{3.186}$$

From (3.180), (3.185), (3.186) and the fact that v_0^σ is S^1-valued in $\mathbb{R}^2\setminus B(0,\kappa^{-1}\Omega^{-1/2})$, we deduce that

$$\limsup_{\varepsilon\to 0}\left\{\mathcal{E}_\varepsilon^{\eta_\varepsilon}(\hat{v}_\varepsilon, \mathbb{R}^2\setminus B(0,\kappa^{-1}\Omega^{-1/2})) - \pi a_0 d^2\ln(\kappa\Omega^{1/2})\right\} \leq$$
$$\leq -\frac{\pi a_0 d^2}{2} + \frac{\pi a_0 d^2}{2}\ln a_0 + \frac{\delta'}{2}. \tag{3.187}$$

Step 3. We are going to extend $\hat{v}_\varepsilon$ to $B(0,\kappa^{-1}\Omega^{-1/2})$. As in [20], we may write in a neighborhood of 0 (using polar coordinates),

$$v_0^\sigma(x) = \exp\big(i(d\theta + \psi_\sigma(x))\big)$$

where ψ_σ is a smooth function in that neighborhood. We choose κ sufficiently small such that $\max|b_j| \leq 1/4\kappa$. We set $b_j^{(\varepsilon)} = \Omega^{-1/2}\, b_j$. We proceed exactly as in in the proof of Lemma 2.6 in [12]. In $A_{\kappa,\varepsilon} = B(0, \kappa^{-1}\Omega^{-1/2}) \setminus B(0, (2\kappa)^{-1}\Omega^{-1/2})$, we write

$$e^{i\psi_\sigma(0)} \prod_{j=1}^{d} \frac{x - b_j^{(\varepsilon)}}{|x - b_j^{(\varepsilon)}|} = \exp\bigl(i(d\theta + \phi_\varepsilon(x))\bigr)$$

for a smooth function ϕ_ε satisfying

$$|\nabla \phi_\varepsilon(x)| = \mathcal{O}(\kappa^2 \Omega^{1/2}) \quad \text{and} \quad |\phi_\varepsilon(x) - \psi_\sigma(0)| = \mathcal{O}(\kappa^2) \quad \text{for } x \in A_{\kappa,\varepsilon}.$$

We define in $A_{\kappa,\varepsilon}$,

$$\hat{v}_\varepsilon(x) = \exp\bigl(i(d\theta + \hat{\psi}_\varepsilon(x))\bigr)$$

with

$$\hat{\psi}_\varepsilon(x) = \bigl(2 - 2\kappa\Omega^{1/2}|x|\bigr)\phi_\varepsilon(x) + \bigl(2\kappa\Omega^{1/2}|x| - 1\bigr)\psi_\sigma(x).$$

As in [12], we get that

$$\limsup_{\varepsilon \to 0} \left\{ \frac{1}{2} \int_{A_{\kappa,\varepsilon}} a_\sigma |\nabla \hat{v}_\varepsilon|^2 - \pi a_0 d^2 \ln 2 \right\} \leq \mathcal{O}(\kappa^2).$$

and hence (using (3.180))

$$\limsup_{\varepsilon \to 0} \bigl\{ \mathcal{E}_\varepsilon^{\eta_\varepsilon}(\hat{v}_\varepsilon, A_{\kappa,\varepsilon}) - \pi a_0 d^2 \ln 2 \bigr\} \leq \mathcal{O}(\kappa^2). \tag{3.188}$$

Next we define $\hat{v}_\varepsilon$ in

$$\Xi_{\kappa,\varepsilon} = B(0, (2\kappa)^{-1}\Omega^{-1/2}) \setminus \cup_{j=1}^{d} B(b_j^{(\varepsilon)}, 2\kappa\Omega^{-1/2})$$

by

$$\hat{v}_\varepsilon(x) = e^{i\psi_\sigma(0)} \prod_{j=1}^{d} \frac{x - b_j^{(\varepsilon)}}{|x - b_j^{(\varepsilon)}|}.$$

Once more as in [12], we have (using (3.180))

$$\begin{aligned} \limsup_{\varepsilon \to 0} \mathcal{E}_\varepsilon^{\eta_\varepsilon}(\hat{v}_\varepsilon, \Xi_{\kappa,\varepsilon}) &\leq \limsup_{\varepsilon \to 0} \frac{1}{2} \int_{\Xi_{\kappa,\varepsilon}} a_\sigma |\nabla \hat{v}_\varepsilon|^2 \\ &\leq \pi a_0 (d^2 + d) \ln \frac{1}{2\kappa} - \pi a_0 \sum_{i \neq j} \ln|b_i - b_j| + \mathcal{O}(\kappa). \end{aligned} \tag{3.189}$$

Finally, in each $B_j^{(\varepsilon)} := B(b_j^{(\varepsilon)}, 2\kappa\Omega^{-1/2})$, we set

$$\hat{v}_\varepsilon(x) = e^{i\psi_\sigma(0)} \tilde{w}_\varepsilon^j \left(\frac{x - b_j^{(\varepsilon)}}{2\kappa\Omega^{-1/2}} \right) \tag{3.190}$$

where $\tilde{w}_\varepsilon^j$ realizes

$$\text{Min}\left\{\frac{1}{2}\int_{B(0,1)}|\nabla v|^2+\frac{1}{2\hat{\varepsilon}^2}(1-|v|^2)^2\,,\ v(y)=\prod_{i=1}^{d}\frac{2\kappa y+b_j-b_i}{|2\kappa y+b_j-b_i|}\ \text{ on }\partial B(0,1)\right\} \quad (3.191)$$

with

$$\hat{\varepsilon}=\frac{\varepsilon}{2\kappa\sqrt{a_0}\,\Omega^{-1/2}}.$$

As in the proof of Lemma 2.3 in [12], we derive

$$\lim_{\varepsilon\to 0}\left\{\frac{1}{2}\int_{B(0,1)}|\nabla\tilde{w}_\varepsilon^j|^2+\frac{1}{2\hat{\varepsilon}^2}(1-|\tilde{w}_\varepsilon^j|^2)^2-\pi|\ln\hat{\varepsilon}|\right\}=\gamma_0+X(\kappa)$$

where γ_0 is defined in (3.165) and $X(\kappa)$ denotes a quantity satisfying $X(\kappa)\to 0$ as $\kappa\to 0$. By scaling, we then obtain

$$\lim_{\varepsilon\to 0}\left\{\frac{1}{2}\int_{B_j^{(\varepsilon)}}|\nabla\hat{v}_\varepsilon|^2+\frac{a_0}{2\varepsilon^2}(1-|\hat{v}_\varepsilon|^2)^2-\pi\ln\frac{2\kappa\Omega^{-1/2}}{\varepsilon}\right\}=\frac{\pi}{2}\ln a_0+\gamma_0+X(\kappa).$$

Notice that in $B_j^{(\varepsilon)}$, we have

$$a_\sigma(x)=a(x)\le a_0-(|\ln\varepsilon|+\omega_1\ln|\ln\varepsilon|)^{-1}\min_{y\in B(b_j,2\kappa)}\frac{a_0|y|^2}{2}$$

and therefore

$$\limsup_{\varepsilon\to 0}\Big\{\frac{1}{2}\int_{B_j^{(\varepsilon)}}a_\sigma|\nabla\hat{v}_\varepsilon|^2+\frac{a_0a_\sigma}{2\varepsilon^2}(1-|\hat{v}_\varepsilon|^2)^2-\pi a_0\ln\frac{2\kappa\Omega^{-1/2}}{\varepsilon}\Big\}\le$$
$$\le\frac{\pi a_0}{2}\ln a_0+a_0\gamma_0-\frac{\pi a_0|b_j|^2}{2}+X(\kappa)$$

and we deduce (using (3.180))

$$\limsup_{\varepsilon\to 0}\Big\{\mathcal{E}_\varepsilon^{\eta_\varepsilon}(\hat{v}_\varepsilon,B_j^{(\varepsilon)})-\pi a_0\ln\frac{2\kappa\Omega^{-1/2}}{\varepsilon}\Big\}\le\frac{\pi a_0}{2}\ln a_0+a_0\gamma_0-\frac{\pi a_0|b_j|^2}{2}+X(\kappa). \quad (3.192)$$

Combining (3.187), (3.188), (3.189) and (3.192), we conclude that choosing κ small enough,

$$\limsup_{\varepsilon\to 0}\left\{\mathcal{E}_\varepsilon^{\eta_\varepsilon}(\hat{v}_\varepsilon)-\pi a_0 d|\ln\varepsilon|-\frac{\pi a_0}{2}(d^2-d)\ln|\ln\varepsilon|\right\}\le \quad (3.193)$$
$$\le-\pi a_0\sum_{i\ne j}\ln|b_i-b_j|-\frac{\pi a_0}{2}\sum_{j=1}^{d}|b_j|^2+Q_d+\delta'.$$

Step 4. Now we are going to estimate $\mathcal{R}_\varepsilon^{\eta_\varepsilon}(\hat{v}_\varepsilon)$. We have

$$|\mathcal{R}_\varepsilon^{\eta_\varepsilon}(\hat{v}_\varepsilon,\mathbb{R}^2\setminus\mathcal{D}_\varepsilon)|\le C\Omega\left(\int_{\mathbb{R}^2\setminus\mathcal{D}_\varepsilon}|x|^2\eta_\varepsilon^2\right)^{1/2}\left(\mathcal{E}_\varepsilon^{\eta_\varepsilon}(\hat{v}_\varepsilon,\mathbb{R}^2\setminus\mathcal{D}_\varepsilon)\right)^{1/2} \quad (3.194)$$

and by *3.1.b)* in Proposition 3.1, (3.32) and (3.186), we derive

$$\lim_{\varepsilon\to 0}\left|\mathcal{R}_\varepsilon^{\eta_\varepsilon}(\hat{v}_\varepsilon)-\mathcal{R}_\varepsilon^{\eta_\varepsilon}(\hat{v}_\varepsilon,\mathcal{D}_\varepsilon)\right|=0. \tag{3.195}$$

By the results in Chap. IX in [20], for $\hat{\varepsilon}$ sufficiently small, for $j=1,\ldots,d$ there exists exactly one disc $\hat{D}_\varepsilon^j\subset B(0,1)$ with $\mathrm{diam}(\hat{D}_\varepsilon^j)\le C\hat{\varepsilon}$ such that $|\tilde{w}_\varepsilon^j|\ge 1/2$ in $B(0,1)\backslash\hat{D}_\varepsilon^j$. By scaling, we infer that exist exactly d discs $D_\varepsilon^1,\ldots,D_\varepsilon^d$ with $D_\varepsilon^j\subset B_j^{(\varepsilon)}$ and $\mathrm{diam}(D_\varepsilon^j)\le C\varepsilon$ such that

$$|\hat{v}_\varepsilon|\ge\frac{1}{2}\quad\text{in }\mathcal{D}_\varepsilon\setminus\cup_{j=1}^d D_\varepsilon^j.$$

By (3.192) we have

$$\left|\mathcal{R}_\varepsilon^{\eta_\varepsilon}(\hat{v}_\varepsilon,\cup_{j=1}^d D_\varepsilon^j)\right|\le C\Omega\,\varepsilon\sum_{j=1}^d\left(\mathcal{E}_\varepsilon^{\eta_\varepsilon}(\hat{v}_\varepsilon,B_j^{(\varepsilon)})\right)^{1/2}\underset{\varepsilon\to 0}{\to}0,$$

and by (3.195), it leads to

$$\lim_{\varepsilon\to 0}\left|\mathcal{R}_\varepsilon^{\eta_\varepsilon}(\hat{v}_\varepsilon)-\mathcal{R}_\varepsilon^{\eta_\varepsilon}(\hat{v}_\varepsilon,\mathcal{D}_\varepsilon\setminus\cup_{j=1}^d D_\varepsilon^j)\right|=0.$$

Arguing as in the proof of Proposition 3.6, we infer that

$$\lim_{\varepsilon\to 0}\left|\mathcal{R}_\varepsilon^{\eta_\varepsilon}(\hat{v}_\varepsilon,\mathcal{D}_\varepsilon\setminus\cup_{j=1}^d D_\varepsilon^j)-\mathcal{R}_\varepsilon^{a}(\hat{v}_\varepsilon,\mathcal{D}_\varepsilon\setminus\cup_{j=1}^d D_\varepsilon^j)\right|=0$$

and hence

$$\lim_{\varepsilon\to 0}\left|\mathcal{R}_\varepsilon^{\eta_\varepsilon}(\hat{v}_\varepsilon)-\mathcal{R}_\varepsilon^{a}(\hat{v}_\varepsilon,\mathcal{D}_\varepsilon\setminus\cup_{j=1}^d D_\varepsilon^j)\right|=0. \tag{3.196}$$

To compute $\mathcal{R}_\varepsilon^a(\hat{v}_\varepsilon,\mathcal{D}\setminus\cup_{j=1}^d D_\varepsilon^j)$, we may proceed as in the proof of Proposition 3.8(here we use that $\mathcal{E}_\varepsilon^{\eta_\varepsilon}(\hat{v}_\varepsilon)\le C|\ln\varepsilon|$ by (3.193)). It yields

$$\lim_{\varepsilon\to 0}\left(\mathcal{R}_\varepsilon^a(\hat{v}_\varepsilon,\mathcal{D}_\varepsilon\setminus\cup_{j=1}^d D_\varepsilon^j)-\frac{\pi\Omega}{2}\sum_{j=1}^d a^2(b_j^{(\varepsilon)})\right)=0$$

since $\deg(\hat{v}_\varepsilon/|\hat{v}_\varepsilon|,\partial D_\varepsilon^j)=+1$ for $j=1,\ldots,d$. Expanding $a^2(b_j^{(\varepsilon)})$ and Ω, we deduce from (3.196) that

$$\lim_{\varepsilon\to 0}\left(\mathcal{R}_\varepsilon^{\eta_\varepsilon}(\hat{v}_\varepsilon)-\pi a_0 d|\ln\varepsilon|-\pi a_0\omega(\varepsilon)d\ln|\ln\varepsilon|\right)=-\pi a_0\sum_{j=1}^d|b_j|^2. \tag{3.197}$$

Combining (3.193) and (3.197), we obtain the announced result. ■

3.8.2 Proof of Theorem 3.5

We consider the map $\hat{v}_\varepsilon$ given in Proposition 3.21 and we set

$$\tilde{v}_\varepsilon = m_\varepsilon^{-1}\hat{v}_\varepsilon \quad \text{and} \quad \tilde{u}_\varepsilon = \tilde{\eta}_\varepsilon \tilde{v}_\varepsilon \quad \text{with} \quad m_\varepsilon = \|\tilde{\eta}_\varepsilon \hat{v}_\varepsilon\|_{L^2(\mathbb{R}^2)},$$

where $\tilde{\eta}_\varepsilon$ is given by Theorem 3.3. Using the characterization of $\tilde{\eta}_\varepsilon$ given in Theorem 3.3 (see (3.44)), we check easily that $\tilde{u}_\varepsilon \in \mathcal{H}$ (and obviously $\|\tilde{u}_\varepsilon\|_{L^2(\mathbb{R}^2)} = 1$). We are going to prove that the map $\tilde{u}_\varepsilon$ satisfies the required property. We proceed in several steps.

