The relaxed energy for S^2-valued maps and measurable weights

L'énergie relaxée pour des applications à valeurs dans S^2 et poids mesurables

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Abstract. We compute explicitly a relaxed type energy for maps $u : \Omega \subset \mathbb{R}^3 \to S^2$. The explicit formula involves the length of a minimal connection relative to some specific distance connecting the topological singularities of u and associated to a measurable weight function. This result generalizes a previous result of F. Bethuel, H. Brezis et J.M. Coron.

Résumé. Nous calculons explicitement une énergie de type relaxée pour des applications $u : \Omega \subset \mathbb{R}^3 \to S^2$. La formule explicite fait intervenir la longueur d'une connexion minimale relative à une certaine distance, connectant les singularités topologiques de u et associée à une fonction de poids mesurable. Ce résultat généralise un résultat antérieur de F. Bethuel, H. Brezis and J.M. Coron.

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1 Introduction and Main Results

Let Ω be a smooth bounded and connected open set of $\mathbb{R}^3$ and let $w : \Omega \to \mathbb{R}$ be a measurable function such that

$$0 < \lambda \leq w \leq \Lambda \quad \text{a.e. in } \Omega \tag{1.1}$$

for some constant λ and Λ. We set $H^1_g\left(\Omega, S^2\right) = \left\{u \in H^1\left(\Omega, S^2\right),\, u = g \text{ on } \partial\Omega\right\}$, where $g : \partial\Omega \to S^2$ is a given smooth boundary data such that $\deg(g) = 0$. Our main goal in this paper is to obtain an explicit formula for the relaxed functional

$$E_w(u) = \operatorname{Inf}\left\{\liminf_{n\to+\infty}\int_\Omega |\nabla u_n(x)|^2 w(x)dx,\ u_n \in H^1_g(\Omega, S^2)\cap\mathcal{C}^1(\overline{\Omega}),\ u_n \rightharpoonup u \text{ weakly in } H^1\right\},$$

defined for $u \in H^1_g(\Omega, S^2)$. By a result of F. Bethuel (see [1]), $H^1_g(\Omega, S^2) \cap \mathcal{C}^1(\overline{\Omega})$ is sequentially dense for the weak topology in $H^1_g(\Omega, S^2)$ and then the functional E_w is well defined.

In [4], F. Bethuel, H. Brezis and J.M. Coron have proved that for $w \equiv 1$,

$$E_1(u) = \int_\Omega |\nabla u(x)|^2 dx + 8\pi L(u),$$

where $L(u)$ denotes the *length of a minimal connection* relative to the Euclidean geodesic distance d_Ω in $\overline{\Omega}$ connecting the singularities of u (see also M. Giaquinta, G. Modica, J. Souček [13]). If $u \in H^1_g(\Omega, S^2)$ is smooth on $\overline{\Omega}$ except at a finite number of points in Ω, the length of a minimal connection relative to d_Ω connecting the singularities of u is given by

$$L(u) = \operatorname*{Min}_{\sigma \in \mathcal{S}_K} \sum_{i=1}^{K} d_\Omega(P_i, N_{\sigma(i)})$$

where $(P_1, \ldots, P_K)$ and $(N_1, \ldots, N_K)$ are respectively the singularities of positive and negative degree counted according to their multiplicity (since $\deg(g) = 0$, the number of positive singularities is equal to the number of negative ones) and $\mathcal{S}_K$ denotes the set of all permutations of K indices. For the definition of $L(u)$ when u is arbitrary in $H^1_g(\Omega, S^2)$, we refer to (1.6)-(1.7) below. The notion of length of a minimal connection between singularities has its origin in [9]. We also refer to the results of J. Bourgain, H. Brezis, P. Mironescu [5] and H. Brezis, P. Mironescu, A.C. Ponce [10] for similar problems involving S^1-valued maps.

For $u \in H^1(\Omega, S^2)$, the vector field $D(u)$ first introduced in [9] and defined by

$$D(u) = \left(u \cdot \frac{\partial u}{\partial x_2} \wedge \frac{\partial u}{\partial x_3}, u \cdot \frac{\partial u}{\partial x_3} \wedge \frac{\partial u}{\partial x_1}, u \cdot \frac{\partial u}{\partial x_1} \wedge \frac{\partial u}{\partial x_2} \right) \tag{1.2}$$

plays a crucial role. Indeed, if u is smooth except at a finite number of points $(P_i, N_i)_{i=1}^K$ in Ω, then (see [9], Appendix B)

$$\operatorname{div} D(u) = 4\pi \sum_{i=1}^{K} (\delta_{P_i} - \delta_{N_i}) \quad \text{in } \mathcal{D}'(\Omega) \tag{1.3}$$

and if in addition $u_{|\partial\Omega} = g$, we have (since $\deg(g) = 0$, see [9], Section IV)

$$L(u) = \operatorname{Sup} \left\{ \sum_{i=1}^{K} (\zeta(P_i) - \zeta(N_i)) \right\} \tag{1.4}$$

where the supremum is taken over all functions $\zeta : \overline{\Omega} \to \mathbb{R}$ which are 1-Lipschitz with respect to distance d_Ω i.e., $|\zeta(x) - \zeta(y)| \leq d_\Omega(x, y)$. Note that for any real Lipschitz function ζ,

$$\sum_{i=1}^{K} \zeta(P_i) - \zeta(N_i) = \frac{1}{4\pi} \int_\Omega \operatorname{div} D(u)\, \zeta = -\frac{1}{4\pi} \int_\Omega D(u) \cdot \nabla\zeta + \frac{1}{4\pi} \int_{\partial\Omega} (D(u) \cdot \nu)\, \zeta \tag{1.5}$$

where ν denotes the outward normal to $\partial\Omega$. We recall that $D(u) \cdot \nu$ is equal to the 2×2 Jacobian determinant of u restricted to $\partial\Omega$ and then it only depends on g. In view of (1.4) and (1.5), $L(u)$ has been defined in [4] for $u \in H^1_g(\Omega, S^2)$ by

$$L(u) = \frac{1}{4\pi} \operatorname{Sup} \left\{ \langle T(u), \zeta \rangle,\ \zeta : \overline{\Omega} \to \mathbb{R} \text{ 1-Lipschitz with respect to } d_\Omega \right\} \tag{1.6}$$

where $T(u) \in \mathcal{D}'(\Omega)$ denotes the distribution defined by its action on real Lipschitz functions through the formula:

$$\langle T(u), \zeta \rangle = \int_\Omega D(u) \cdot \nabla \zeta \, - \int_{\partial\Omega} (D(u) \cdot \nu) \, \zeta \, . \tag{1.7}$$