Step 1. We recall that $\tilde{\eta}_\varepsilon$ satisfies equation (3.50) and then, exactly as in Lemma 3.4, the functional $\tilde{E}_\varepsilon$ defined in (3.54) splits into two independent pieces. More precisely, for any $u \in \mathcal{H}$ we have

$$\mathcal{E}_\varepsilon^{\tilde{\eta}_\varepsilon}\left(\frac{u}{\tilde{\eta}_\varepsilon}\right) < +\infty$$

and

$$\tilde{E}_\varepsilon(u) = \tilde{E}_\varepsilon(\tilde{\eta}_\varepsilon) + \mathcal{E}_\varepsilon^{\tilde{\eta}_\varepsilon}\left(\frac{u}{\tilde{\eta}_\varepsilon}\right).$$

From (3.63), we infer that

$$\tilde{E}_\varepsilon(u) = E_\varepsilon(\eta_\varepsilon) + \mathcal{E}_\varepsilon^{\tilde{\eta}_\varepsilon}\left(\frac{u}{\tilde{\eta}_\varepsilon}\right) + o(\varepsilon) \tag{3.198}$$

where the "error term" $o(\varepsilon)$ is independent of u. Moreover, if $\|u\|_{L^2(\mathbb{R}^2)} = 1$ we may also rewrite $\tilde{E}_\varepsilon(u)$ as

$$\tilde{E}_\varepsilon(u) = E_\varepsilon(u) - \frac{k_\varepsilon}{2} + \frac{1}{4\varepsilon^2}\int_{\mathbb{R}^2} (a_\varepsilon^+(x))^2 - (a^+(x))^2.$$

By (3.58) and (3.43), we have

$$-\frac{k_\varepsilon}{2} + \frac{1}{4\varepsilon^2}\int_{\mathbb{R}^2} (a_\varepsilon^+(x))^2 - (a^+(x))^2 = o(\varepsilon)$$

and using (3.198), we conclude that

$$E_\varepsilon(\tilde{u}_\varepsilon) = E_\varepsilon(\eta_\varepsilon) + \mathcal{E}_\varepsilon^{\tilde{\eta}_\varepsilon}(\tilde{v}_\varepsilon) + o(\varepsilon). \tag{3.199}$$

Step 2. We claim that

$$\mathcal{E}_\varepsilon^{\tilde{\eta}_\varepsilon}(\tilde{v}_\varepsilon) = \mathcal{E}_\varepsilon^{\tilde{\eta}_\varepsilon}(\hat{v}_\varepsilon) + o(1). \tag{3.200}$$

First we estimate m_ε. Since $|\hat{v}_\varepsilon| = 1$ in $\mathbb{R}^2 \setminus \cup_{j=1}^d B_j^{(\varepsilon)}$ and $\|\tilde{\eta}_\varepsilon\|_{L^2(\mathbb{R}^2)} = 1$, we have

$$m_\varepsilon^2 = \int_{\mathbb{R}^2} \tilde{\eta}_\varepsilon^2 + \int_{\cup_{j=1}^d B_j^{(\varepsilon)}} \tilde{\eta}_\varepsilon^2(|\hat{v}_\varepsilon|^2 - 1) = 1 + \int_{\cup_{j=1}^d B_j^{(\varepsilon)}} \tilde{\eta}_\varepsilon^2(|\hat{v}_\varepsilon|^2 - 1).$$

Using Cauchy-Schwarz inequality, we derive from (3.190), (3.191) and Theorem III.2 in [20] that

$$\Big|\int_{\cup_{j=1}^d B_j^{(\varepsilon)}} \tilde{\eta}_\varepsilon^2(|\hat{v}_\varepsilon|^2 - 1)\Big| \leq C\varepsilon|\ln\varepsilon|^{-1/2} \tag{3.201}$$

and thus

$$m_\varepsilon^2 = 1 + \mathcal{O}(\varepsilon|\ln\varepsilon|^{-1/2}). \tag{3.202}$$

By Theorem 3.3 and (3.193), we derive that

$$\mathcal{E}_\varepsilon^{\tilde{\eta}_\varepsilon}(\hat{v}_\varepsilon) \leq C|\ln\varepsilon|. \tag{3.203}$$

and thus

$$\int_{\mathbb{R}^2} \tilde{\eta}_\varepsilon^2|\nabla\tilde{v}_\varepsilon|^2 = m_\varepsilon^{-2}\int_{\mathbb{R}^2} \tilde{\eta}_\varepsilon^2|\nabla\hat{v}_\varepsilon|^2 = \int_{\mathbb{R}^2} \tilde{\eta}_\varepsilon^2|\nabla\hat{v}_\varepsilon|^2 + \mathcal{O}(\varepsilon|\ln\varepsilon|^{1/2}). \tag{3.204}$$

Since $|\hat{v}_\varepsilon| = 1$ in $\mathbb{R}^2 \setminus \cup_{j=1}^d B_j^{(\varepsilon)}$, we may write

$$\begin{aligned}\frac{1}{\varepsilon^2}\int_{\mathbb{R}^2} \tilde{\eta}_\varepsilon^4(1-|\tilde{v}_\varepsilon|^2)^2 = \frac{1}{\varepsilon^2}\int_{\mathbb{R}^2} \tilde{\eta}_\varepsilon^4(1-|\hat{v}_\varepsilon|^2)^2 + \frac{2(1-m_\varepsilon^{-2})}{\varepsilon^2}\int_{\cup_{j=1}^d B_j^{(\varepsilon)}} \tilde{\eta}_\varepsilon^4|\hat{v}_\varepsilon|^2(1-|\hat{v}_\varepsilon|^2)\\ + \frac{(1-m_\varepsilon^{-2})^2}{\varepsilon^2}\int_{\mathbb{R}^2} \tilde{\eta}_\varepsilon^4|\hat{v}_\varepsilon|^4.\end{aligned} \tag{3.205}$$

We infer from (3.201), (3.202) and (3.203) that

$$\frac{(1-m_\varepsilon^{-2})^2}{\varepsilon^2}\int_{\mathbb{R}^2} \tilde{\eta}_\varepsilon^4|\hat{v}_\varepsilon|^4 \leq C|\ln\varepsilon|^{-1}, \tag{3.206}$$

and

$$0 \leq \frac{|1-m_\varepsilon^{-2}|}{\varepsilon^2}\int_{\cup_{j=1}^d B_j^{(\varepsilon)}} \tilde{\eta}_\varepsilon^4|\hat{v}_\varepsilon|^2(1-|\hat{v}_\varepsilon|^2) \leq C|\ln\varepsilon|^{-1}. \tag{3.207}$$

Combining (3.204), (3.205), (3.206) and (3.207), we deduce that (3.200) holds.

Step 3. By (3.199) and (3.200), we have

$$E_\varepsilon(\tilde{u}_\varepsilon) = E_\varepsilon(\eta_\varepsilon) + \mathcal{E}_\varepsilon^{\tilde{\eta}_\varepsilon}(\hat{v}_\varepsilon) + o(1). \tag{3.208}$$

Moreover, (3.197) and Theorem 3.3 imply $|\mathcal{R}_\varepsilon^{\tilde{\eta}_\varepsilon}(\hat{v}_\varepsilon)| \leq C|\ln\varepsilon|$. Hence

$$\mathcal{R}_\varepsilon^{\tilde{\eta}_\varepsilon}(\tilde{v}_\varepsilon) = m_\varepsilon^{-2}\mathcal{R}_\varepsilon^{\tilde{\eta}_\varepsilon}(\hat{v}_\varepsilon) = \mathcal{R}_\varepsilon^{\tilde{\eta}_\varepsilon}(\hat{v}_\varepsilon) + \mathcal{O}(\varepsilon|\ln\varepsilon|^{1/2})$$

and since $R_\varepsilon(\tilde{u}_\varepsilon) = \mathcal{R}_\varepsilon^{\tilde{\eta}_\varepsilon}(\tilde{v}_\varepsilon)$, we conclude

$$F_\varepsilon(\tilde{u}_\varepsilon) = E_\varepsilon(\eta_\varepsilon) + \mathcal{F}_\varepsilon^{\tilde{\eta}_\varepsilon}(\hat{v}_\varepsilon) + o(1).$$

In view of Proposition 3.21, to prove Theorem 3.5, it suffices to show

$$\lim_{\varepsilon\to 0}\left|\mathcal{F}_\varepsilon^{\tilde{\eta}_\varepsilon}(\hat{v}_\varepsilon) - \mathcal{F}_\varepsilon^{\eta_\varepsilon}(\hat{v}_\varepsilon)\right| = 0. \tag{3.209}$$

By (3.44), we may obtain exactly as for (3.186) and (3.195),

$$\lim_{\varepsilon\to 0}\mathcal{E}_\varepsilon^{\tilde{\eta}_\varepsilon}(\hat{v}_\varepsilon, \mathbb{R}^2\setminus\mathcal{D}_\varepsilon) = 0 \quad \text{and} \quad \lim_{\varepsilon\to 0}\mathcal{R}_\varepsilon^{\tilde{\eta}_\varepsilon}(\hat{v}_\varepsilon, \mathbb{R}^2\setminus\mathcal{D}_\varepsilon) = 0.$$

As in the proof of Proposition 3.6, we derive using (3.193),

$$\lim_{\varepsilon\to 0}\left|\mathcal{E}_\varepsilon^{\eta_\varepsilon}(\hat{v}_\varepsilon,\mathcal{D}_\varepsilon)-\mathcal{E}_\varepsilon^{a}(\hat{v}_\varepsilon,\mathcal{D}_\varepsilon)\right|=0 \quad\text{and}\quad \lim_{\varepsilon\to 0}\left|\mathcal{R}_\varepsilon^{\eta_\varepsilon}(\hat{v}_\varepsilon,\mathcal{D}_\varepsilon)-\mathcal{R}_\varepsilon^{a}(\hat{v}_\varepsilon,\mathcal{D}_\varepsilon)\right|=0.$$

To get (3.209) it suffices to prove that

$$\lim_{\varepsilon\to 0}\left|\mathcal{E}_\varepsilon^{\tilde{\eta}_\varepsilon}(\hat{v}_\varepsilon,\mathcal{D}_\varepsilon)-\mathcal{E}_\varepsilon^{a}(\hat{v}_\varepsilon,\mathcal{D}_\varepsilon)\right|=0 \quad\text{and}\quad \lim_{\varepsilon\to 0}\left|\mathcal{R}_\varepsilon^{\tilde{\eta}_\varepsilon}(\hat{v}_\varepsilon,\mathcal{D}_\varepsilon)-\mathcal{R}_\varepsilon^{a}(\hat{v}_\varepsilon,\mathcal{D}_\varepsilon)\right|=0. \tag{3.210}$$

From *3.1.c)* in Proposition 3.1 and Theorem 3.3, we infer that (see Remark 3.4)

$$\left\|\frac{a-\tilde{\eta}_\varepsilon^2}{\tilde{\eta}_\varepsilon^2}\right\|_{L^\infty(\mathcal{D}_\varepsilon)}\le C\varepsilon^{1/3} \quad\text{and}\quad \left\|\frac{a^2-\tilde{\eta}_\varepsilon^4}{\tilde{\eta}_\varepsilon^4}\right\|_{L^\infty(\mathcal{D}_\varepsilon)}\le C\varepsilon^{1/3}$$

and we may proceed as in the proof of Proposition 3.6 (using (3.193)) to obtain (3.210) which ends the proof. ■

Chapitre 4

On a Ginzburg-Landau energy with ε-depending weight

4.1 Introduction and main results

Let G be a smooth bounded and simply connected domain in $\mathbb{R}^2$ and let $g : \partial G \to S^1$ be a fixed smooth map of topological degree $d > 0$. For $\varepsilon > 0$, we consider the Ginzburg-Landau type functional

$$E_\varepsilon(u) = \frac{1}{2}\int_G |\nabla u(x)|^2 dx + \frac{1}{4\varepsilon^2}\int_G a_\varepsilon(x)(1-|u(x)|^2)^2 dx,$$

defined for $u \in H^1(G, \mathbb{C})$ such that $u = g$ on ∂G. The weight function $a_\varepsilon(x)$ is given by

$$a_\varepsilon(x) = \varepsilon^{-\alpha} \text{ if } x \in G^+ \text{ and } a_\varepsilon(x) = 1 \text{ if } x \in G^-,$$

where G^+ and G^- are two open subsets of G such that $\overline{G^+} \cup \overline{G^-} = \overline{G}$ and $\Sigma = \overline{G^+} \cap \overline{G^-}$ defines a smooth curve as in Figure 4.1, and α is a positive constant.

In this chapter, we study the asymptotic behavior as ε goes to 0 of minimizers u_ε of the energy E_ε. Each minimizer u_ε satisfies the associated Euler equation

$$\begin{cases} -\Delta u_\varepsilon = \dfrac{1}{\varepsilon^2}\, a_\varepsilon(x)(1-|u_\varepsilon|^2)u_\varepsilon & \text{in } G, \\ u_\varepsilon = g & \text{on } \partial G. \end{cases} \tag{4.1}$$

In the case $a_\varepsilon \equiv 1$, F. Bethuel, H. Brezis and F. Hélein have proved that for each sequence $\varepsilon_n \to 0$, there exist a subsequence $\varepsilon_{n_k} \to 0$ and d distinct points $a_1, \dots, a_d$ in G such that $u_{\varepsilon_{n_k}}$ converges in certain topologies to u_0 the canonical harmonic map with values into S^1 associated to $\{a_1, \dots, a_d\}$ with degrees $+1$ and to the boundary data g (see [20]). The map u_0 is given by

$$u_0(z) = \frac{z-a_1}{|z-a_1|} \cdots \frac{z-a_d}{|z-a_d|}\, e^{i\varphi(z)} \quad \text{in } G \setminus \{a_1, \dots, a_d\}$$

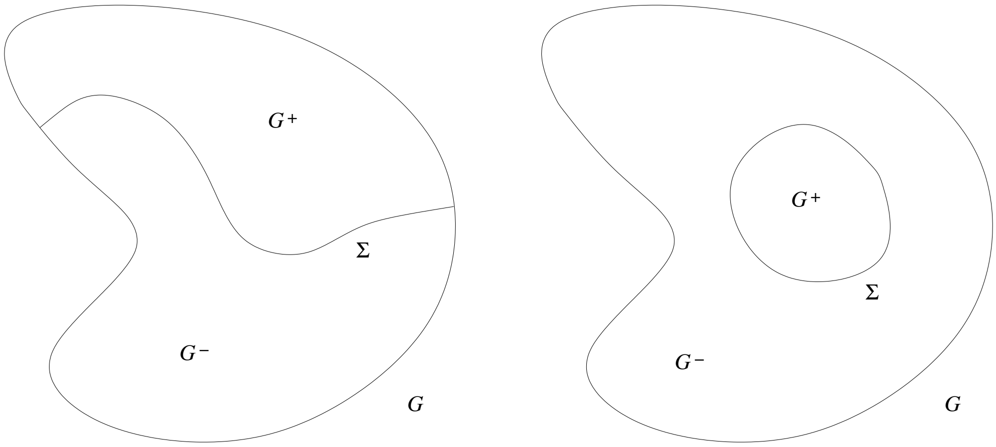

FIG. 4.1 – Admissible geometries for G^+ and G^-

with

$$\begin{cases} \Delta\varphi = 0 & \text{in } G, \\ u_0 = g & \text{on } \partial G. \end{cases}$$

They also show that the singularities $a_1, \ldots, a_d$ can be localized in G as a minimizing configuration of the *renormalized energy* $W(\cdot)$ associated to the boundary data g and the configurations of d points in G of degree $+1$ (cf. Section 4.4 for the definition of W and we refer to Chapter II in [20] for more details).

In our situation, we prove a similar result of convergence and we show that all the singularities are confined in $G^- \cup \Sigma$, the less penalized part of the domain. This result can be stated as follows.

Theorem 4.1. *For each sequence $\varepsilon_n \to 0$, there exist a subsequence also denoted by (ε_n) and d distinct points $a_1, \ldots, a_d$ in $G^- \cup \Sigma$ such that u_{ε_n} converges to u_0 as $\varepsilon_n \to 0$ in the spaces $H^1_{\text{loc}}\left(G \setminus \cup_{i=1}^d \{a_i\}\right)$, $C^0_{\text{loc}}\left(\overline{G} \setminus \cup_{i=1}^d \{a_i\}\right)$, $C^{1,\beta}_{\text{loc}}\left(\overline{G} \setminus (\cup_{i=1}^d \{a_i\} \cup \overline{\Sigma})\right)$ for any $\beta < 1$, $C^k_{\text{loc}}\left(G \setminus (\cup_{i=1}^d \{a_i\} \cup \Sigma)\right)$ for any $k \in \mathbb{N}$ where u_0 is the canonical harmonic map with values into S^1 associated to $\{a_1, \ldots, a_d\}$ with degrees $+1$ and to the boundary data g.*

We also prove that the location of the singularities $a_1, \ldots, a_d$ is governed by the *renormalized energy* $W(\cdot)$ restricted to $(G^- \cup \Sigma)^d$. More precisely, we have :

Theorem 4.2. *The limiting configuration $(a_1, \ldots, a_d)$ minimizes the renormalized energy $W(\cdot)$ over all configurations in $(G^- \cup \Sigma)^d$.*

Remark 4.1. Since $W(a_1, \ldots, a_d) \to +\infty$ as one point a_i tends to ∂G, any minimal configuration $(a_1, \ldots, a_d)$ for $W/_{(G^- \cup \Sigma)^d}$ satisfies $a_i \notin \partial G$ but some of the a_i's might be on Σ. Indeed, if $g(z) = z$, G is the unit disc of $\mathbb{R}^2$ and $G^- = G \cap \{(x_1, x_2) \in \mathbb{R}^2,\ x_2 > 0\}$, then $a = 0$ minimizes W on $G^- \cup \Sigma$.

The proofs of Theorem 4.1 and Theorem 4.2 are given in Section 4.3 and Section 4.4 respectively. In Section 4.2, we present the analogue result of Theorem 2 in [19]. This result is the main tool in the proof of the convergence near a point of Σ.

4.2 A preliminary study in degree zero

Let B be the unit disc of $\mathbb{R}^2$ and let f be a smooth real function defined on a neighborhood of $[-1,1]$ such that $f(0)=0$ and $\|f'\|_\infty \ll 1$. We denote by Γ the smooth curve $\{(x_1, f(x_1)),\, x_1 \in]-1,1[\} \cap B$ and we define for any set $E \subset \overline{B}$,

$$E^+ = \{(x_1,x_2) \in E,\, x_2 > f(x_1)\} \quad \text{and} \quad E^- = \{(x_1,x_2) \in E,\, x_2 < f(x_1)\}\,.$$

For $\varepsilon > 0$, we consider the following minimization problem :

$$\underset{u \in H^1_{g_\varepsilon}(B,\mathbb{C})}{\mathrm{Min}} \; \frac{1}{2}\int_B |\nabla u(x)|^2 dx + \frac{1}{4\varepsilon^2}\int_B \tilde{a}_\varepsilon(x)(1-|u(x)|^2)^2 dx, \tag{4.2}$$

where $g_\varepsilon : \partial B \to \mathbb{C}$ is a smooth given map, $H^1_{g_\varepsilon}(B,\mathbb{C})$ denotes the set of all maps $u \in H^1(B,\mathbb{C})$ such that $u = g_\varepsilon$ on ∂B and the function $\tilde{a}_\varepsilon(x)$ is given by

$$\tilde{a}_\varepsilon(x) = \varepsilon^{-\alpha} \quad \text{if } x \in B^+ \text{ and } \quad \tilde{a}_\varepsilon(x) = 1 \quad \text{if } x \in B^-.$$

For any $\varepsilon > 0$, this problem admits at least one solution u_ε which satisfies

$$\begin{cases} -\Delta u_\varepsilon = \dfrac{1}{\varepsilon^2}\,\tilde{a}_\varepsilon(x)(1-|u_\varepsilon|^2)u_\varepsilon & \text{in } B, \\ u_\varepsilon = g_\varepsilon & \text{on } \partial B. \end{cases} \tag{4.3}$$

Our goal in this section is to study the asymptotic behavior of u_ε as ε goes to 0 in the following context : we suppose that exists a function $g : \partial B \to \mathbb{C}$ such that

$$g_\varepsilon \to g \quad \text{uniformly on } \partial B \text{ as } \varepsilon \to 0, \tag{4.4}$$

$$\|g_\varepsilon\|_{L^\infty(\partial B)} \le 1, \tag{4.5}$$

$$\|g_\varepsilon\|_{H^1(\partial B)} \le C, \tag{4.6}$$

$$\int_{\partial B} \tilde{a}_\varepsilon(x)(1-|g_\varepsilon(x)|^2)^2 dx \le C\varepsilon^2. \tag{4.7}$$

We notice that (4.4) and (4.7) imply that $|g| \equiv 1$ on ∂B. Therefore the topological degree of g is well defined. We assume that

$$\deg(g, \partial B) = 0. \tag{4.8}$$

From (4.8) we infer that exists a function $\varphi_0 \in H^1(\partial B, \mathbb{R})$ such that $g = e^{i\varphi_0}$ on ∂B. Moreover (4.8) implies that

$$H^1_g(B,S^1) = \big\{u \in H^1(B,\mathbb{C}),\, |u| = 1 \text{ a.e. in } B,\, u = g \text{ on } \partial B\big\} \neq \emptyset$$

and the following minimization problem makes sense

$$\operatorname*{Min}_{u\in H^1_g(B,S^1)} \int_B |\nabla u|^2. \tag{4.9}$$

By the results in [19], we know that (4.9) admits a unique solution $u_\star$ which satisfies

$$\begin{cases} -\Delta u_\star = |\nabla u_\star|^2 u_\star & \text{in } B, \\ u_\star = g & \text{on } \partial B. \end{cases}$$

In addition, $u_\star$ is characterized by $u_\star = e^{i\varphi_\star}$ where $\varphi_\star$ is the unique solution of the equation

$$\begin{cases} \Delta\varphi_\star = 0 & \text{in } B, \\ \varphi_\star = \varphi_0 & \text{on } \partial B. \end{cases} \tag{4.10}$$

Theorem 4.3. *Under the hypothesis (4.4)-(4.8), we have as* $\varepsilon \to 0$ *:*

$$u_\varepsilon \to u_\star \quad \text{in } H^1(B), \tag{4.11}$$

$$u_\varepsilon \to u_\star \quad \text{uniformly on } \overline{B}, \tag{4.12}$$

$$u_\varepsilon \to u_\star \quad \text{in } C^k_{\text{loc}}(B \setminus \Gamma)\ \forall k, \tag{4.13}$$

$$\frac{\tilde{a}_\varepsilon(x)\,(1-|u_\varepsilon|^2)}{\varepsilon^2} \to |\nabla u_\star|^2 \quad \text{in } C^k_{\text{loc}}(B \setminus \Gamma)\ \forall k. \tag{4.14}$$

We split the proof into several steps.

Step 1 : Proof of (4.11). As in [19], we use a comparison method. We consider $v_\varepsilon : B \to \mathbb{C}$ defined by

$$v_\varepsilon = \eta_\varepsilon e^{i\psi_\varepsilon},$$

where η_ε is the solution of

$$\begin{cases} -\varepsilon^2\Delta\eta_\varepsilon + \tilde{a}_\varepsilon(x)(\eta_\varepsilon - 1) = 0 & \text{in } B, \\ \eta_\varepsilon = |g_\varepsilon| & \text{on } \partial B, \end{cases} \tag{4.15}$$

and ψ_ε the solution of

$$\begin{cases} \Delta\psi_\varepsilon = 0 & \text{in } B, \\ \psi_\varepsilon = \varphi_\varepsilon & \text{on } \partial B, \end{cases} \tag{4.16}$$

where $\varphi_\varepsilon : \partial B \to \mathbb{R}$ is given by

$$e^{i\varphi_\varepsilon} = \frac{g_\varepsilon}{|g_\varepsilon|},$$

(which is possible since $\deg(g_\varepsilon, \partial B) = 0$ for ε sufficiently small by (4.4) and (4.8)). By (4.4), we may choose φ_ε such that $\varphi_\varepsilon \to \varphi_0$ uniformly on ∂B. We claim that

$$\int_B |\nabla\eta_\varepsilon|^2 + \frac{1}{\varepsilon^2}\int_B \tilde{a}_\varepsilon(x)(\eta_\varepsilon - 1)^2 \le C\varepsilon. \tag{4.17}$$

Proof of (4.17) : The function η_ε minimizes on $H^1_{|g_\varepsilon|}(B,\mathbb{R})$ the functional

$$\eta \longmapsto \int_B |\nabla \eta|^2 + \frac{1}{\varepsilon^2}\int_B \tilde{a}_\varepsilon(x)(\eta-1)^2.$$

We use a comparison function. We construct $\tilde{\eta}_\varepsilon$ an extension in B of $|g_\varepsilon|$. We proceed as follows. We define the map Φ on a neighborhood of $\overline{B}$ by

$$\Phi(x_1,x_2) = (x_1, x_2 - f(x_1)).$$

By the assumptions on f, Φ defines a smooth change of variables in a neighborhood of $\overline{B}$. Since Φ is a small perturbation of the identity, $\Phi(B)$ can be parametrized using polar coordinates :

$$\Phi(B) = \left\{se^{i\theta},\ s\in[0,R(\theta)[\right\}.$$

We remark that for any set $E\subset\overline{B}$,

$$\Phi(E^+) = \Phi(E)\cap\left\{(y_1,y_2)\in\mathbb{R}^2,\ y_2>0\right\}$$

and

$$\Phi(E^-) = \Phi(E)\cap\left\{(y_1,y_2)\in\mathbb{R}^2,\ y_2<0\right\}.$$

We denote by $(r(x),\theta(x))$ the polar coordinates of $\Phi(x)$ for $x\in\overline{B}$ and we define $\tilde{\eta}_\varepsilon$ by

$$\tilde{\eta}_\varepsilon(x) = \left(\left|g_\varepsilon(\Phi^{-1}\left(R(\theta(x))e^{i\theta(x)}\right)\right|-1\right)\gamma\left(|x|\right)+1,$$

where γ is a smooth real function with small support near 1 with $\gamma(1)=1$. By (4.6) we infer that

$$\int_B |\nabla\tilde{\eta}_\varepsilon|^2 \le C$$

and using the change of variables $y=\Phi(x)$, we obtain

$$\begin{aligned}\int_{B^+}(\tilde{\eta}_\varepsilon-1)^2 &\le C\int_{\Phi(B^+)}(\tilde{\eta}_\varepsilon(\Phi^{-1}(y))-1)^2dy\\ &\le C\int_0^\pi\int_0^{R(\theta)}(|g_\varepsilon(\Phi^{-1}(R(\theta)e^{i\theta})|-1)^2 s ds d\theta\\ &\le C\int_0^\pi(|g_\varepsilon(\Phi^{-1}(R(\theta)e^{i\theta})|-1)^2 R(\theta) ds d\theta\\ &\le C\int_{(\partial B)^+}(|g_\varepsilon|-1)^2\end{aligned}$$

and in the same way,

$$\int_{B^-}(\tilde{\eta}_\varepsilon-1)^2 \le C\int_{(\partial B)^-}(|g_\varepsilon|-1)^2.$$

We derive from (4.7) that

$$\int_B |\nabla\tilde{\eta}_\varepsilon|^2 + \frac{1}{\varepsilon^2}\int_B \tilde{a}_\varepsilon(x)(\tilde{\eta}_\varepsilon-1)^2 \le C.$$

Hence we conclude

$$\int_B |\nabla \eta_\varepsilon|^2 + \frac{1}{\varepsilon^2}\int_B \tilde{a}_\varepsilon(x)(\eta_\varepsilon - 1)^2 \leq C. \tag{4.18}$$

Now, we multiply (4.15) by $V \cdot \nabla(\eta_\varepsilon - 1)$ with $V(x) = \dfrac{\partial x}{\partial r}\gamma(|x|)$ (recall that (r,θ) is defined as the polar coordinates of $\Phi(x)$) and we integrate by parts. Estimates (4.6) and (4.18) yield

$$\int_B \Delta\eta_\varepsilon(V.\nabla(\eta_\varepsilon - 1)) = \frac{1}{2}\int_{\partial B}\left|\frac{\partial \eta_\varepsilon}{\partial \nu}\right|^2 (V\cdot\nu) + \int_{\partial B}\frac{\partial \eta_\varepsilon}{\partial \nu}\frac{\partial |g_\varepsilon|}{\partial \tau}(V\cdot\tau) + O(1)$$

and

$$\begin{aligned}\int_B \tilde{a}_\varepsilon(x)(\eta_\varepsilon - 1).(V.\nabla(\eta_\varepsilon - 1)) &= \varepsilon^{-\alpha}\int_{B^+}(\eta_\varepsilon - 1).(V.\nabla(\eta_\varepsilon - 1)) \\ &\quad + \int_{B^-}(\eta_\varepsilon - 1).(V.\nabla(\eta_\varepsilon - 1)) \\ &= \frac{\varepsilon^{-\alpha}}{2}\int_{\partial(B^+)}(|g_\epsilon| - 1)^2(V\cdot\nu) \\ &\quad + \frac{1}{2}\int_{\partial(B^-)}(|g_\epsilon| - 1)^2(V\cdot\nu) + O(1).\end{aligned}$$

By construction, $V(x)$ is tangent to Γ at $x \in \Gamma$ and since $V(x)$ is close to $x/|x|$ on ∂B, we can find $c > 0$ such that $V(x)\cdot\nu \geq c$ on ∂B. Therefore we obtain

$$\frac{c}{2}\int_{\partial B}\left|\frac{\partial \eta_\varepsilon}{\partial \nu}\right|^2 + \frac{c}{2\varepsilon^2}\int_{\partial B}\tilde{a}_\varepsilon(x)(|g_\varepsilon| - 1)^2 \leq -\int_{\partial B}\frac{\partial \eta_\varepsilon}{\partial \nu}\frac{\partial |g_\varepsilon|}{\partial \tau}(V\cdot\tau) + O(1).$$

From (4.6) and (4.7), we conclude that

$$\int_{\partial B}\left|\frac{\partial \eta_\varepsilon}{\partial \nu}\right|^2 \leq C.$$

Now we multiply (4.15) by $(\eta_\varepsilon - 1)$ and we integrate by parts. This yields

$$\begin{aligned}\varepsilon^2\int_B |\nabla\eta_\varepsilon|^2 + \int_B \tilde{a}_\varepsilon(x)(\eta_\varepsilon - 1)^2 &\leq \varepsilon^2\int_{\partial B}\left|\frac{\partial \eta_\varepsilon}{\partial \nu}\right||\eta_\varepsilon - 1| \\ &\leq \varepsilon^2\left\|\frac{\partial \eta_\varepsilon}{\partial \nu}\right\|_{L^2(\partial B)} \|\, |g_\varepsilon| - 1\,\|_{L^2(\partial B)} \\ &\leq C\varepsilon^3\end{aligned}$$

which ends the proof of (4.17). ■

End of Step 1. Now we claim that

$$\frac{1}{2}\int_B |\nabla u_\varepsilon|^2 + \frac{1}{4\varepsilon^2}\int_B \tilde{a}_\varepsilon(x)(1 - |u_\varepsilon|^2)^2 \leq \frac{1}{2}\int_B |\nabla\psi_\epsilon|^2 + C\varepsilon. \tag{4.19}$$

By construction of v_ε and (4.17), we have

$$\frac{1}{\varepsilon^2}\int_B \tilde{a}_\varepsilon(x)(1-|v_\varepsilon|^2)^2 = \frac{1}{\varepsilon^2}\int_B \tilde{a}_\varepsilon(x)(1-|\eta_\varepsilon|^2)^2 \le C\varepsilon$$

and

$$\int_B |\nabla v_\varepsilon|^2 = \int_B |\nabla \eta_\varepsilon|^2 + \eta_\varepsilon^2|\nabla\psi_\varepsilon|^2 \le C\varepsilon + \int_B |\nabla\psi_\varepsilon|^2$$

(by the maximum principle we get easily that $\eta_\varepsilon \le 1$). This prove (4.19) since u_ε is a solution of problem (4.2).