In a previous paper [14], we have studied the following variational problem: given two distinct points P and N in Ω,

$$E_w(P, N) = \operatorname{Inf} \left\{ \int_\Omega |\nabla v(x)|^2 w(x) dx, \; v \in \mathcal{E}(P, N) \right\}$$

where

$$\mathcal{E}(P, N) = \left\{ v \in H^1(\Omega, S^2) \cap \mathcal{C}^1\left(\overline{\Omega} \setminus \{P, N\}\right), \; v = \text{const on } \partial\Omega, \; T(v) = 4\pi(\delta_P - \delta_N) \text{ in } \mathcal{D}'(\Omega) \right\}.$$

In the case $w \equiv 1$, H. Brezis, J.M. Coron and E. Lieb have shown that (see [9])

$$E_1(P, N) = 8\pi d_\Omega(P, N).$$

For an arbitrary function w, we have proved (see [14]) that $E_w(\cdot, \cdot)$ defines a distance function satisfying

$$8\pi\lambda \, d_\Omega(\cdot, \cdot) \leq E_w(\cdot, \cdot) \leq 8\pi\Lambda \, d_\Omega(\cdot, \cdot). \tag{1.8}$$

From (1.8), we infer that E_w extends to $\overline{\Omega} \times \overline{\Omega}$ into a distance on $\overline{\Omega}$. In what follows, we set for $x, y \in \overline{\Omega}$,

$$d_w(x, y) = \frac{1}{8\pi} \, E_w(x, y).$$

When w is continuous, we also have shown that the distance d_w can be characterized in the following way: for any $x, y \in \overline{\Omega}$,

$$d_w(x, y) = \operatorname{Min} \int_0^1 w(\gamma(t)) |\dot{\gamma}(t)| dt$$

where the minimum is taken over all Lipschitz curve $\gamma : [0, 1] \to \overline{\Omega}$ verifying $\gamma(0) = x$ and $\gamma(1) = y$. For an arbitrary measurable function w, the previous formula is meaningless since w is not well defined on curves but a similar characterization of d_w actually holds. We refer to [14] for more details. We also recall the general result in [14]:

Theorem 1.1. *Let $(P_i)_{i=1}^K$ and $(N_i)_{i=1}^K$ be two lists of points in Ω and consider*

$$\mathcal{E}\left((P_i, N_i)_{i=1}^K\right) = \Big\{ \, v \in H^1(\Omega, S^2) \cap \mathcal{C}^1\left(\overline{\Omega} \setminus \{(P_i, N_i)_{i=1}^K\}\right),$$
$$v = \text{const } \textit{on } \partial\Omega \textit{ and } T(v) = 4\pi \sum_{i=1}^K \delta_{P_i} - \delta_{N_i} \textit{ in } \mathcal{D}'(\Omega) \Big\}.$$

Then we have

$$\operatorname{Inf} \left\{ \int_\Omega |\nabla v(x)|^2 w(x) dx, \; v \in \mathcal{E}\left((P_i, N_i)_{i=1}^K\right) \right\} = 8\pi L_w$$

where L_w is the length of a minimal connection relative to distance d_w connecting the points (P_i) and (N_i) i.e.,

$$L_w = \operatorname*{Min}_{\sigma \in \mathcal{S}_K} \sum_{i=1}^K d_w(P_i, N_{\sigma(i)}).$$

By analogy with the case $w \equiv 1$, we define for $u \in H^1_g(\Omega, S^2)$,

$$L_w(u) = \frac{1}{4\pi} \operatorname{Sup} \left\{ \langle T(u), \zeta \rangle,\ \zeta : \overline{\Omega} \to \mathbb{R} \text{ 1-Lipschitz with respect to } d_w \right\}$$

(note that any real function ζ which is 1-Lipschitz with respect to d_w, is a Lipschitz function with respect to d_Ω since d_w is strongly equivalent to d_Ω and then $\langle T(u), \zeta \rangle$ is well defined). When u is smooth except at a finite number of points $(P_i, N_i)_{i=1}^K$ in Ω, it follows as in [9] that $L_w(u)$ is equal to the length of a minimal connection relative to distance d_w connecting the points (P_i) and (N_i). Our main result is the following.

Theorem 1.2. *For any $u \in H^1_g(\Omega, S^2)$, we have*

$$E_w(u) = \int_\Omega |\nabla u(x)|^2 w(x) dx + 8\pi L_w(u) .$$

The proof of Theorem 1.2 is presented in Section 3 and is based on a method similar to the one used in [4] and on a *Dipole Removing Technique* exposed in the next section. This technique is mostly inspired from [1] but involves some tools developed in [14] in order to treat the problem for a non smooth function w.

In Section 4, we prove a stability property of E_w. More precisely, we give some conditions on a sequence $(w_n)_{n\in\mathbb{N}}$ under which one can conclude that the sequence of functionals $(E_{w_n})_{n\in\mathbb{N}}$ converges pointwise to E_w on $H^1_g(\Omega, S^2)$. The results are obtained using previous ones in [14]. In Section 5, we present similar results for a relaxed type functional in which we do not prescribed any boundary data.