We infer from (4.6), φ_ε is bounded in $H^1(\partial B)$, $\varphi_\varepsilon \to \varphi_0$ uniformly on ∂B and $\varphi_\varepsilon \to \varphi_0$ strongly in $H^{1/2}(\partial G)$. From Equation (4.16) we derive that $\psi_\varepsilon \to \varphi_\star$ strongly in $H^1(G)$ and by (4.19), u_ε is bounded in $H^1(B)$. Therefore we can find a sequence $\varepsilon_n \to 0$ and $u \in H^1(B)$ such that

$$u_{\varepsilon_n} \rightharpoonup u \quad \text{weakly in } H^1 .$$

From (4.19) and a lower semi-continuity argument, we deduce that the map u satisfies

$$\int_B |\nabla u|^2 \le \int_B |\nabla \varphi_\star|^2 = \int_B |\nabla u_\star|^2. \tag{4.20}$$

Since we have

$$\int_B (1-|u_{\varepsilon_n}|^2)^2 \le C{\varepsilon_n}^2,$$

we conclude that $|u| = 1$ a.e. in B. Thus $u \in H^1_g(G, S^1)$ and u is a solution of (4.9). By uniqueness, it implies that $u = u_\star$. We obtain the strong convergence of u_ε as $\varepsilon \to 0$ to $u_\star$ in $H^1(B)$ from (4.19) and the uniqueness of the limit. ■

Remark 4.2. Note that we also obtain from (4.19),

$$\frac{1}{\varepsilon^2}\int_B \tilde{a}_\varepsilon(x)(1-|u_\varepsilon|^2)^2 \to 0. \tag{4.21}$$

Step 2 : Proof of (4.12). As in [19], we derive from the maximum principle that

$$|u_\varepsilon| \le 1 \quad \text{in } B. \tag{4.22}$$

By Lemma A.1 in [19] and (4.3), we infer that the following estimates hold

$$|\nabla u_\varepsilon| \le C_{K_1}\varepsilon^{-(1+\alpha/2)} \quad \text{in any compact set } K_1 \subset B, \tag{4.23}$$

$$|\nabla u_\varepsilon| \le C_{K_2}\varepsilon^{-1} \quad \text{in any compact set } K_2 \subset B^-. \tag{4.24}$$

Arguing as in Step A.1-A.2, Section 2 in [19], we show that $|u_\varepsilon| \to 1$ uniformly in any compact set $K_1 \subset (B^+ \cup \Gamma)$ or $K_2 \subset B^-$ (here we make use of (4.21)). Following Step 2, Section 3 in [19], we also prove that $|u_\varepsilon| \to 1$ on $\overline{B^+}$ and in any compact set $K \subset (\overline{B^-} \backslash \overline{\Gamma})$. We claim that

$$|u_\varepsilon| \to 1 \quad \text{uniformly on } \overline{B^-}. \tag{4.25}$$

We argue by contradiction. Assume that exist a sequence $\varepsilon_n \to 0$, a sequence $(x_n) \subset B^-$ and $\delta > 0$ such that

$$|u_{\varepsilon_n}(x_n)| \le 1 - \delta \quad \text{for every } n \in \mathbb{N}. \tag{4.26}$$

We may also assume that

$$x_n \underset{n\to+\infty}{\longrightarrow} \tilde{x} \in \overline{\Gamma}.$$

We set $u_n = u_{\varepsilon_n}$ and $d_n = \text{dist}(x_n, \partial B^-)$ where "dist" denotes the Euclidean distance in $\mathbb{R}^2$. Following Step 2, Section 3 in [19] and using (4.21), we obtain

$$\frac{d_n}{\varepsilon_n} \to 0 \quad \text{as } n \to +\infty. \tag{4.27}$$

Now we use a blow-up argument with $v_n = |u_n|^2$. The function v_n satisfies the equation

$$\begin{cases} -\Delta v_n = \dfrac{1}{\varepsilon_n^2}(1 - |u_n|^2)|u_n|^2 - 2|\nabla u_n|^2 & \text{in } B^-, \\ \quad v_n = |g_{\varepsilon_n}|^2 & \text{on } \partial B^- \cap \partial B, \\ \quad v_n = |u_n|^2 & \text{sur } \partial B^- \cap \Gamma, \end{cases} \tag{4.28}$$

and

$$\int_{B^-} |\nabla v_n|^2 \le 4 \int_B |\nabla u_n|^2 \le C < +\infty. \tag{4.29}$$

We set

$$w_n(y) = v_n\left(d_n y + x_n\right) \quad \text{for } y \in \Omega_n = \frac{1}{d_n}\left(G^- - \{x_n\}\right).$$

The function w_n satisfies the equation

$$-\Delta w_n = \frac{d_n^2}{\varepsilon_n^2}(1 - w_n)w_n - 2d_n^2 \left|\nabla u_n(d_n y + x_n)\right|^2 \quad \text{in } \Omega_n \tag{4.30}$$

and

$$\int_{\Omega_n} |\nabla w_n|^2 = \int_{B^-} |\nabla v_n|^2 \le C. \tag{4.31}$$

We may assume that $\Omega_n \to \Omega$ as $n \to +\infty$ where Ω is an angular sector of $\mathbb{R}^2$ and the convergence is defined in the following sense : for any compact set $K \subset \Omega$, resp. $K' \subset \mathbb{R}^2 \setminus \overline{\Omega}$, there exists $N \in \mathbb{N}$ such that for any $n \ge N$, $K \subset \Omega_n$, resp. $K' \subset \mathbb{R}^2 \setminus \overline{\Omega}_n$. We claim that

$$\int_\Omega \Delta w_n \,\varphi \underset{n\to+\infty}{\longrightarrow} 0 \quad \text{for every } \varphi \in \mathcal{D}(\Omega). \tag{4.32}$$

Indeed, let $\varphi \in \mathcal{D}(\Omega)$ and $K = \text{supp } \varphi$. For n sufficiently large, $K \subset \Omega_n$ and then we have

$$\int_\Omega -\Delta w_n \, \varphi = \int_K \left(\frac{d_n^2}{\varepsilon_n^2} (1 - w_n(y)) \, w_n(y) - 2d_n^2 |\nabla u_n(d_n y + x_n)|^2 \right) \varphi(y) dy$$

Since $0 \le w_n \le 1$, we derive that

$$I_1 = \left| \frac{d_n^2}{\varepsilon_n^2} \int_K (1 - w_n) w_n \varphi \right| \le \frac{d_n^2}{\varepsilon_n^2} |K| \, \|\varphi\|_\infty$$

and by (4.27), we conclude that $I_1 \to 0$ as $n \to +\infty$. Next we have

$$\begin{aligned} I_2 &= \left| \int_K 2d_n^2 |\nabla u_n(d_n y + x_n)|^2 \varphi(y) dy \right| \\ &= \left| \int_{d_n K + x_n} 2|\nabla u_n(z)|^2 \varphi\left(\frac{z - x_n}{d_n} \right) dz \right| \\ &\le \left(\int_{d_n K + x_n} 2|\nabla u_n(z)|^2 dz \right) \|\varphi\|_\infty . \end{aligned}$$

Since $u_n \to u_\star$ strongly in $H^1(B)$ and $|d_n K + x_n| \to 0$, we deduce that

$$\int_{d_n K + x_n} |\nabla u_n(z)|^2 dz \underset{n \to +\infty}{\to} 0.$$

Therefore $I_2 \to 0$ as $n \to +\infty$ and (4.32) is proved.

Going back to (4.31), we may assume that $w_n \to w$ strongly in $L^p(K)$ for any compact set $K \subset \Omega$. From (4.31) and (4.32), we infer that

$$\Delta w = 0 \quad \text{in } \mathcal{D}'(\Omega) \tag{4.33}$$

and

$$\int_\Omega |\nabla w|^2 < +\infty. \tag{4.34}$$

We claim that

$$w_n \to w \quad \text{uniformly in any compact set } K \subset \Omega. \tag{4.35}$$

Fix $y \in \Omega$ and $\sigma > 0$ such that $\text{dist}(y, \partial\Omega) \ge 2\sigma$. For n sufficiently large, $B_\sigma(y) \subset \Omega_n$. By Lemma A.1 in [19], we have

$$|\nabla u_n(x)|^2 \le C \left(\frac{1}{\varepsilon_n^2} + \frac{4}{d_n^2 \sigma^2} \right) \quad \text{for every } x \in B_{\frac{d_n \sigma}{2}}(d_n y + x_n).$$

Using (4.22), (4.27) and (4.30), we deduce that

$$|\Delta w_n| \le C_\sigma \quad \text{in } B_{\frac{\sigma}{2}}(y).$$

Therefore $(w_n)_{n \in \mathbb{N}}$ is compact in $C^0(\overline{B}_{\frac{\sigma}{4}}(y))$. We conclude that $w_n \to w$ uniformly on $B_{\frac{\sigma}{4}}(y)$ which ends the proof of (4.35).

We claim that

$$w = 1 \quad \text{on } \partial\Omega. \tag{4.36}$$

It suffices to prove that $|w-1| \le \tilde{\varepsilon}$ a.e. on $\partial\Omega$ for any $\tilde{\varepsilon} > 0$. Since all cases can be treated in the same way, we just consider the case

$$\Omega = (-\infty, a] \times (-\infty, b] \quad \text{with } a, b < +\infty.$$

Let $z \in \partial\Omega$, for instance $z = (a, z_2)$. For $\tilde{\delta} > 0$, let $y = (a - \tilde{\delta}, z_2)$. For n sufficiently large, the projection $\Pi_n(z)$ of z on $\partial\Omega_n$ in the direction (z, y) is well defined. From (4.31) and (4.34), we get that for almost every x_2,

$$\int_{(\mathbb{R}\times\{x_2\})\cap\Omega_n} |\nabla w_n|^2 \le C, \tag{4.37}$$

and

$$\int_{(\mathbb{R}\times\{x_2\})\cap\Omega} |\nabla w|^2 < +\infty. \tag{4.38}$$

We may assume that z_2 satisfies (4.37) and (4.38) and we deduce that for n sufficiently large and $\tilde{\delta}$ sufficiently small,

$$|w(z) - w(y)| \le \frac{\tilde{\varepsilon}}{4} \quad \text{and} \quad |w_n(y) - w_n(\Pi_n(z))| \le \frac{\tilde{\varepsilon}}{4}. \tag{4.39}$$

Since $|u_n| \to 1$ uniformly on $\partial(B^+)$, and by (4.4), (4.35), we have for n sufficiently large

$$|w(y) - w_n(y)| \le \frac{\tilde{\varepsilon}}{4} \quad \text{and} \quad |w_n(\Pi_n(z)) - 1| \le \frac{\tilde{\varepsilon}}{4}.$$

We finally obtain, choosing $\tilde{\delta}$ sufficiently small and n sufficiently large,

$$\begin{aligned} |w(z) - 1| &\le |w(z) - w(y)| + |w(y) - w_n(y)| \\ &\quad + |w_n(y) - w_n(\Pi_n(z))| + |w_n(\Pi_n(z)) - 1| \\ &\le \tilde{\varepsilon}. \end{aligned}$$

Therefore $|w(z) - 1| \le \tilde{\varepsilon}$ for almost every $z \in \partial\Omega$.

By classical arguments (see [54] for instance), we deduce from (4.33), (4.34) and (4.36) that $w \equiv 1$ in Ω. Then (4.35) implies that $w_n(0) \to 1$ which contradicts $v_n(x_n) = w_n(0) \le 1 - \delta$ for every $n \in \mathbb{N}$, and the proof of (4.25) is complete.

Since $|u_\varepsilon| \to 1$ uniformly on $\overline{B}$, we have $|u_\varepsilon| \ge \frac{1}{2}$ for ε sufficiently small and we can write

$$u_\varepsilon = \rho_\varepsilon e^{i\varphi_\varepsilon} \quad \text{with } \rho_\varepsilon = |u_\varepsilon|. \tag{4.40}$$

As in [19], we get from (4.3),

$$-\Delta\rho_\varepsilon + \rho_\varepsilon|\nabla\varphi_\varepsilon|^2 = \frac{1}{\varepsilon^2}\,\tilde{a}_\varepsilon(x)(1 - \rho_\varepsilon^2)\rho_\varepsilon \quad \text{in } B, \tag{4.41}$$

and

$$\operatorname{div}\left(\rho_\varepsilon^2 \nabla \varphi_\varepsilon\right) = 0 \quad \text{in } B. \tag{4.42}$$

By (4.10) and (4.42), we have

$$-\operatorname{div}\left(\rho_\varepsilon^2 \nabla(\varphi_\varepsilon - \varphi_\star)\right) = \operatorname{div}\left((\rho_\varepsilon^2 - 1)\nabla\varphi_\star\right) \quad \text{in } B. \tag{4.43}$$

Since $\rho_\varepsilon \to 1$ uniformly, the equation (4.43) is uniformly elliptic for ε sufficiently small. By classical estimates (see [54]), we obtain for $p > 2$,

$$\| \varphi_\varepsilon - \varphi_\star \|_{L^\infty(B)} \leq C\left(\| \varphi_\varepsilon - \varphi_\star \|_{L^\infty(\partial B)} + \| (\rho_\varepsilon^2 - 1)\nabla\varphi_\star \|_{L^p(G)}\right). \tag{4.44}$$

Since $g \in H^1(\partial B)$, we infer from (4.10) that $\varphi_* \in H^{3/2}(B)$ and therefore $\nabla\varphi_* \in H^{1/2}(B) \subset L^4(B)$. Choosing $p = 4$ in (4.44), we get that

$$\| \varphi_\varepsilon - \varphi_\star \|_{L^\infty(B)} \leq C\left(\| \varphi_\varepsilon - \varphi_\varepsilon \|_{L^\infty(\partial B)} + \| (\rho_\varepsilon^2 - 1) \|_{L^\infty(B)}\right) \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

Then $\varphi_\varepsilon \to \varphi_\star$ uniformly on $\overline{B}$ which end the proof of (4.12). ■

Step 3 : End of the proof. To prove (4.13) and (4.14), we consider u_ε on B^+ and B^- separately. We have

$$-\Delta u_\varepsilon = \frac{1}{\varepsilon^{2+\alpha}}(1 - |u_\varepsilon|^2)u_\varepsilon \quad \text{in } B^+,$$

and $u_\varepsilon \to u_\star$ in $H^1(B^+)$ and uniformly on $\overline{B^+}$. Applying Step 3, Section 3 in [19], we obtain

$$u_\varepsilon \to u_\star \quad \text{in } C^k_{\text{loc}}(B^+) \text{ for any } k \geq 1$$

and

$$\frac{1 - |u_\varepsilon|^2}{\varepsilon^{2+\alpha}} \to |\nabla u_\star|^2 \quad \text{in } C^k_{\text{loc}}(B^+) \text{ for any } k \geq 0.$$

Using the same arguments on B^- (with ε^2 instead of $\varepsilon^{2+\alpha}$), we obtain the announced result. ■

Remark 4.3. A consequence of Theorem 4.3 is that u_ε converges in B^+ faster than in B^-. To illustrate this fact, one can consider the function

$$w_\varepsilon(x) = \frac{\ln(1 - |u_\varepsilon(x)|^2)}{\ln\varepsilon}.$$

If we assume that $|\nabla u_\star|$ does not vanish in B, we derive from (4.14) that for each $x \in B$,

$$\ln(1 - |u_\varepsilon(x)|^2) = \begin{cases} 2\ln\varepsilon + \ln(|\nabla u_\star(x)|^2) + \mathcal{O}(1) & \text{if } x \in B^+, \\ (2+\alpha)\ln\varepsilon + \ln(|\nabla u_\star(x)|^2) + \mathcal{O}(1) & \text{if } x \in B^-. \end{cases}$$

and we conclude that $w_\varepsilon \to 2 + \alpha\chi_{B^+}$ uniformly on every compact subset of $B \setminus \Gamma$ as $\varepsilon \to 0$ (here χ_{B^+} denotes the characteristic function of the set B^+). The precise behavior of the profile of u_ε across Γ remains an open question.

4.3 The convergence result, proof of Theorem 4.1.

We begin the proof of Theorem 4.1 by some fundamental estimates.

Lemma 4.1. *There exist $\varepsilon_0 > 0$ and $C_1 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$,*

$$E_\varepsilon(u_\varepsilon) \leq \pi d \ln \frac{1}{\varepsilon} + C_1. \tag{4.45}$$

Proof. We fix d distinct points $b_1, \ldots, b_d$ in G^- and $R > 0$ such that

$$\overline{B}(b_i, R) \cap \overline{B}(b_j, R) = \emptyset \quad \text{and} \quad \overline{B}(b_i, R) \cap \partial G^- = \emptyset \quad \forall i \neq j.$$

Let $\Omega = G \setminus \bigcup_{i=1}^d \overline{B}(b_i, R)$ and $\overline{g} : \partial\Omega \to S^1$ defined by

$$\overline{g}(z) = \begin{cases} g(z) & \text{if } z \in \partial G, \\ \dfrac{z - b_i}{|z - b_i|} & \text{if } z \in \partial B(b_i, R). \end{cases}$$

By construction $\deg(\overline{g}, \partial\Omega) = 0$ and then there exists a smooth function $\overline{v} : \Omega \to S^1$ such that $\overline{v} = \overline{g}$ on $\partial\Omega$. We set

$$v(z) = \begin{cases} \overline{v}(z) & \text{if } z \in \Omega, \\ w(z - b_i) & \text{if } z \in B(b_i, R), \end{cases}$$

where w realizes

$$I(\varepsilon, R) = \underset{v \in H^1_{\frac{z}{|z|}}(B_R, \mathbb{C})}{\operatorname{Min}} \frac{1}{2} \int_{B_R} |\nabla v(x)|^2 dx + \frac{1}{4\varepsilon^2} \int_{B_R} (1 - |v(x)|^2)^2 dx.$$

Since u_ε is a minimizer of E_ε, we have

$$E_\varepsilon(u_\varepsilon) \leq E_\varepsilon(v) = \frac{1}{2} \int_\Omega |\nabla \overline{v}(x)|^2 dx + d I(\varepsilon, R).$$

By the results in [20], we know that for $\varepsilon < R$,

$$I(\varepsilon, R) \leq \pi \ln \frac{1}{\varepsilon} + I(1, 1),$$

which leads to (4.45). ■

Lemma 4.2. *There exists a constant $C_2 > 0$ such that for any $\varepsilon > 0$,*

$$\frac{1}{4\varepsilon^2} \int_G a_\varepsilon(x)(1 - |u_\varepsilon(x)|^2)^2 dx \leq C_2. \tag{4.46}$$

Proof . We follow the method in [44]. For any $\varepsilon > 0$, we have

$$E_\varepsilon(u_\varepsilon) - E_{2\varepsilon}(u_\varepsilon) = \frac{3}{16\varepsilon^2}\int_{G^-}(1-|u_\varepsilon|^2)^2 + \frac{2^{2+\alpha}-1}{2^{4+\alpha}\varepsilon^{2+\alpha}}\int_{G^+}(1-|u_\varepsilon|^2)^2. \tag{4.47}$$

By the results in [20], there exists a constant C_0 such that for any $v \in H^1_g(G,\mathbb{C})$ and any $\varepsilon > 0$,

$$\frac{1}{2}\int_G |\nabla v|^2 + \frac{1}{4\varepsilon^2}\int_G (1-|v|^2)^2 \geq \pi d \ln\frac{1}{\varepsilon} - C_0.$$

For ε small enough, $a_{2\varepsilon} \geq 1$ and consequently

$$E_{2\varepsilon}(u_\varepsilon) \geq \pi d \ln\frac{1}{2\varepsilon} - C_0. \tag{4.48}$$

Combining (4.45), (4.47) and (4.48), we derive

$$\frac{3}{16\varepsilon^2}\int_{G^-}(1-|u_\varepsilon|^2)^2 + \frac{2^{2+\alpha}-1}{2^{4+\alpha}\varepsilon^{2+\alpha}}\int_{G^+}(1-|u_\varepsilon|^2)^2 \leq \pi d \ln 2 + C_1 + C_0$$

which ends the proof. ■

We deduce directly from Lemma 4.1 and Theorem 4 in [73], the first convergence result which can be stated as follows :

Proposition 4.1. *For each sequence $\varepsilon_n \to 0$, there exist a subsequence (also denoted by ε_n) and k distinct points $a_1,\ldots,a_k$ in $\overline{G}$ with $k \leq d$ such that u_{ε_n} converges weakly in $H^1_{\text{loc}}(\overline{G}\setminus\{a_1,\ldots,a_k\})$ to an S^1-valued map $u_\star$.*

Now we can precise the convergence result.

Proposition 4.2. *We have*

$$u_\star \in C^\infty\left(G\setminus\{a_1,\ldots,a_k\}\right), \tag{4.49}$$

$$\begin{cases} -\Delta u_\star = |\nabla u_\star|^2 u_\star & \text{in } G\setminus\{a_1,\ldots,a_k\}, \\ u_\star = g & \text{on } \partial G, \end{cases} \tag{4.50}$$

and the following convergences hold as $\varepsilon_n \to 0$:

$$u_{\varepsilon_n} \to u_\star \quad \text{in } C^0_{loc}\left(\overline{G}\setminus\{a_1,\ldots,a_k\}\right), \tag{4.51}$$

$$u_{\varepsilon_n} \to u_\star \quad \text{in } C^{1,\beta}_{loc}\left(\overline{G}\setminus\left(\overline{\Sigma}\cup\{a_1,\ldots,a_k\}\right)\right)\ \forall\beta<1, \tag{4.52}$$

$$u_{\varepsilon_n} \to u_\star \quad \text{in } C^k_{loc}\left(G\setminus(\Sigma\cup\{a_1,\ldots,a_k\})\right)\ \forall k, \tag{4.53}$$

$$\frac{a_{\varepsilon_n}(x)\,(1-|u_{\varepsilon_n}|^2)}{\varepsilon_n^2} \to |\nabla u_\star|^2 \quad \text{in } C^k_{loc}\left(G\setminus(\Sigma\cup\{a_1,\ldots,a_k\})\right)\ \forall k. \tag{4.54}$$

Proof. We fix $x_0 \in G \setminus \{a_1, \dots, a_k\}$ and consider $R > 0$ satisfying

$$\overline{B(x_0, 2R)} \subset G \setminus (\Sigma \cup \{a_1, \dots, a_k\}) \quad \text{if } x_0 \notin \Sigma$$

or

$$\overline{B(x_0, 2R)} \subset G \setminus \{a_1, \dots, a_k\} \quad \text{if } x_0 \in \Sigma.$$

From Proposition 4.1 and Lemma 4.2, we can find $R' \in (R, 2R)$ such that

$$\int_{\partial B(x_0,R')} |\nabla u_{\varepsilon_n}|^2 \leq C \tag{4.55}$$

and

$$\int_{\partial B(x_0,R')} a_{\varepsilon_n}(x)(1 - |u_{\varepsilon_n}|^2)^2 \leq C\varepsilon_n^2. \tag{4.56}$$

From (4.55), we infer that (extracting a subsequence if necessary)

$$u_{\varepsilon_n} \to u_\star \quad \text{uniformly on } \partial B(x_0, R').$$

Since $u_\star \in H^1(B(x_0, R'), S^1)$, we have

$$\deg(u_\star, \partial B(x_0, R')) = 0.$$

For n sufficiently large, $|u_{\varepsilon_n}| \geq 1/2$ on $\partial B(x_0, R')$ and extracting a subsequence if necessary, we may assume that

$$\deg(u_{\varepsilon_n}, \partial B(x_0, R')) = 0.$$

If $x_0 \notin \Sigma$, we apply Theorem 2 in [19]. If $x_0 \in \Sigma$, choosing R' sufficiently small, we may assume that $\Sigma \cap B(x_0, R')$ can represented, in local coordinates, by the graph of a function f as in Section 2. Then we apply Theorem 4.3. We obtain (4.49), (4.50), (4.53), (4.54) and convergence in $C^0_{loc}(G \setminus \{a_1, \dots, a_d\})$ and $C^{1,\beta}_{loc}(G \setminus (\Sigma \cup \{a_1, \dots, a_d\}))$. Now we consider $x_0 \in \partial G \setminus \{a_1, \dots, a_d\}$. If $x_0 \notin \partial G \cap \overline{\Sigma}$, we apply Theorem A.3 in [20] and we get (4.52). If $x_0 \in \partial G \cap \overline{\Sigma}$, we use a simple modification of Theorem 4.3 in order to obtain (4.51). ■

Lemma 4.3. *We have*

$$\deg(u_\star, a_i) = 1 \quad \forall i \quad \textit{and then } k = d, \tag{4.57}$$

$$a_i \in G^- \cup \Sigma \quad \forall i \in \{1, \dots, d\}. \tag{4.58}$$

Proof. As in [20], we will extend our maps to a largest domain G'such that $G \subset\subset G'$. We fix a smooth map $\overline{g} : G' \setminus G \to S^1$ verifying $\overline{g} = g$ on ∂G. Then we extend all maps $u : G \to S^1$ into a map defined on G' and also denoted by u letting $u = \overline{g}$ on ∂G.

We may assume that for any $i = 1, \dots, k$, $\deg(u_\star, a_i) \neq 0$. Indeed, suppose that $\deg(u_*, a_i) = 0$. Then, for $R > 0$ sufficiently small, u_{ε_n} is bounded in $H^1(B(a_i, R))$ and

a_i is not a singularity. Since $E_{\varepsilon_n}(u_{\varepsilon_n}) \to +\infty$ as $n \to +\infty$, there is at least one singular point.

We fix $\rho > 0$ such that

$$4\rho < \frac{1}{8} \min \{\operatorname{dist}(a_i, \partial G'), |a_i - a_j|\} .$$

From Proposition 4.2 we infer

$$\int_{\partial B(a_i,\rho)} |\nabla u_{\varepsilon_n}|^2 \leq C(\rho).$$

Then for n sufficiently large, we have $|u_{\varepsilon_n}| \geq 1/2$ on $\partial B(a_i, \rho)$ and

$$\deg(u_{\varepsilon_n}, \partial B(a_i, \rho)) = \deg(u_\star, \partial B(a_i, \rho)) := k_i \ .$$

Applying the Corollary in [73], we obtain for $i = 1, \ldots, k$,

$$\frac{1}{2} \int_{B(a_i,\rho)} |\nabla u_{\varepsilon_n}|^2 + \frac{1}{4\varepsilon_n^2} \int_{B(a_i,\rho)} (1 - |u_{\varepsilon_n}|^2)^2 \geq \pi |k_i| \ln \frac{1}{\varepsilon_n} - C(\rho).$$

Summing these inequalities in i and then combining with (4.45), we get that

$$\sum_{i=1}^{k} |k_i| \leq d + \frac{C(\rho)}{|\ln \varepsilon_n|}.$$

Letting $n \to +\infty$, we derive

$$\sum_{i=1}^{k} |k_i| \leq d.$$

Since $\sum k_i = d$, we deduce that $k_i > 0$ for each $i \in \{1, \ldots, k\}$. By Lemma 4.1, we can apply Theorem 3 in [73]. Then we find a constant $C_3 > 0$ such that for any $n \geq N(\rho)$, there exists a collection of points in $\overline{G}$ $(x_1^n, \ldots, x_{\tilde{k}(n)}^n)$, $\tilde{k}(n) \leq d$, satisfying

$$\frac{1}{2} \int_{G' \setminus \bigcup_j B\left(x_j^n, \frac{\rho}{2}\right)} |\nabla u_{\varepsilon_n}|^2 \leq \pi d \ln \frac{1}{\rho} + C_3.$$

Extracting a subsequence if necessary, we may assume that

$$\tilde{k}(n) \equiv K = \text{constant} \quad \text{and} \quad x_j^n \to l_j \in \overline{G} \quad \text{as } n \to +\infty.$$

Then, for n sufficiently large, we have $\bigcup_j B(x_j^n, \frac{\rho}{2}) \subset \bigcup_j B(l_j, \rho)$ and therefore

$$\frac{1}{2} \int_{G' \setminus \bigcup_j B(l_j,\rho)} |\nabla u_{\varepsilon_n}|^2 \leq \pi d \ln \frac{1}{\rho} + C_3. \tag{4.59}$$

We set $J = \big\{j \in \{1, \dots, K\},\, \overline{B}(l_j, \rho) \cap \big(\bigcup_i \overline{B}(a_i, \rho)\big) \neq \emptyset\big\}$. We have

$$\bigcup_{j \in J} B(l_j, \rho) \subset \bigcup_{i=1}^{k} B(a_i, 4\rho).$$

From (4.59) and Proposition 4.1, we infer that

$$\frac{1}{2} \int_{G' \setminus \bigcup_i B(a_i, 4\rho)} |\nabla u_{\varepsilon_n}|^2 \leq \pi d \ln \frac{1}{\rho} + C.$$

Letting $n \to +\infty$, we get

$$\frac{1}{2} \int_{G' \setminus \bigcup_i B(a_i, 4\rho)} |\nabla u_\star|^2 \leq \pi d \ln \frac{1}{\rho} + C. \tag{4.60}$$

By Corollary II.2 in [20], we have

$$\frac{1}{2} \int_{G' \setminus \bigcup_i B(a_i, 4\rho)} |\nabla u_\star|^2 \geq \pi \left(\sum_{i=1}^{k} k_j^2 \right) \ln \frac{1}{\rho} - C.$$

Combining the last inequality with (4.60), and letting $\rho \to 0$, we obtain

$$\sum_i k_i^2 - k_i \leq 0$$

and then $k_i = 1$ for each $i \in \{1, \dots, k\}$. Since $\sum k_i = d$, we deduce $k = d$.

It remains to prove that $a_i \notin \partial G$ for any i. We argue by contradiction. Suppose that exists $i_0 \in \{1, \dots, d\}$ such that $a_{i_0} \in \partial G$ and fix $R > 0$ verifying

$$\overline{B}(a_i, R) \cap \overline{B}(a_j, R) = \emptyset \quad \forall i \neq j \quad \text{and} \quad \overline{B}(a_i, R) \subset G' \quad \forall i.$$

By Lemma VI.1 in [20], we have for any $\rho \in \big(0, \frac{R}{4}\big)$,

$$\frac{1}{2} \int_{B\left(a_{i_0}, R\right) \setminus B\left(a_{i_0}, 4\rho\right)} |\nabla u_\star|^2 \geq 2\pi \ln \frac{1}{\rho} - C,$$

and by Lemma 1.1 in [73], for any $i \neq i_0$,

$$\frac{1}{2} \int_{B(a_i, R) \setminus B(a_i, 4\rho)} |\nabla u_\star|^2 \geq \pi \ln \frac{1}{\rho} - C.$$

From this two inequalities, we obtain

$$\frac{1}{2} \int_{G' \setminus \bigcup_i B(a_i, 4\rho)} |\nabla u_\star|^2 \geq \pi (d+1) \ln \frac{1}{\rho} - C,$$

which contradicts (4.60) for ρ sufficiently small. Therefore $a_i \in G$ for any i.