Throughout the paper, a sequence of smooth mollifiers means any sequence $(\rho_n)_{n\in\mathbb{N}}$ satisfying

$$\rho_n \in \mathcal{C}^\infty(\mathbb{R}^3, \mathbb{R}), \quad \operatorname{Supp} \rho_n \subset B_{1/n}, \quad \int_{\mathbb{R}^3} \rho_n = 1, \quad \rho_n \geq 0 \text{ on } \mathbb{R}^3.$$

2 The Dipole Removing Technique

In this section, we first give a technical result which will be used for the *dipole removing technique* in Section 2.2.

2.1 Preliminaries

Let α and β be two distinct points in Ω. We denote by $p_{\alpha,\beta}(\xi)$ the projection of $\xi \in \mathbb{R}^3$ on the straight line passing by α and β and $r_{\alpha,\beta}(\xi) = \operatorname{dist}(x, [\alpha, \beta])$, where "dist" denotes the Euclidean distance in $\mathbb{R}^3$. For $m \in \mathbb{N}^*$, we set

$$a_m^{\alpha,\beta} = \frac{|\alpha - \beta|}{m} \quad \text{and} \quad s_j^{\alpha,\beta} = j\, a_m^{\alpha,\beta} \quad \text{for } j = 0, \dots, m.$$

For $\xi \in \mathbb{R}^3$ such that $p_{\alpha,\beta}(\xi) \in [\alpha, \beta]$, we define

$$h_m^{\alpha,\beta}(\xi) = \min_{0 \leq j \leq m} \left| |p_{\alpha,\beta}(\xi) - \alpha| - s_j^{\alpha,\beta} \right|,$$

and we set

$$\Theta_m([\alpha,\beta]) = \left\{\xi \in \mathbb{R}^3,\ p_{\alpha,\beta}(\xi) \in [\alpha,\beta] \text{ and } r_{\alpha,\beta}(\xi) \leq a_m^{\alpha,\beta} h_m^{\alpha,\beta}(\xi)\right\}.$$

For two points x and y in Ω, we consider the class $\mathcal{Q}(x,y)$ of all finite collections of segments $\mathcal{F} = ([\alpha_k,\beta_k])_{k=1}^{n(\mathcal{F})}$ such that $\beta_k = \alpha_{k+1}$, $\alpha_1 = x$, $\beta_{n(\mathcal{F})} = y$, $[\alpha_k,\beta_k] \subset \Omega$ and $\alpha_k \neq \beta_k$. We define the "length" of an element $\mathcal{F} \in \mathcal{Q}(x,y)$ by

$$\bar{\ell}_w(\mathcal{F}) = \sum_{k=1}^{n(\mathcal{F})} \liminf_{m\to+\infty} \frac{1}{\pi} \int_{\Theta_m([\alpha_k,\beta_k])\cap\Omega} \varepsilon_{\alpha_k,\beta_k}^m(\xi)\, w(\xi)\, d\xi$$

with

$$\varepsilon_{\alpha_k,\beta_k}^m(\xi) = \frac{\left(h_m^{\alpha_k,\beta_k}(\xi)\right)^2 \left(a_m^{\alpha_k,\beta_k}\right)^4}{\left(\left(h_m^{\alpha_k,\beta_k}(\xi)\right)^2 \left(a_m^{\alpha_k,\beta_k}\right)^4 + r_{\alpha_k,\beta_k}^2(\xi)\right)^2}.$$

Lemma 2.1. *Let $\mathbb{P}$ be a finite collection of distinct points in Ω or $\mathbb{P} = \emptyset$. For any distinct points x_0, y_0 in $\Omega\setminus\mathbb{P}$ and $\delta>0$, there exists $\mathcal{F}_\delta = ([\alpha_1,\beta_1],\ldots,[\alpha_n,\beta_n]) \in \mathcal{Q}(x_0,y_0)$ such that $(\mathbb{P}\cup\{y_0\}) \cap \left(\cup_{k=1}^{n-1}[\alpha_k,\beta_k] \cup [\alpha_n,\beta_n[\right) = \emptyset$ and*

$$\bar{\ell}_w(\mathcal{F}) \leq d_w(x_0,y_0) + \delta.$$

Proof. Step 1. Assume that w is smooth on Ω. We are going to prove that for every element $\mathcal{F} = ([\alpha_1,\beta_1],\ldots,[\alpha_n,\beta_n]) \in \mathcal{Q}(x,y)$, we have

$$\bar{\ell}_w(\mathcal{F}) = \int_{\cup_{k=1}^n [\alpha_k,\beta_k]} w(s)ds.$$

It suffices to prove that for any distinct points $\alpha,\beta \in \Omega$,

$$\lim_{m\to+\infty} \frac{1}{\pi} \int_{\Theta_m([\alpha,\beta])\cap\Omega} \varepsilon_k^m(\xi)\, w(\xi)\, d\xi = \int_{[\alpha,\beta]} w(s)ds. \tag{2.1}$$

Without loss of generality, we may assume that $[\alpha,\beta] = \{(0,0)\}\times[0,R]$ and we drop the indices α and β for simplicity. We set for $j = 0,\ldots,m-1$,

$$C_m^{j+} = \left\{\xi = (\xi_1,\xi_2,\xi_3) \in \Theta_m([\alpha,\beta]),\ \xi_3 \in \left[s_j, s_j + \frac{a_m}{2}\right]\right\},$$

and for $j = 1,\ldots,m$,

$$C_m^{j-} = \left\{\xi = (\xi_1,\xi_2,\xi_3) \in \Theta_m([\alpha,\beta]),\ \xi_3 \in \left[s_j - \frac{a_m}{2}, s_j\right]\right\}.$$

For $\xi \in C_m^{j+} \cup C_m^{j-}$, we have $h_m(\xi) = |\xi_3 - s_j|$ and we get that for m large enough,

$$\int_{\Theta_m([\alpha,\beta])\cap\Omega} \varepsilon_k^m(\xi)\, w(\xi)\, d\xi = \sum_{j=0}^{m-1} I_m^{j+} + \sum_{j=1}^{m} I_m^{j-} \tag{2.2}$$

with

$$I_m^{j+} = \int_{C_m^{j+}} \frac{|\xi_3 - s_j|^2 a_m^4 w(\xi)}{\left(|\xi_3 - s_j|^2 a_m^4 + r^2(\xi)\right)^2} d\xi \quad \text{for } j = 0, \ldots, m-1,$$

$$I_m^{j-} = \int_{C_m^{j-}} \frac{|\xi_3 - s_j|^2 a_m^4 w(\xi)}{\left(|\xi_3 - s_j|^2 a_m^4 + r^2(\xi)\right)^2} d\xi \quad \text{for } j = 1, \ldots, m.$$