Now suppose that exists $a_{i_0} \in G^+$ and fix $R > 0$ such that

$$\overline{B}(a_i, R) \cap \overline{B}(a_j, R) = \emptyset \quad \forall i \neq j, \quad \overline{B}(a_i, R) \subset G \quad \forall i \quad \text{and} \quad \overline{B}(a_{i_0}, R) \subset G^+.$$

For n sufficiently large, $|u_{\varepsilon_n}| \geq 1/2$ on $\partial B(a_i, R)$ and

$$\deg(u_{\varepsilon_n}, \partial B(a_i, R) = \deg(u_\star, \partial B(a_i, R) = 1.$$

Applying the Corollary of [73], we obtain for any $i \neq i_0$,

$$\frac{1}{2}\int_{B(a_i,R)} |\nabla u_{\varepsilon_n}|^2 + \frac{1}{4\varepsilon_n^2}\int_{B(a_i,R)} (1 - |u_{\varepsilon_n}|^2)^2 \geq \pi \ln \frac{1}{\varepsilon_n} - C,$$

and

$$\frac{1}{2}\int_{B(a_{i_0},R)} |\nabla u_{\varepsilon_n}|^2 + \frac{1}{4\varepsilon_n^{2+\alpha}}\int_{B(a_{i_0},R)} (1 - |u_{\varepsilon_n}|^2)^2 \geq \pi \left(1 + \frac{\alpha}{2}\right) \ln \frac{1}{\varepsilon_n} - C.$$

We deduce that

$$E_{\varepsilon_n}(u_{\varepsilon_n}) \geq \pi \left(d + \frac{\alpha}{2}\right) \ln \frac{1}{\varepsilon_n} - C,$$

which contradicts Lemma 4.1 for ε_n sufficiently small. ■

Lemma 4.4. *We have*

$$u_\star \equiv u_0$$

the canonical harmonic map relative to the singularities $(a_1, \ldots, a_d)$ *with associated degrees* $(+1, \ldots, +1)$ *and to the boundary data* g.

Proof. Taking the exterior product between equation (4.1) and u_{ε_n} and letting $n \to +\infty$ we obtain

$$\operatorname{div}(\nabla u_\star \times u_\star) = \lim_{n\to\infty} \operatorname{div}(\nabla u_{\varepsilon_n} \times u_{\varepsilon_n}) = 0 \quad \text{in } \mathcal{D}'(G).$$

Using the method in [82], we infer that (u_{ε_n}) is bounded in $W^{1,p}(G)$ for any $p < 2$ and then $u_\star \in W^{1,p}(G)$ for $p < 2$. From the results in [20] Chapter I, we conclude that $u_\star \equiv u_0$. ■

4.4 The Renormalized Energy

In section I.4 in [20], the authors introduce the function called *renormalized energy*, $W = W(b, d, g)$, associated to a general configuration of distinct points in G, $b = (b_i)_{i=1}^n$, of degrees $(d_i)_{i=1}^n \subset \mathbb{Z}^n$ with $\sum_i d_i = d$. In our setting, we consider only the configurations of d distinct points of degree $+1$. Then the *renormalized energy* W is given by

$$W(b) = -\pi \sum_{i\neq j} \ln |b_i - b_j| + \frac{1}{2}\int_{\partial G} \Phi_0 \left(g \times \frac{\partial g}{\partial \tau}\right) - \pi \sum_{i=1}^{d} R_0(b_i),$$

where Φ_0 is the solution of the Neumann problem

$$\begin{cases} \Delta\Phi_0 = \sum_{i=1}^{d} 2\pi\delta_{b_i} & \text{in } G, \\ \dfrac{\partial\Phi_0}{\partial\nu} = g \times \dfrac{\partial g}{\partial\tau} & \text{on } \partial G, \end{cases}$$

such that $\int_{\partial G} \Phi_0 = 0$ and $R_0(x) = \Phi_0(x) - \sum_{i=1}^{d} \ln|x - b_i|$. The proof of Theorem 4.2 is based on the two following lemmae.

Lemma 4.5. *Let $b = (b_i)$ be a configuration of d distinct points in G^-. There exists $\rho_b > 0$ such that for any $0 < \rho < \rho_b$ and for any $\varepsilon > 0$,*

$$E_\varepsilon(u_\varepsilon) \leq dI(\varepsilon,\rho) + W(b) + \pi d \ln\frac{1}{\rho} + \mathcal{O}(\rho), \tag{4.61}$$

where

$$I(\varepsilon,\rho) = \underset{v\in H^1_{\frac{x}{|x|}}(B_\rho)}{\text{Min}} \ \frac{1}{2}\int_{B_\rho} |\nabla v|^2 + \frac{1}{4\varepsilon^2}\int_{B_\rho} (1-|v|^2)^2.$$

Lemma 4.6. *For any $\rho > 0$ sufficiently small, there exists an integer $N(\rho)$ such that for any $n \geq N(\rho)$,*

$$E_{\varepsilon_n}(u_{\varepsilon_n}) \geq dI(\varepsilon_n,\rho) + W(a) + \pi \ln\frac{1}{\rho} + o_\rho(n), \tag{4.62}$$

where $o_\rho(n)$ denotes a quantity verifying $\lim_{\rho\to 0}\big(\limsup_{n\to\infty} o_\rho(n)\big) = 0$.

Proof of Theorem 4.2. Let $b = (b_i)$ be a configuration of d distinct points in G^-. From Lemma 4.5 and Lemma 4.6, we infer that for ρ sufficiently small and any $n \geq N(\rho)$,

$$W(a) \leq W(b) + o_\rho(n) + \mathcal{O}(\rho).$$

Letting $n \to +\infty$ and then $\rho \to 0$, we get that

$$W(a) \leq W(b).$$

Since b est arbitrary in $(G^-)^d$, we conclude that a realizes

$$\underset{b\in(G^-)^d}{\text{Inf}} W(b) = \underset{b\in(G^-\cup\Sigma)^d}{\text{Min}} W(b)$$

which ends the proof of Theorem 4.2. ∎

Proof of Lemma 4.5. We apply Theorem I.9 in [20] to the configuration of points b. For any $0 < \rho < \frac{1}{2}\min_{i\neq j}\{|b_i - b_j|, \text{dist}(b_i,\partial G)\}$, there exists a map $\hat{u}_\rho : G \setminus \bigcup_i B(b_i,\rho) \to S^1$ such that $\hat{u}_\rho = g$ on ∂G and $\hat{u}_\rho(z) = \alpha_i \dfrac{z - b_i}{|z - b_i|}$ on $\partial B(b_i,\rho)$ with $|\alpha_i| = 1$ and

$$\frac{1}{2}\int_{G\setminus\bigcup_i B(b_i,\rho)} |\nabla\hat{u}_\rho|^2 = \pi d \ln\frac{1}{\rho} + W(b) + \mathcal{O}(\rho).$$

Let w be a map realizing $I(\varepsilon,\rho)$. We define

$$v(z) = \begin{cases} \hat{u}_\rho(z) & \text{in } G \setminus \bigcup_i B(b_i,\rho), \\ \alpha_i w(z-b_i) & \text{in } B(b_i,\rho) \text{ for } i=1,\dots,d. \end{cases}$$

We easily check that $v \in H^1_g(G)$ and

$$E_\varepsilon(v) = dI(\varepsilon,\rho) + W(b) + \pi d \ln \frac{1}{\rho} + \mathcal{O}(\rho).$$

Then (4.61) directly follows from $E_\varepsilon(u_\varepsilon) \leq E_\varepsilon(v)$. ■

Proof of Lemma 4.6. By Theorem 4.1, for any $\rho > 0$ fixed sufficiently small, there exists $N_1(\rho) \in \mathbb{N}$ such that for any $n \geq N_1(\rho)$,

$$\frac{1}{2}\int_{\Omega_\rho} |\nabla u_{\varepsilon_n}|^2 \geq \frac{1}{2}\int_{\Omega_\rho} |\nabla u_0|^2 - \rho^2, \tag{4.63}$$

where $\Omega_\rho = G \setminus \bigcup_i B(a_i,\rho)$. By the results in [20], we know that

$$\frac{1}{2}\int_{\Omega_\rho} |\nabla u_0|^2 = W(a) + \pi \ln \frac{1}{\rho} + \mathcal{O}(\rho^2). \tag{4.64}$$

Combining (4.63) and (4.64), we obtain for any $n \geq N_1(\rho)$,

$$\frac{1}{2}\int_{\Omega_\rho} |\nabla u_{\varepsilon_n}|^2 + \frac{1}{4\varepsilon_n^2}\int_{\Omega_\rho} a_{\varepsilon_n}(x)(1-|u_{\varepsilon_n}|^2)^2 \geq W(a) + \pi \ln \frac{1}{\rho} + \mathcal{O}(\rho^2).$$

Then it suffices to prove that for every $i \in \{1,\dots,d\}$,

$$\frac{1}{2}\int_{B(a_i,\rho)} |\nabla u_{\varepsilon_n}|^2 + \frac{1}{4\varepsilon_n^2}\int_{B(a_i,\rho)} a_{\varepsilon_n}(x)(1-|u_{\varepsilon_n}|^2)^2 \geq I(\varepsilon_n,\rho) + o_\rho(n). \tag{4.65}$$

We use the method in [65] and [66]. In the annulus

$$A^i_{\rho,\lambda\rho} = \{x \in \mathbb{C},\ \rho \leq |z-a_i| \leq \lambda\rho\}$$

with $\lambda = 1+\rho$, we can write for n sufficiently large,

$$u_{\varepsilon_n}(z) = |u_{\varepsilon_n}(z)| \frac{z-a_i}{|z-a_i|}\, e^{i\psi_n(z)},$$

where ψ_n is a smooth real function. We consider the test function

$$w(z) = \begin{cases} u_{\varepsilon_n}(z) & \text{in } B(a_i,\rho), \\ \xi_n(z-a_i) & \text{in } A^i_{\rho,\lambda\rho}, \end{cases}$$

with

$$\xi_n(z) = \left(\frac{|u_{\varepsilon_n}(\rho\frac{z}{|z|} + a_i)| - 1}{\rho(1-\lambda)} + \frac{1 - \lambda|u_{\varepsilon_n}(\rho\frac{z}{|z|} + a_i)|}{(1-\lambda)|z|} \right) z \exp\left(i\frac{|z| - \lambda\rho}{\rho(1-\lambda)}\, \psi_n(\rho\frac{z}{|z|} + a_i) \right).$$

We easily check that

$$\xi_n(z - a_i) = \begin{cases} u_{\varepsilon_n}(z) & \text{on } \partial B(a_i, \rho), \\ \dfrac{z - a_i}{|z - a_i|} & \text{on } \partial B(a_i, \lambda\rho), \end{cases}$$

and $|u_{\varepsilon_n}(z)| \leq |\xi_n(z - a_i)| \leq 1$ in $B(a_i, \lambda\rho)$. Then we have $w \in H^1(B(a_i, \lambda\rho)$ and $w(z) = \dfrac{z - a_i}{|z - a_i|}$ on $\partial B(a_i, \lambda\rho)$. Therefore we deduce that

$$I(\varepsilon_n, \lambda\rho) \leq \frac{1}{2} \int_{B(a_i, \lambda\rho)} |\nabla w|^2 + \frac{1}{4\varepsilon_n^2} \int_{B(a_i, \lambda\rho)} (1 - |w|^2)^2,$$

Since $I(\varepsilon_n, \lambda\rho) \geq I(\varepsilon_n, \rho)$, it remains to prove that

$$\frac{1}{2} \int_{A^i_{\rho,\lambda\rho}} |\nabla \xi_n(z - a_i)|^2 + \frac{1}{4\varepsilon_n^2} \int_{A^i_{\rho,\lambda\rho}} (1 - |\xi_n(z - a_i)|^2)^2 = o_\rho(n).$$

From (4.21) we derive

$$\frac{1}{\varepsilon_n^2} \int_{A^i_{\rho,\lambda\rho}} (1 - |\xi_n(z - a_i)|^2)^2 \leq \frac{1}{\varepsilon_n^2} \int_{B(a_i, \rho)} a_{\varepsilon_n}(x)(1 - |u_{\varepsilon_n}|^2)^2 = o_\rho(n).$$

By Theorem 4.1, we have

$$\int_{A^i_{\rho,\lambda\rho}} |\nabla \xi_n(z - a_i)|^2 \to \int_{A^i_{\rho,\lambda\rho}} |\nabla \xi|^2 \quad \text{as } n \to +\infty$$

where

$$\xi(z) = \frac{z - a_i}{|z - a_i|} \exp\left(i\frac{|z - a_i| - \lambda\rho}{\rho(1-\lambda)}\, \psi(\rho\frac{z - a_i}{|z - a_i|} + a_i) \right),$$

and ψ is a smooth function in a neighborhood of a_i such that

$$u_0(z) = \frac{z - a_i}{|z - a_i|}\, e^{i\psi(z)}.$$

Since ψ is smooth, we infer

$$\int_{A^i_{\rho,\lambda\rho}} |\nabla \xi|^2 = \mathcal{O}(\rho),$$

and we conclude that

$$\int_{A^i_{\rho,\lambda\rho}} |\nabla \xi_n(z - a_i)|^2 = o_\rho(n),$$

which ends the proof of Lemma 6. ■

Chapitre 5

Stabilization in finite time for a system of damped oscillators

5.1 Introduction

The purpose of this work is to made a first presentation of the study made by the authors on the dynamics of the finite-dimensional system corresponding to vibration of N-particles of equal mass m located along the interval $(0,1)$ of the x axis. Each particle is connected to its neighbors by two harmonic springs of strength k, the elongation of the left one is given by $x_i(t)$ and we assume the motion subject to a resultant friction force which is the composition of a Coulomb (or solid) friction and other type of frictions such as, for instance, the one due to the viscosity of an surrounding fluid. The equations of motion for this system are

$$(P_N)\begin{cases} m\ddot{x}_i(t) + k(-x_{i-1}(t) + 2x_i(t) - x_{i+1}(t)) + \mu_\beta \beta(\dot{x}_i(t)) + \mu_g g(\dot{x}_i(t)) \ni 0 \\ x_i(0) = u_{0,i} \\ \dot{x}_i(0) = v_{0,i} \end{cases}$$

$i = 1, \ldots, N$, where we are assuming that $x_0(t) = 0$, $x_{N+1}(t) = 0$ for any $t \in (0, +\infty)$, μ_β, μ_g are positive constants, the term $\mu_\beta \beta(\dot{x}_i(t))$ represents the Coulomb friction, with β given by the maximal monotone graph in $\mathbb{R}^2$

$$\beta(r) = \begin{cases} \{-1\} & \text{if } r < 0, \\ [-1,1] & \text{if } r = 0, \\ \{1\} & \text{if } r > 0, \end{cases}$$

g is a Lipschitz continuous function such that $g(0) = 0$, $\mu_\beta \beta(r) + \mu_g g(r) > 0$ for any $r > 0$ and the reverse inequality for $r < 0$. The *internal* initial data $(u_{0,i})$, $(v_{0,i})$ are given in $\mathbb{R}^N$.

It is well known that, if we write, for simplicity, $k = \frac{1}{h^2}$ (with $h = 1/(N+1)$) and $m = 1$, then problem (P_N) arises in the spatial discretization, by finite differences, of the damped string equation

$$(P_\infty)\begin{cases} u_{tt} - u_{xx} + \mu_\beta \beta(u_t) + \mu_g g(u_t) \ni 0 & \text{in } (0,1)\times(0,+\infty), \\ u(0,t) = u(1,t) = 0, & t \in (0,+\infty), \\ u(x,0) = u_0(x) & x \in (0,1), \\ u_t(x,0) = v_0(x) & x \in (0,1). \end{cases}$$

In fact, it was by passing to the limit, $N \to \infty$ in (P_N), how the wave equation (without friction) was obtained by Jean Le Rond D'Alembert in 1746.

Our main goal is to give several criteria in order to have the stabilization in a finite time for this mechanical system. The study of the special case of a single oscillator, $N = 1$, without viscous friction,

$$m\ddot{x} + 2kx + \mu_\beta \beta(\dot{x}) \ni 0,$$

can be found in many textbooks (see, for instance, [67]). It is easy to see then that the motion stops definitively after a finite time, i.e., there exists $T_e < +\infty$ and $x_\infty \in [-\frac{\mu_\beta}{2k}, \frac{\mu_\beta}{2k}]$ such that $x(t) \equiv x_\infty$ for any $t \geq T_e$. There are, also, some partial results on the stabilization to an equilibrium state in a finite time for the solutions of the wave equation (see [36] and [37] for some particular initial data). The case of arbitrary initial data $u_0(x)$ and $v_0(x)$ seems to be, still, an open problem.

Concerning the case of $N-$particles we can mention the work by Bamberger and Cabannes [14] in which they prove the stabilization in a finite time in absence of viscous friction ($\mu_g = 0$). We point out that this type of friction arises very often in the applications and that its consideration was already proposed by Lord Rayleigh (see, e.g. [72]). Concrete expressions for g can be found also in [67]. The case of a linear damping $g(\dot{x}_i) = \lambda \dot{x}_i$ and the absence of stabilization in a finite time for λ large enough was commented at the end of the paper [14] but no mention to the possibility of a simultaneous dichotomy of behaviors was made there.

One of our main goals is to prove that the presence of a viscous friction may originate a qualitative distinction among the orbits in the sense that the state of the system $\mathbf{x}(t) := (x_1(t), x_2(t), ..., x_N(t))^T$ (here $\mathbf{h}^T$ means, in general, the transposed vector of $\mathbf{h}$) may reach an equilibrium state in a finite time or merely in an asymptotic way (as $t \to +\infty$), according the initial data $\mathbf{x}(0) = \mathbf{x}_0 := (u_{0,1}, u_{0,2}, ..., u_{0,N})^T$ and $\dot{\mathbf{x}}(0) = \mathbf{v}_0 := (v_{0,1}, v_{0,2}, ..., v_{0,N})^T$. This dichotomy seems to be new in the literature and contrasts with the phenomena of *finite extinction time* for first order (in time) ordinary and parabolic nonlinear equations (see, for instance, the exposition made in [13]). Some results exhibiting this alternative, but for the case of a single particle with a non-Lipschitz friction term $\beta(u) = |u|^{\alpha-1} u$ ($\alpha \in (0,1)$), can be found in [46], [47] and [9] (problem raised, many year ago, by Haïm Brezis). In the last section we show that this alternative may occur also in the case of the wave equation (P_∞) in all dimension in space and under suitable conditions.

5.2 The dichotomy for the N-dimensional system

The system under study can be written, in short, as a vectorial problem

$$(\mathbf{P}_N)\begin{cases} m\ddot{\mathbf{x}}(t) + k\mathbf{A}\mathbf{x}(t) + \mu_\beta \mathbf{B}(\dot{\mathbf{x}}(t)) + \mu_\beta \mathbf{G}(\dot{\mathbf{x}}(t)) \ni \mathbf{0}, \\ \mathbf{x}(0) = \mathbf{x}_0, \\ \dot{\mathbf{x}}(0) = \mathbf{v}_0 \end{cases}$$

where $\mathbf{x}(t) := (x_1(t), x_2(t), ..., x_N(t))^T$, $\mathbf{A}$ is the symmetric positive definite matrix of $\mathbb{R}^{N\times N}$ given by

$$\mathbf{A} = \begin{pmatrix} 2 & -1 & 0 & ... & 0 \\ -1 & 2 & -1 & 0 & ... \\ 0 & -1 & 2 & -1 & 0 \\ ... & 0 & -1 & 2 & -1 \\ 0 & ... & 0 & -1 & 2 \end{pmatrix},$$

$\mathbf{B}$:$\mathbb{R}^N \to \mathcal{P}(\ \mathbb{R}^N)$ denotes the (multivalued) maximal monotone operator given by

$$\mathbf{B}(y_1, \ldots, y_N) = (\beta(y_1), \ldots, \beta(y_N))^T$$

and $\mathbf{G} : \mathbb{R}^N \to \mathbb{R}^N$ is the Lipschitz continuous function defined by

$$\mathbf{G}(y_1, \ldots, y_N) = (g(y_1), \ldots, g(y_N))^T.$$

In what follows, $\mathbf{a} \cdot \mathbf{b}$ denotes the Euclidian scalar product of $\mathbf{a}, \mathbf{b} \in \mathbb{R}^N$ and $\parallel . \parallel$ the Euclidean norm.

Our first result deals with the existence, uniqueness and asymptotic behavior of solutions of $(\mathbf{P}_N)$

Theorem 5.1. *For any initial datum* $(\mathbf{x}_0, \mathbf{v}_0) \in \mathbb{R}^{2N}$*, the Cauchy problem* $(\mathbf{P}_N)$ *admits a unique weak solution* $\mathbf{x} \in C^1([0, +\infty) : \mathbb{R}^N)$*. Moreover, there exists a unique equilibrium state* $\mathbf{x}_\infty \in \mathbb{R}^N$ *satisfying that* $\mathbf{A}\mathbf{x}_\infty \in ([-\frac{\mu_\beta}{2k}, \frac{\mu_\beta}{2k}]^N)^T$ *such that*

$$\parallel \dot{\mathbf{x}}(t) \parallel + \parallel \mathbf{x}(t) - \mathbf{x}_\infty \parallel \to 0 \quad \text{as } t \to +\infty. \tag{5.1}$$

Concerning the dichotomy mentioned at the introduction, the following result shows that the stabilization in a finite time depends of the structural behavior of the viscous friction g near 0.

Theorem 5.2. i) *Suppose that* $g(r)r \leq 0$ *in some neighborhood of* 0*. Then all solutions of* $(\mathbf{P}_N)$ *stabilize in a finite time.*
ii) *Suppose that* $g(r) = \lambda r$ *with* $\lambda \geq \frac{2\sqrt{\lambda_1 mk}}{\mu_g}$*, where* λ_1 *denotes the first eigenvalue of* $\mathbf{A}$*. Then there exist solutions of* $(\mathbf{P}_N)$ *which do not stabilize in any finite time.*
iii) *Suppose that* $N = 1$*,* $A = 1 \in \mathbb{R}$ *and* g *is* C^1 *in some neighborhood of* 0*. Then, if* $g'(0) < \frac{2\sqrt{mk}}{\mu_g}$*, all solutions stabilize in finite time but if* $g'(0) \geq \frac{2\sqrt{mk}}{\mu_g}$ *there exist some solutions which do not stabilize in any finite time.*

Remark 5.1. Notice that the growth condition on $g(r)$, near $r = 0$, is independent on μ_β. In the case of a single particle (notice that then $\lambda_1 = 1$) more precise results can be obtained by using, as in [46], [47], [9], the trajectory equation in the phase space $y_x \in \frac{-kx-\mu_\beta\beta(y)-\mu_g g(y)}{y}$ but they will not be presented here.

Remark 5.2. The positive results on stabilization in a finite time remain true for a general symmetric and positive definite matrix $\mathbf{A}$ as well as under the presence of some *impulsive forces* $\mathbf{f}(t)$ leading to the system

$$m\ddot{\mathbf{x}}(t) + k\mathbf{A}\mathbf{x}(t) + \mu_\beta\mathbf{B}(\dot{\mathbf{x}}(t)) + \mu_\beta\mathbf{G}(\dot{\mathbf{x}}(t)) \ni \mathbf{f}(t)$$

assuming that their amplitude is small enough : more precisely if

$$\exists\alpha > 0 \text{ such that } \mu_\beta\beta(r) + \mu_g g(r) \geq \alpha \text{ and } g(-r) = g(r) \text{ for any } r > 0$$

then we have to we assume that

$$\mathbf{f}(t) \in ([-\alpha+\epsilon, \alpha-\epsilon]^N)^T \text{ for some } \epsilon \in [0,\alpha) \text{ and for a.e. } t \geq T_f, \text{ for some } T_f \geq 0.$$

This behavior face up to with the case in which the amplitude of $\mathbf{f}(t)$ becomes large and $g'(v) < 0$ for any $v \neq 0$. Then, the dynamics generates a wide range of events leading to the chaos (see [39]).

Remark 5.3. The simultaneous possibility of the occurrence of stabilization in a finite or infinite time does not hold for solutions of scalar first order in time equations of the form

$$u_t - d\Delta u + \beta(u) \ni 0 \tag{5.2}$$

for $\beta(u)$ multivalued at $u = 0$ and $d \geq 0$ (see, for instance, [31], [45] and their references). We assume given homogeneous Dirichlet boundary conditions and an initial datum. Moreover, if we add an extra term, $g(u)$, such that, $g(u)u \geq 0$ for any $u \in \mathbb{R}$, then the solutions of

$$U_t - d\Delta U + \beta(U) + g(U) \ni 0 \tag{5.3}$$

satisfy that $\|u(t,.)\|_{L^p(\Omega)} \geq \|U(t,.)\|_{L^p(\Omega)}$ and so, the extinction in a finite time of $u(t,.)$ implies the same property for $U(t,.)$. The opposite comparison holds when $g(u)u \leq 0$. This explain the important different behaviors among the solutions of problems of first and second order in time. Notice that if we assume $k = 0$ in $(\mathbf{P}_1)$ then we get that $U(t) = \dot{x}(t)$ satisfies an equation similar to (5.3) with $d = 0$. Notice, also, that if m is very small then problem (P_1) becomes a *quasi-static problem* (in the terminology of [49]) and then the solutions are closed to the solutions of the first order in time problem

$$(QSP_1)\begin{cases} 2kx + \mu_\beta\beta(\dot{x}) + \mu_g g(\dot{x}(t)) \ni 0, \\ x(0) = x_0 \end{cases}$$

In that case, $g(u)u \geq 0$ implies an opposite comparison to the above mentioned one with respect the solutions with $g = 0$. Nevertheless, the multivalued character of β at $u = 0$ does not imply, now, the stabilization in a finite time for the solutions of (QSP_1).

Proof of Theorem 5.1. In order to reformulate $(\mathbf{P}_N)$ in the framework of nonlinear semi-group operators theory we introduce the *phase space* $\mathbf{H} = (\mathbb{R}^N, <,>_{\mathbf{A}}) \times (\mathbb{R}^N, \cdot)$, with $< \mathbf{a}, \mathbf{b} >_{\mathbf{A}} = \mathbf{A}\mathbf{a} \cdot \mathbf{b}$, and we define the operator $\mathbf{L}$ in $\mathbf{H}$ by

$$\mathbf{L}(\mathbf{x}, \mathbf{y}) = \{-\mathbf{y}\} \times \{\frac{k}{m}\mathbf{A}\mathbf{x} + \frac{\mu_\beta}{m}\mathbf{B}(\mathbf{y})\} \text{ for } (\mathbf{x}, \mathbf{y}) \in \mathbf{H}. \tag{5.4}$$

It is easy to prove that $\mathbf{L}$ is maximal monotone in H and since $\frac{\mu_g}{m}\mathbf{G}(\mathbf{y})$ is Lipschitz continuous, by using the results on Lipschitz perturbations of maximal monotone operators (see [25]) we get the existence and uniqueness of a solution of $(\mathbf{P}_N)$. Multiplying the equation by $\dot{\mathbf{x}}(t)$ and integrating in time we get the *energy relation*

$$E(t) + \int_0^t \Big[\sum_{i=1}^N \frac{\mu_\beta}{m}\,|\dot{x}_i(s)| + \frac{\mu_g}{m}\, g\big(\dot{x}_i(s)\big)\dot{x}_i(s)\Big] ds = E(0), \tag{5.5}$$

where

$$E(t) = \frac{1}{2} \parallel \dot{\mathbf{x}}(t) \parallel^2 + \frac{k}{2m}\,\mathbf{A}\mathbf{x}(t) \cdot \mathbf{x}(t). \tag{5.6}$$

By (5.5), the trajectory $(\mathbf{x}(t), \dot{\mathbf{x}}(t))_{t \geq 0}$ is compact in $\mathbf{H}$, so, we can find $\alpha > 0$ such that

$$\mu_\beta|\dot{x}_i(t)| + \mu_g g(\dot{x}_i(t))\dot{x}_i(t) \geq \alpha|\dot{x}_i(t)| \quad \text{for } i = 1, \ldots, N \text{ and all } t \geq 0.$$

By (5.5), we conclude that $\dot{\mathbf{x}} \in L^1(\mathbb{R}_+)$ which leads to the existence of the limit

$$\mathbf{x}_\infty := \lim_{t \to +\infty} \mathbf{x}(t)$$

and to $\lim_{t \to +\infty} \dot{\mathbf{x}}(t) = 0$. ■

In order to prove Theorem 5.2, it is useful to reformulate the problem in its nondimensional form.

Lemma 5.1. *The change of scales* $\mathbf{x}(t) := \widetilde{\mathbf{x}}(\widetilde{t}\,)x^*$, $\widetilde{t} = \frac{t}{t^*}$, $x^* = \frac{\mu_\beta}{k}$, $t^* = \sqrt{\frac{m}{k}}$, *transforms* $(\mathbf{P}_N)$ *in the nondimensional problem*

$$(\widetilde{\mathbf{P}}_N) \begin{cases} \ddot{\widetilde{\mathbf{x}}}(\widetilde{t}\,) + \mathbf{A}\widetilde{\mathbf{x}}(\widetilde{t}\,) + \mathbf{B}\big(\dot{\widetilde{\mathbf{x}}}(\widetilde{t}\,)\big) + \frac{\mu_g}{\mu_\beta}\,\mathbf{G}\big(\frac{\mu_\beta}{\sqrt{mk}}\,\dot{\widetilde{\mathbf{x}}}(\widetilde{t}\,)\big) \ni \mathbf{0}, \\ \widetilde{\mathbf{x}}(0) = \widetilde{\mathbf{x}}_0, \\ \dot{\widetilde{\mathbf{x}}}(0) = \widetilde{\mathbf{v}}_0, \end{cases}$$

with $\widetilde{\mathbf{x}}_0 = \frac{k}{\mu_\beta}\mathbf{x}_0$ and $\widetilde{\mathbf{v}}_0 = \frac{\sqrt{mk}}{\mu_\beta}\mathbf{v}_0$.

Proof. It is enough to check that $\dot{\mathbf{x}}(t) = \frac{x^*}{t^*}\frac{d\widetilde{\mathbf{x}}}{d\widetilde{t}}$ and to use that $\mathbf{B}(\theta\dot{\mathbf{x}}(t)) = \mathbf{B}(\dot{\mathbf{x}}(t))$ for any $\theta > 0$. ■

We come back to the proof of part i) of Theorem 5.2. In the following we shall identify $(\widetilde{\mathbf{P}}_N)$ with $(\mathbf{P}_N)$ if no confusion may arises. In view of Theorem 5.1 and Lemma 5.1, we have to prove that there exists $T_e \geq 0$ such that $\mathbf{x}(t) \equiv \mathbf{x}_\infty$ for any $t \geq T_e$. In what follows we shall adopt some notation similar to the introduced by Bamberger and Cabannes in [14]

$$\Delta_i(t) := (\mathbf{A}\mathbf{x}(t))_i \text{ and } \Delta_i^* := (\mathbf{A}\mathbf{x}_\infty)_i\,, \quad \text{for } i \in \{1, \dots, N\}.$$

We recall that, since $\mathbf{x}_\infty$ is an stationary point, we have $(\Delta_i^*)_{i=1}^N \in [-1,1]^N$. We need an auxiliary lemma describing the behavior of $\mathbf{x}(t)$ for large time. In the statement, the constants may depend on the initial data.