Using the change of variable $z_1 = \frac{\xi_1}{|\xi_3 - s_j|}$, $z_2 = \frac{\xi_2}{|\xi_3 - s_j|}$ and $z_3 = \xi_3$, we derive that

$$\begin{aligned}
I_m^{j+} &= \int_{s_j}^{s_j + \frac{a_m}{2}} \left(\int_{B_{a_m}(0)} \frac{a_m^4 w(|z_3 - s_j| z_1, |z_3 - s_j| z_2, z_3)}{(a_m^4 + z_1^2 + z_2^2)^2} dz_1 dz_2 \right) dz_3 \\
&= \int_{s_j}^{s_j + \frac{a_m}{2}} (w(0,0,z_3) + \mathcal{O}(a_m)) \left(\int_{B_{a_m}(0)} \frac{a_m^4}{(a_m^4 + z_1^2 + z_2^2)^2} dz_1 dz_2 \right) dz_3 \\
&= \pi \int_{s_j}^{s_j + \frac{a_m}{2}} w(0,0,z_3) dz_3 + \mathcal{O}(a_m^2).
\end{aligned}$$

By similar computations we get that

$$I_m^{j-} = \pi \int_{s_j - \frac{a_m}{2}}^{s_j} w(0,0,z_3) dz_3 + \mathcal{O}(a_m^2).$$

Combining this equalities with (2.2), we obtain that

$$\int_{\Theta_m([\alpha,\beta]) \cap \Omega} \varepsilon_k^m(\xi) w(\xi) d\xi = \pi \int_0^R w(0,0,z_3) dz_3 + \mathcal{O}(a_m)$$

which ends the proof of (2.1).

Step 2. We fix two distinct points $x_0, y_0 \in \Omega \setminus \mathbb{P}$. For any points x, y in $\Omega \setminus (\mathbb{P} \cup \{y_0\})$, let $\mathcal{Q}'(x,y)$ be the class of elements $\mathcal{F} = ([\alpha_1, \beta_1], \ldots, [\alpha_n, \beta_n]) \in \mathcal{Q}(x,y)$ such that

$$\cup_{k=1}^n [\alpha_k, \beta_k] \subset \Omega \setminus (\mathbb{P} \cup \{y_0\}).$$

We consider the function $\mathcal{D}_w : \Omega \setminus (\mathbb{P} \cup \{y_0\}) \times \Omega \setminus (\mathbb{P} \cup \{y_0\}) \to \mathbb{R}_+$ defined by

$$\mathcal{D}_w(x,y) = \operatorname*{Inf}_{\mathcal{F} \in \mathcal{Q}'(x,y)} \overline{\ell}(\mathcal{F}).$$

We are going to show that $\mathcal{D}_w$ defines a distance function which can be extended to $\overline{\Omega} \times \overline{\Omega}$. Let $x, y \in \Omega \setminus (\mathbb{P} \cup \{y_0\})$ and let $\mathcal{F} = ([\alpha_1, \beta_1], \ldots, [\alpha_n, \beta_n])$ be an element of $\mathcal{Q}'(x,y)$. Assumption (1.1) and similar computations to those in Step 1 lead to

$$\lambda \sum_{k=1}^n |\alpha_k - \beta_k| \leq \overline{\ell}_w(\mathcal{F}) \leq \Lambda \sum_{k=1}^n |\alpha_k - \beta_k|.$$

Taking the infimum over all $\mathcal{F} \in \mathcal{Q}'(x,y)$, we infer that

$$\lambda\, d_\Omega(x,y) \leq \mathcal{D}_w(x,y) \leq \Lambda\, d_\Omega(x,y). \tag{2.3}$$

From (2.3), we deduce that $\mathcal{D}_w(x,y)=0$ if and only if $x=y$. Let us now prove that $\mathcal{D}_w$ is symmetric. Let $x,y\in\Omega\setminus(\mathbb{P}\cup\{y_0\})$ and $\delta>0$ arbitrary small. By definition, we can find $\mathcal{F}_\delta=([\alpha_1,\beta_2],\dots,[\alpha_n,\beta_n])$ in $\mathcal{Q}'(x,y)$ satisfying

$$\overline{\ell}_w\left(\mathcal{F}_\delta\right)\le\mathcal{D}_w(x,y)+\delta.$$

Then for $\mathcal{F}'_\delta=([\beta_n,\alpha_n],\dots,[\beta_1,\alpha_1])\in\mathcal{Q}'(y,x)$, we have

$$\mathcal{D}_w(y,x)\le\overline{\ell}_w\left(\mathcal{F}'_\delta\right)=\overline{\ell}_w\left(\mathcal{F}_\delta\right)\le\mathcal{D}_w(x,y)+\delta.$$

Since δ is arbitrary, we obtain $\mathcal{D}_w(y,x)\le\mathcal{D}_w(x,y)$ and we conclude that $\mathcal{D}_w(y,x)=\mathcal{D}_w(x,y)$ inverting the roles of x and y. The triangle inequality is immediate since the juxtaposition of $\mathcal{F}_1\in\mathcal{Q}'(x,z)$ with $\mathcal{F}_2\in\mathcal{Q}'(z,y)$ is an element of $\mathcal{Q}'(x,y)$. Hence $\mathcal{D}_w$ defines a distance on $\Omega\setminus(\mathbb{P}\cup\{y_0\})$ verifying (2.3). Therefore distance $\mathcal{D}_w$ extends uniquely to $\overline{\Omega}\times\overline{\Omega}$ into a distance function that we still denote by $\mathcal{D}_w$. By continuity, $\mathcal{D}_w$ satisfies (2.3) for any $x,y\in\overline{\Omega}$.

Step 3. We consider the function $\zeta:\overline{\Omega}\to\mathbb{R}$ defined by

$$\zeta(x)=\mathcal{D}_w(x,x_0).$$

Note that function ζ is 1-Lipschitz with respect to distance $\mathcal{D}_w$ and therefore Λ-Lipschitz with respect to the Euclidean geodesic distance on $\overline{\Omega}$ by (2.3). We fix an arbitrary point $z_0\in\Omega\setminus(\mathbb{P}\cup\{y_0\})$ and some $R>0$ such that $B_{3R}(z_0)\subset\Omega\setminus(\mathbb{P}\cup\{y_0\})$. Let $(\rho_n)_{n\in\mathbb{N}}$ be a sequence of smooth mollifiers. For $n>1/R$, we consider the smooth function $\zeta_n=\rho_n*\zeta:B_R(z_0)\to\mathbb{R}$. We write

$$\zeta_n(x)=\int_{B_{1/n}}\rho_n(-z)\zeta(x+z)dz$$

and therefore for all $x,y\in B_R(z_0)$,

$$\begin{aligned}|\zeta_n(x)-\zeta_n(y)|&\le\int_{B_{1/n}}\rho_n(-z)\left|\zeta(x+z)-\zeta(y+z)\right|dz\\&\le\int_{B_{1/n}}\rho_n(-z)\,\mathcal{D}_w(x+z,y+z)dz\\&\le\int_{B_{1/n}}\rho_n(-z)\,\overline{\ell}_w\left([x+z,y+z]\right)dz.\end{aligned}$$

We remark that $\Theta_m\left([x+z,y+z]\right)=z+\Theta_m\left([x,y]\right)$. For m large enough $z+\Theta_m\left([x,y]\right)\subset B_{3R}(z_0)$ and then for any vector $\xi\in\Theta_m\left([x,y]\right)$, we have $\varepsilon^m_{x+z,y+z}\left(\xi+z\right)=\varepsilon^m_{x,y}\left(\xi\right)$. Hence we obtain for all $z\in B_{1/n}(0)$ and m sufficiently large,

$$\overline{\ell}_w\left([x+z,y+z]\right)=\liminf_{m\to+\infty}\frac{1}{\pi}\int_{\Theta_m([x,y])}\varepsilon^m_{x,y}\left(\xi\right)w\left(\xi+z\right)\,d\xi.$$

Using Fatou's lemma, we get that

$$\begin{aligned}|\zeta_n(x)-\zeta_n(y)|&\le\int_{B_{1/n}}\rho_n(-z)\left(\liminf_{m\to+\infty}\frac{1}{\pi}\int_{\Theta_m([x,y])}\varepsilon^m_{x,y}\left(\xi\right)w(\xi+z)d\xi\right)dz\\&\le\liminf_{m\to+\infty}\frac{1}{\pi}\int_{B_{1/n}}\int_{\Theta_m([x,y])}\rho_n(-z)\,\varepsilon^m_{x,y}\left(\xi\right)w(\xi+z)d\xi dz.\end{aligned}$$

For each $m \in \mathbb{N}$ sufficiently large we have

$$\frac{1}{\pi}\int_{B_{1/n}}\int_{\Theta_m([x,y])}\rho_n(-z)\,\varepsilon^m_{x,y}(\xi)\,w(\xi+z)d\xi dz=\frac{1}{\pi}\int_{\Theta_m([x,y])}\varepsilon^m_{x,y}(\xi)\,\rho_n*w(\xi)d\xi,$$

and since $\rho_n * w$ is smooth, we obtain as in Step 1,

$$\frac{1}{\pi}\int_{\Theta_m([x,y])}\varepsilon^m_{x,y}(\xi)\,\rho_n*w(\xi)d\xi\to\int_{[x,y]}\rho_n*w(s)ds\quad\text{as } m\to+\infty.$$

Thus for each $x, y \in B_R(z_0)$ we have

$$|\zeta_n(x)-\zeta_n(y)|\le\int_{[x,y]}\rho_n*w(s)ds.$$

Then for $x \in B_R(z_0)$, $h \in S^2$ fixed and $\delta > 0$ small, we infer that

$$\frac{|\zeta_n(x+\delta h)-\zeta_n(x)|}{\delta}\le\frac{1}{\delta}\int_{[x,x+\delta h]}\rho_n*w(s)ds\underset{\delta\to0^+}{\to}\rho_n*w(x)$$

and we conclude, letting $\delta \to 0$, that $|\nabla\zeta_n(x)\cdot h| \le \rho_n * w(x)$ for each $x \in B_R(z_0)$ and $h \in S^2$ which implies that $|\nabla\zeta_n| \le \rho_n*w$ on $B_R(z_0)$. Since $\nabla\zeta_n \to \nabla\zeta$ and $\rho_n*w \to w$ a.e. on $B_R(z_0)$ as $n \to +\infty$, we deduce that $|\nabla\zeta| \le w$ a.e. on $B_R(z_0)$. Since z_0 is arbitrary in $\Omega \setminus (\mathbb{P} \cup \{y_0\})$, we derive

$$|\nabla\zeta|\le w\quad\text{a.e. on }\Omega.$$

By Proposition 2.3. in [14], it follows that $|\zeta(x) - \zeta(y)| \le d_w(x, y)$ for any $x, y \in \overline{\Omega}$ and in particular, we obtain choosing $y = x_0$,

$$\mathcal{D}_w(x,x_0)\le d_w(x,x_0)\quad\text{for all } x\in\overline{\Omega}.$$

Step 4. End of the Proof. Let $\delta > 0$ be given. We choose some $\tilde{y}_0 \in \Omega \setminus (\mathbb{P} \cup \{y_0\})$ such that $[\tilde{y}_0, y_0] \subset \Omega \setminus \mathbb{P}$ and $|\tilde{y}_0 - y_0| \le \frac{\delta}{3\Lambda}$. By the previous step, we can find an element $\mathcal{F}' = ([\alpha_1, \beta_1], \dots, [\alpha_n, \beta_n]) \in \mathcal{Q}'(x_0, \tilde{y}_0)$ verifying

$$\bar{\ell}_w\left(\mathcal{F}'\right)\le d_w(x_0,\tilde{y}_0)+\frac{\delta}{3}.$$