Lemma 5.2. 1) *Suppose that for some $i \in 1, \dots, N$, $|\Delta_i^*| < 1$. Then there exists $T_i \geq 0$ such that $\forall t \geq T_i$, $\dot{x}_i(t) = 0$.*

2) *If, for some $i \in 1, \dots, N$, $\Delta_i^* = 1$ (resp. $\Delta_i^* = -1$). Then there exists $T_i \geq 0$ such that $\forall t \geq T_i$, $\dot{x}_i(t) \leq 0$ (resp. $\dot{x}_i(t) \geq 0$).*

Proof. Let $0 < \delta \ll 1$ be fixed. By Theorem 5.1 we can find $t_0 \geq 0$ such that

$$\forall t \geq t_0, \quad |\Delta_i(t)| \leq (1 - 2\delta) \quad \text{and} \quad |g(\frac{\mu_\beta}{\sqrt{mk}}\,\dot{x}_i(t))| \leq \frac{\mu_\beta}{\mu_g}\,\delta. \tag{5.7}$$

If $\dot{x}_i(t_0) = 0$, we conclude that $x_i(t) \equiv x_i(t_0) = (x_\infty)_i$ for any $t \geq t_0$ since $\Delta_i(t) \in [-1,1]$ for any $t \geq t_0$. If not, let

$$T = \sup\big\{s \geq t_0,\ |\dot{x}_i(t)| > 0\ \forall t \in [t_0, s[\big\}.$$

Multiplying the ith component of $(\mathbf{P}_N)$ by $\dot{x}_i(t)$ and using (5.7) we obtain

$$\frac{1}{2}\frac{d}{dt}(|\dot{x}_i(t)|^2) + \delta|\dot{x}_i(t)| \leq 0 \quad \text{for a.e. } t \in [t_0, T[. \tag{5.8}$$

Dividing (5.8) by $|\dot{x}_i(t)|$ we get

$$\frac{d}{dt}(|\dot{x}_i(t)|) + \delta \leq 0 \quad \text{for a.e. } t \in [t_0, T[. \tag{5.9}$$

Integrating, we see that

$$\dot{x}_i\big(t_0 + \frac{|\dot{x}_i(t_0)|}{\delta}\big) = 0.$$

Thus $T < +\infty$ and we conclude, as before, that $x_i(t) \equiv x_i(T) = (x_\infty)_i$ for any $t \geq T$.

To prove part 2) we consider, again, $0 < \delta \ll 1$ and suppose that $\Delta_i^* = 1$ (the case $\Delta_i^* = -1$ is similar). By Theorem 5.1 we can find $t_0 \geq 0$ such that

$$\Delta_i(t) \geq \delta \quad \text{and} \quad |g(\frac{\mu_\beta}{\sqrt{mk}}\,\dot{x}_i(t))| \leq \frac{\mu_\beta}{\mu_g}\,\delta \quad \text{for a.e. } t \geq t_0. \tag{5.10}$$

Suppose that $\dot{x}_i(t_0) > 0$ and let

$$\tau = \sup\big\{s > t_0,\ \dot{x}_i(t) > 0 \quad \forall t \in [t_0, s[\big\}.$$

In $[t_0, \tau[$ we have

$$\ddot{x}_i(t) + \Delta_i(t) + 1 + \frac{\mu_g}{\mu_\beta} g(\frac{\mu_\beta}{\sqrt{mk}}\,\dot{x}_i(t)) = 0.$$

From (5.10), we get that $\ddot{x}_i(t) \leq -1$ in $[t_0, \tau[$ and by integration

$$\dot{x}_i(t) \leq \dot{x}_i(t_0) - (t - t_0) \quad \text{in } [t_0, \tau[.$$

Thus $\tau < +\infty$ and we conclude that we can find $T \geq t_0$ such that $\dot{x}_i(T) \leq 0$. Now suppose that there exists $t_1 > T$ such that $\dot{x}_i(t_1) > 0$. From the continuity of $\dot{x}_i$, there exists some interval $]t_2, t_3[$ with $t_2 > T$ and $\dot{x}_i(t_2) > 0$, where $\dot{x}_i$ is strictly increasing. In $]t_2, t_3[$ we have $\ddot{x}_i = -1 - \Delta_i - \frac{\mu_g}{\mu_\beta} g(\frac{\mu_\beta}{\sqrt{mk}}\,\dot{x}_i)$. Thus form the choice of δ, $\dot{x}$ is strictly decreasing in $]t_2, t_3[$, which is a contradiction. ■

Proof of Theorem 5.2 (continuation), proof of i). We set

$$I^+ = \{i \in \{1, \ldots, N\},\ \Delta_i^* = 1\} \quad \text{and} \quad I^- = \{i \in \{1, \ldots, N\},\ \Delta_i^* = -1\}.$$

In view of Lemma 5.1, we can find $T \geq 0$ such that for any $t \geq T$ we have that :

a) $\forall i \in \{1, \ldots, N\}, \quad g(\frac{\mu_\beta}{\sqrt{mk}}\,\dot{x}_i(t))\dot{x}_i(t) \leq 0,$

b) $\forall i \in I^+, \quad \dot{x}_i(t) \leq 0,$

c) $\forall i \in I^-, \quad \dot{x}_i(t) \geq 0,$

d) $\forall i \notin I^+ \cup I^-, \quad \dot{x}_i(t) = 0.$

We write the equations of $(\mathbf{P}_N)$ as

$$\ddot{x}_i(t) + \Delta_i(t) - \Delta_i^* + 1 + \beta(\dot{x}_i(t)) + \frac{\mu_g}{\mu_\beta}\, g(\frac{\mu_\beta}{\sqrt{mk}}\,\dot{x}_i(t)) \ni 0, \text{ for } i \in I^+, \tag{5.11}$$

(and analogy for $i \in I^-$). Multiplying by $\dot{x}_i(t)$ and summing over i, we get

$$\ddot{\mathbf{x}}(t) \cdot \dot{\mathbf{x}}(t) + \mathbf{A}(\mathbf{x}(t) - \mathbf{x}_\infty) \cdot \dot{\mathbf{x}}(t) + \frac{\mu_g}{\mu_\beta}\, \mathbf{G}(\frac{\mu_\beta}{\sqrt{mk}}\,\dot{\mathbf{x}}(t)) \cdot \dot{\mathbf{x}}(t) = 0, \forall t \geq T,$$

Integrating in time, we infer that for $t \geq T$,

$$\| \dot{\mathbf{x}}(t) \|^2 + \mathbf{A}(\mathbf{x}(t) - \mathbf{x}_\infty) \cdot (\mathbf{x}(t) - \mathbf{x}_\infty) \geq \| \dot{\mathbf{x}}(T) \|^2 + \mathbf{A}(\mathbf{x}(T) - \mathbf{x}_\infty) \cdot (\mathbf{x}(T) - \mathbf{x}_\infty) \geq 0$$

Letting $t \to +\infty$ we obtain $\| \dot{\mathbf{x}}(T) \|^2 + \mathbf{A}(\mathbf{x}(T) - \mathbf{x}_\infty) \cdot (\mathbf{x}(T) - \mathbf{x}_\infty) = \mathbf{0}$. Since $\mathbf{A}$ is a positive definite matrix, we conclude that $\mathbf{x}(T) = \mathbf{x}_\infty$ and thus $\mathbf{x}(t) = \mathbf{x}_\infty$ for any $t \geq T$.

Proof of ii). Assume now that $g(r) = \lambda r$ with $\lambda \geq \frac{2\sqrt{\lambda_1 mk}}{\mu_g}$. In order to construct a solution of $(\mathbf{P}_N)$ which does not stabilize in finite time we seek a particular solution of the vectorial linear ODE

$$\ddot{\mathbf{X}} + \mathbf{A}\mathbf{X} + \frac{\lambda \mu_g}{\sqrt{mk}}\,\dot{\mathbf{X}} = \mathbf{0}. \tag{5.12}$$

Since $\mathbf{A}$ is a symmetric definite positive matrix, we can find a matrix $\mathbf{P} \in \mathbb{R}^{N\times N}$ such that $\mathbf{A} = \mathbf{P}^T \mathrm{diag}(\lambda_1, \cdots, \lambda_N)\mathbf{P}$ with $0 < \lambda_1 \le \lambda_2 \le \ldots \le \lambda_N$ and $\mathbf{P}^T\mathbf{P} = \mathbf{I}$, the identity matrix. Writing $\mathbf{X} = \mathbf{P}^T\mathbf{Y}$, system (5.12) is equivalent to the system

$$\ddot{y}_i + \lambda_i y_i + \frac{\lambda\mu_g}{\sqrt{mk}}\,\dot{y}_i = 0 \quad \text{ for } i = 1, \ldots, N. \tag{5.13}$$

The equation $\ddot{y}_1 + \lambda_1 y_1 + \frac{\lambda\mu_g}{\sqrt{mk}}\,\dot{y}_1 = 0$ admits a solution $y_1(t)$ such that $\dot{y}_1(t) < 0$ for any $t \ge 0$ since $\lambda \ge \frac{2\sqrt{\lambda_1 mk}}{\mu_g}$. We define $\mathbf{Y}(t) = (y_1(t), 0, \ldots, 0)$ which satisfies (5.13). Then, $\mathbf{X}(t) := \mathbf{P}^T\mathbf{Y}(t)$ satisfies (5.12) and is such that $\dot{x}_i(t)$ has a constant sign and never vanishes or $\dot{x}_i(t) \equiv 0$. If we denote by $\mathbf{\Delta}^*$ the constant vector of $\mathbb{R}^N$ defined by $\Delta_i^* = \beta_0(\dot{x}_i)$, $i = 1, \ldots, N$, with $\beta_0(r) = \beta(r)$ if $r \ne 0$ and $\beta_0(0) = 0$, and consider $\mathbf{x}_\infty$ as the solution of $\mathbf{A}\mathbf{x}_\infty = -\mathbf{\Delta}^*$. Summing $\mathbf{X}$ and $\mathbf{x}_\infty$, we get a solution of $(\mathbf{P}_N)$ which never stops.

Proof of iii). We suppose $N = 1$ (and take $\mathbf{A} = 1$). The problem becomes

$$\ddot{x} + x + \beta(\dot{x}) + \frac{\mu_g}{\mu_\beta}\,g(\frac{\mu_\beta}{\sqrt{mk}}\,\dot{x}) \ni 0. \tag{5.14}$$

Firstly, suppose that $g'(0) < \frac{2\sqrt{mk}}{\mu_g}$. We want to prove that all solutions of (5.14) stabilize in finite time. In view of the previous steps, we only have to consider the case $|x(t)| \to 1$. By analogy, it is enough to consider the case $x(t) \to 1$. We know that there exists a time T such that $\dot{x}(t) \le 0$ and $|x(t) - 1| \ll 1$ for any $t \ge T$. If the process does not stop at a time T, then there exists a $t_0 \ge T$ such that $\dot{x}(t_0) < 0$. Let $\tau = \sup\{t \ge t_0,\ \dot{x}(t) < 0\}$. Since g is regular near 0 and $g'(0) < \frac{2\sqrt{mk}}{\mu_g}$ we know by Hartman's Theorem ([59]) that the point $(1, 0)$ is a *center* or a *focus* for the equation

$$\ddot{u} + u - 1 + \frac{\mu_g}{\mu_\beta}\,g(\frac{\mu_\beta}{\sqrt{mk}}\,\dot{u}) = 0. \tag{5.15}$$

Since $x(t)$ satisfies this equation in (t_0, τ), we deduce that $\tau < \infty$ and $x(\tau) < 1$ with $\dot{x}(\tau) = 0$, thus the process stops at time τ which contradicts that $x(t) \to 1$ as $t \to +\infty$. If we assume, now, that $g'(0) \ge \frac{2\sqrt{mk}}{\mu_g}$, since g is regular near 0, by Hartman's Theorem, the point $(1, 0)$ is a *node* for equation (5.15) and we can find a solution $u(t)$ such that $\dot{u}(t) < 0$ for any $t \ge 0$. Such solution is also a solution of (5.14) which does not stabilize in any finite time. ∎

Remark 5.4. Similar results also hold for other N-dimensional systems arising when the spatial discretization of the wave equation is taken by finite elements instead of finite differences.

5.3 The dichotomy for the damped wave equation

As an illustration of possible extensions of ii) of Theorem 5.2 to other dynamical systems, we consider the damped wave equation in a bounded regular open set $\Omega \subset \mathbb{R}^N$

$$u_{tt} - \Delta u + \beta(u_t) + \lambda u_t \ni 0 \quad \text{in } \Omega \times (0, +\infty), \tag{5.16}$$

with Dirichlet boundary conditions $u(., t) = 0$ on $\partial\Omega$ for $t \in (0, +\infty)$. Let us assume that $\lambda \geq 2\sqrt{\lambda_1}$, with λ_1 the first eigenvalue of the operator $u \to -\Delta u$ associated to homogeneous Dirichlet boundary conditions. Then we can find some solution of (5.16) which does not stabilize in any finite time and also some solution which stabilizes in a finite time. We construct the first type of solution in the form

$$u(x, t) = a(t)v(x) + \xi(x),$$

where v is a solution of the eigenvalue problem

$$\begin{cases} -\Delta v = \lambda_1 v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

such that $v > 0$ in Ω, the function ξ is defined as the solution of

$$\begin{cases} \Delta\xi = 1 & \text{in } \Omega, \\ \xi = 0 & \text{on } \partial\Omega, \end{cases}$$

and $a(t)$ solves the ODE

$$\ddot{a} + \lambda_1 a + \lambda\dot{a} = 0, \tag{5.17}$$

such that $\dot{a}(t) > 0$ for any $t > 0$ (which is possible since $\lambda \geq 2\sqrt{\lambda_1}$). Then, we get a solution which does not stabilize in any finite time.

By the contrary, if we choose $a(t)$ as a solution of (5.17) such that

$$\dot{a}(t) > 0 \quad \text{for } t \in [0, 1), \quad \dot{a}(1) = 0 \quad \text{and} \quad a(1) = K,$$

with $K = \frac{1}{\lambda_1 \|v\|_{L^\infty(\Omega)}}$ and take

$$u(x, t) = b(t)v(x) + \xi(x)$$

where

$$b(t) = \begin{cases} a(t) & \text{if } t \leq 1, \\ K & \text{otherwise}, \end{cases}$$

we get a solution which attains the stationary state $u_\infty(x) = Kv(x) + \xi(x)$ exactly at time $t = 1$. ■

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Résumé

Dans le Chapitre 1, nous calculons l'infimum d'une énergie comportant un poids mesurable, sur des classes d'applications à valeurs dans S^2 ayant des singularités prescrites. Nous montrons qu'une telle quantité induit une distance. Ceci nous permet de calculer dans le Chapitre 2, une énergie de type relaxée pour des applications $u : \Omega \subset \mathbb{R}^3 \to S^2$. La formule fait intervenir la longueur d'une connexion minimale associée à la distance obtenue au Chapitre 1, connectant les singularités topologiques de u. Dans le Chapitre 3, nous étudions le modèle physique d'un condensat de Bose-Einstein bidimensionnel en rotation. Nous estimons la vitesse critique de rotation pour avoir d tourbillons et nous déterminons leur position. Dans le Chapitre 4, nous étudions le comportement asymptotique des minimiseurs d'une énergie de Ginzburg-Landau avec un poids dépendant de ε et nous montrons un phénomène d'ancrage des singularités limites. Dans le Chapitre 5, nous présentons quelques résultats sur la stabilisation en temps fini de processus mécaniques où un frottement de Coulomb coexiste avec d'autres types de forces donnant lieu à des oscillations dans l'absence de frottement.

Mots-clés: singularités topologiques, connexion minimale, énergie relaxée, condensation de Bose-Einstein, fonctionnelle de Ginzburg-Landau, énergie renormalisée, frottement de Coulomb, stabilisation en temps fini

Abstract

In Chapter 1, we compute the infimum of an energy with measurable weight, over classes of S^2-valued maps with prescribed singularities. We prove that such quantity induces a distance. This result allows to compute in Chapter 2 a relaxed type energy for maps $u : \Omega \subset \mathbb{R}^3 \to S^2$. The explicit formula involves the length of a minimal connection relative to the distance defined in Chapter 1 connecting the topological singularities of u. In Chapter 3, we investigate the physical model for a two dimensional rotating Bose-Einstein condensate. We estimate the critical angular velocity for having d vortices and we determine their location. In Chapter 4, we study the asymptotic behavior of minimizers of a Ginzburg-Landau energy ε-depending weight and we prove a pinning effect on the limiting singularities. In Chapter 5, we present a set of results on the stabilization in a finite time of some mechanical processes where a Coulomb friction term coexists with other physical frameworks leading to oscillations in absence of friction.

Keywords: topological singularities, minimal connection, relaxed energy, Bose-Einstein condensation, Ginzburg-Landau functional, renormalized energy, Coulomb friction, stabilization in finite time