Then we consider $\mathcal{F} = ([\alpha_1, \beta_1], \dots, [\alpha_n, \beta_n], [\tilde{y}_0, y_0]) \in \mathcal{Q}(x_0, y_0)$. We have

$$\begin{aligned}\bar{\ell}_w(\mathcal{F})\le\bar{\ell}_w\left(\mathcal{F}'\right)+\Lambda|\tilde{y}_0-y_0|&\le d_w(x_0,\tilde{y}_0)+\frac{2\delta}{3}\\&\le d_w(x_0,y_0)+d_w(y_0,\tilde{y}_0)+\frac{2\delta}{3}\\&\le d_w(x_0,y_0)+\delta\end{aligned}$$

and then $\mathcal{F}$ satisfies the requirement. ∎

2.2 The Dipole Removing Technique

We first present the *dipole removing technique* for a simple dipole. We then treat the case of several point singularities.

Lemma 2.2. *Let P and N be two distinct points in Ω and consider $u \in H^1(\Omega, S^2) \cap \mathcal{C}^1\left(\overline{\Omega} \setminus \{P, N\}\right)$ with $\deg(u, P) = +1$ and $\deg(u, N) = -1$. Let $\mathcal{F} = ([\alpha_1, \beta_1], \ldots, [\alpha_n, \beta_n])$ be an element of $\mathcal{Q}(P, N)$ such that $N \notin \cup_{k=1}^{n-1}[\alpha_k, \beta_k] \cup [\alpha_n, \beta_n[$. Then for any $\delta > 0$ small enough, there exists a map $u_\delta \in \mathcal{C}^1\left(\overline{\Omega}, S^2\right)$ such that:*

$$\int_\Omega |\nabla u_\delta(x)|^2 w(x)dx \leq \int_\Omega |\nabla u(x)|^2 w(x)dx + 8\pi\bar{\ell}_w(\mathcal{F}) + \delta$$

and u_δ coincides with u outside a δ-neighborhood of $\cup_{k=1}^n[\alpha_k, \beta_k]$ included in Ω.

Proof. Let $\mathcal{F} = ([\alpha_1, \beta_1], \ldots, [\alpha_n, \beta_n]) \in \mathcal{Q}(P, N)$ such that $N \notin \cup_{k=1}^{n-1}[\alpha_k, \beta_k] \cup [\alpha_n, \beta_n[$ and fix some $\delta > 0$ small. We proceed in several steps.

Step 1. We consider a small $0 < r_0 < \delta$ verifying $B_{r_0}(\alpha_1) \subset \Omega \setminus \{N\}$. By Lemma A.1 in [1], we can find $v \in \mathcal{C}^1\left(\overline{\Omega} \setminus \{\alpha_1, N\}, S^2\right) \cap H^1(\Omega)$ (recall that $\alpha_1 = P$) satisfying

$$v(x) = \begin{cases} u(x) & \text{on } \Omega \setminus B_{r_0}(\alpha_1), \\ R\left(\dfrac{x - \alpha_1}{|x - \alpha_1|}\right) & \text{on } B_{r_0}(\alpha_1), \end{cases} \tag{2.4}$$

for some rotation R and

$$\int_\Omega |\nabla v(x)|^2 w(x)dx \leq \int_\Omega |\nabla u(x)|^2 w(x)dx + \delta. \tag{2.5}$$

Let $W = \{x \in \mathbb{R}^3,\ \mathrm{dist}(x, [\alpha_1, \beta_1]) < \delta\}$. For δ small enough, we have $\overline{W} \subset \Omega \setminus \{N\}$. We set $d = |\alpha_1 - \beta_1|$. We choose normal coordinates such that $\alpha_1 = (0, 0, 0)$ and $\beta_1 = (0, 0, d)$. Let $0 < r < \frac{r_0}{2}$. Since v is smooth on $W \setminus B_{r_0}(\alpha_1)$, we can find a constant $\sigma(r)$ such that $|\nabla v| \leq \sigma(r)$ on $W \setminus B_{r_0}(\alpha_1)$. For $m \in \mathbb{N}^*$, we consider

$$K_m = \left[-\frac{a_m^{\alpha_1,\beta_1}}{2}, \frac{a_m^{\alpha_1,\beta_1}}{2}\right]^2 \times \left[-\frac{a_m^{\alpha_1,\beta_1}}{2}, d + \frac{a_m^{\alpha_1,\beta_1}}{2}\right].$$

For m large enough, we have $\Theta_m([\alpha_1, \beta_1]) \subset K_m \subset W$. As in [1], we are going to construct in the next step a map $v_1 \in \mathcal{C}^1\left(\overline{W} \setminus \{\beta_1\}, S^2\right) \cap H^1(W)$ verifying $v_1 = v$ in a neighborhood of ∂W and $\deg(v_1, \beta_1) = +1$. For simplicity, we drop the indices α_1 and β_1.

Step 2. We divide K_m in $m + 1$ cubes Q_m^j defined by

$$Q_m^j = \left[-\frac{a_m}{2}, \frac{a_m}{2}\right]^2 \times \left[\left(j - \frac{1}{2}\right)a_m, \left(j + \frac{1}{2}\right)a_m\right] \quad \text{for } j = 0, \ldots, m.$$

Arguing as in [1], we infer from (2.4) that

$$\sum_{j=0}^m \int_{\partial Q_m^j} |\nabla v|^2 \leq C\left(\frac{r}{a_m} + m\,\sigma(r)^2 a_m^2\right). \tag{2.6}$$

We are going to make use of a map $\omega_m : B^2_{a_m}(0) \subset \mathbb{R}^2 \to S^2$ defined by

$$\omega_m(x_1, x_2) = \frac{2a_m^2}{a_m^4 + x_1^2 + x_2^2}\left(x_1, x_2, -a_m^2\right) + (0,0,1)$$

(ω_m was first introduced in [8] and we refer to the proof of Lemma 2 in [8] for its main properties). For $j = 1, \ldots, m$, we choose an orthonormal direct basis (e_1^j, e_2^j, e_3^j) of $\mathbb{R}^3$ such that

$$v(0, 0, (j - 1/2)a_m) = (0, 0, 1) \quad \text{in the basis } (e_1^j, e_2^j, e_3^j),$$

and we define the map $v_1^m : \cup_{j=0}^m \partial Q_m^j \to S^2$ by

1) for $(x_1, x_2, x_3) \in (\cup_{j=0}^m \partial Q_m^j) \setminus (\cup_{j=1}^m B^2_{a_m^2}(0) \times \{(j - 1/2)a_m\})$,

$$v_1^m(x_1, x_2, x_3) = v(x_1, x_2, x_3),$$

2) for $j = 1, \ldots, m$ and $(x_1, x_2, x_3) \in B^2_{\frac{a_m^2}{2}}(0) \times \{(j - 1/2)a_m\}$,

$$v_1^m(x_1, x_2, x_3) = \omega_m\left(\frac{2x_1}{a_m}, \frac{2x_2}{a_m}\right) \quad \text{in the basis } (e_1^j, e_2^j, e_3^j),$$

3) for $j = 1, \ldots, m$, for $(x_1, x_2, x_3) \in (B^2_{a_m^2}(0) \setminus B^2_{\frac{a_m^2}{2}}(0)) \times \{(j - 1/2)a_m\}$ and using cylindrical coordinates $(x_1, x_2, x_3) = (\rho\cos\theta, \rho\sin\theta, z)$,

$$v_1^m(x_1, x_2, x_3) = \left(A_1\rho + B_1, A_2\rho + B_2, \sqrt{1 - (A_1\rho + B_1)^2 - (A_2\rho + B_2)^2}\right)$$

in the basis (e_1^j, e_2^j, e_3^j), where A_1, A_2, B_1, B_2 are determined to make v_1^m continuous. More precisely, if we write $v = v_1 e_1^j + v_2 e_2^j + v_3 e_3^j$ then

$$\begin{cases} a_m^2 A_i(\theta) + B_i(\theta) = v_i(a_m^2\cos\theta, a_m^2\sin\theta, (j - 1/2)a_m) \quad & i = 1, 2, \\[2ex] \dfrac{a_m^2}{2} A_1(\theta) + B_1(\theta) = \dfrac{2a_m^3}{a_m^4 + a_m^2}\cos\theta, \\[2ex] \dfrac{a_m^2}{2} A_2(\theta) + B_2(\theta) = \dfrac{2a_m^3}{a_m^4 + a_m^2}\sin\theta. \end{cases}$$

The map v_1^m satisfies by construction $v_1^m = v$ on ∂K_m. Moreover, it follows exactly as in the proof of Lemma 2 in [1] that $\deg(v_1^m, \partial Q_m^j) = 0$ for $j = 0, \ldots, m-1$ and $\deg(v_1^m, \partial Q_m^m) = +1$. Then we extend v_1^m on each cube Q_m^j by setting

$$v_1^m(x) = v_1^m\left(\frac{a_m(x - b_j)}{2\|x - b_j\|_\infty} + b_j\right) \quad \text{on } Q_m^j \text{ for } j = 0, \ldots, m,$$

where $b_j = (0, 0, s_j)$ is the barycenter of Q_m^j and $\|x - b_j\|_\infty = \max(|x_1|, |x_2|, |x_3 - s_j|)$. We easily check that $v_1^m \in H^1(K_m, S^2)$, $v_1^m = v$ on ∂K_m, v_1^m is continuous except at the points b_j and Lipschitz continuous outside any small neighborhood of the points b_j. We also get that

$$\deg(v_1^m, b_m) = +1 \quad \text{and} \quad \deg(v_1^m, b_j) = 0 \quad \text{for } j = 0, \ldots, m-1 \,. \tag{2.7}$$

We remark that if we set

$$D_m^j = B^2_{\frac{a_m^2}{2}}(0) \times \{(j-1/2)a_m\} \cup B^2_{\frac{a_m^2}{2}}(0) \times \{(j+1/2)a_m\} \quad \text{for } j=1,\dots,m-1,$$

$$D_m^0 = B^2_{\frac{a_m^2}{2}}(0) \times \{1/2\, a_m\} \quad \text{and} \quad D_m^m = B^2_{\frac{a_m^2}{2}}(0) \times \{(m-1/2)a_m\},$$

then we have

$$\bigcup_{j=0}^{m} \left\{ x \in Q_m^j,\ \frac{a_m(x-b_j)}{2\|x-b_j\|_\infty} + b_j \in D_m^j \text{ if } x \neq b_j \text{ or } x = b_j \text{ otherwise} \right\} = \Theta_m([\alpha_1,\beta_1])$$

and if $x \in Q_m^j \cap \Theta_m([\alpha_1,\beta_1])$ for some $j \in \{0,\dots,m\}$ then

$$h_m(x) = |x_3 - s_j| = \|x-b_j\|_\infty \quad \text{and} \quad r(x) = \sqrt{x_1^2+x_2^2}\,. \tag{2.8}$$

Some classical computations (see [1] and [8]) lead to, for $j=0,\dots,m$,

$$\int_{(\partial Q_m^j)\setminus D_m^j} |\nabla v_1^m|^2 \le \int_{\partial Q_m^j} |\nabla v|^2 + \mathcal{O}(a_m^2)$$

and therefore

$$\int_{Q_m^j \setminus \Theta_m([\alpha_1,\beta_1])} |\nabla v_1^m(x)|^2 w(x)dx \le C_1 \Lambda\, a_m \int_{\partial Q_m^j} |\nabla v|^2 + C_2 \Lambda\, a_m^3.$$

Adding these inequalities for $j=0,\dots,m$ and combining with (2.6) we obtain

$$\int_{K_m \setminus \Theta_m([\alpha_1,\beta_1])} |\nabla v_1^m(x)|^2 w(x)dx \le C\Lambda\left(r + m\sigma(r)^2 a_m^3 + a_m^2\right). \tag{2.9}$$

For $x \in Q_m^j \cap \Theta_m([\alpha_1,\beta_1])$ for some $j\in\{0,\dots,m\}$, we have

$$v_1^m(x) = \begin{cases} \omega_m\left(\dfrac{x_1}{|x_3-s_j|}, \dfrac{x_2}{|x_3-s_j|}\right) & \text{in the basis } (e_1^{j+1}, e_2^{j+1}, e_3^{j+1}) \text{ if } x_3 - s_j > 0, \\[2ex] \omega_m\left(\dfrac{x_1}{|x_3-s_j|}, \dfrac{x_2}{|x_3-s_j|}\right) & \text{in the basis } (e_1^{j}, e_2^{j}, e_3^{j}) \text{ otherwise.} \end{cases}$$

Following the computations in [6], we infer that

$$|\nabla v_1^m(x)|^2 \le \frac{1+Ca_m^2}{|x_3-s_j|^2}\left|\nabla\omega_m\left(\frac{x_1}{|x_3-s_j|}, \frac{x_2}{|x_3-s_j|}\right)\right|^2 \quad \text{in } Q_m^j \cap \Theta_m([\alpha_1,\beta_1]).$$

Since we have (see [8])

$$\left|\nabla\omega_m\left(\frac{x_1}{|x_3-s_j|}, \frac{x_2}{|x_3-s_j|}\right)\right|^2 = \frac{8|x_3-s_j|^4 a_m^4}{(|x_3-s_j|^2 a_m^4 + x_1^2 + x_2^2)^2},$$

we derive that

$$\int_{Q_m^j \cap \Theta_m([\alpha_1,\beta_1])} |\nabla v_1^m(x)|^2 w(x)dx \le \int_{Q_m^j \cap \Theta_m([\alpha_1,\beta_1])} \frac{8|x_3-s_j|^2 a_m^4 w(x)}{(|x_3-s_j|^2 a_m^4 + x_1^2+x_2^2)^2} dx + C\Lambda a_m^3.$$

Summing these inequalities for $j = 0, \dots, m$ and using (2.8) we obtain that

$$\int_{\Theta_m([\alpha_1,\beta_1])} |\nabla v_1^m(x)|^2 w(x)dx \le 8 \int_{\Theta_m([\alpha_1,\beta_1])} \varepsilon^m_{\alpha_1,\beta_1}(x)w(x)dx + C\Lambda a_m^2 \tag{2.10}$$

Combining (2.9) with (2.10) we conclude that

$$\int_{K_m} |\nabla v_1^m(x)|^2 w(x)dx \le 8 \int_{\Theta_m([\alpha_1,\beta_1])} \varepsilon^m_{\alpha_1,\beta_1}(x)w(x)dx + C\Lambda \left(r + m\sigma(r)^2 a_m^3 + a_m^2 \right).$$

Taking the lim inf in m, we derive that we can find $m_1 \in \mathbb{N}$ large and r small enough such that

$$\int_{K_{m_1}} |\nabla v_1^{m_1}(x)|^2 w(x)dx \le 8 \liminf_{m\to+\infty} \int_{\Theta_m([\alpha_1,\beta_1])} \varepsilon^m_{\alpha_1,\beta_1}(x)w(x)dx + \delta. \tag{2.11}$$

Since $v_1^{m_1} = v$ on ∂K_{m_1}, we may extend $v_1^{m_1}$ to W by setting $v_1^{m_1} = v$ on $W \setminus K_{m_1}$. Now we recall that $v_1^{m_1}$ is singular only at the points b_j, $j = 0, \dots, m$ (we also recall that $b_m = \beta_1$). From (2.7) and the results in [1, 2, 3], we infer that exists a map $v_1 \in \mathcal{C}^1\left(\overline{W} \setminus \{\beta_1\}, S^2\right) \cap H^1(W)$ satisfying $v_1 = v$ in a neighborhood of ∂W, $\deg(v_1, \beta_1) = +1$ and

$$\int_{W_1} |\nabla v_1(x)|^2 w(x)dx \le \int_{W_1} |\nabla v_1^{m_1}(x)|^2 w(x)dx + \delta. \tag{2.12}$$

Since $v = u$ in a neighborhood of ∂W, we may extend v_1 to $\overline{\Omega}$ by setting $v_1 = u$ on $\overline{\Omega} \setminus W$. Then we conclude that $v_1 \in \mathcal{C}^1\left(\overline{\Omega} \setminus \{\beta_1, N\}, S^2\right) \cap H^1(\Omega)$, $\deg(v_1, \beta_1) = +1$, $\deg(v_1, N) = -1$ and by (2.5)-(2.11)-(2.12),

$$\int_\Omega |\nabla v_1(x)|^2 w(x)dx \le \int_\Omega |\nabla u(x)|^2 w(x)dx + 8 \liminf_{m\to+\infty} \int_{\Theta_m([\alpha_1,\beta_1])} \varepsilon^m_{\alpha_1,\beta_1}(x)w(x)dx + C\delta.$$

Step 3. Applying Step 1 and Step 2 to v_1 instead of u and replacing (α_1, β_1) by (α_2, β_2) (recall that $\beta_1 = \alpha_2$), we obtain a map $v_2 \in \mathcal{C}^1\left(\overline{\Omega} \setminus \{\beta_2, N\}, S^2\right) \cap H^1(\Omega)$ satisfying $v_2 = v_1$ outside a δ-neighborhood of $[\alpha_2, \beta_2]$ included in Ω, $\deg(v_2, \beta_2) = +1$, $\deg(v_2, N) = -1$ and

$$\int_\Omega |\nabla v_2(x)|^2 w(x)dx \le \int_\Omega |\nabla v_1(x)|^2 w(x)dx + 8 \liminf_{m\to+\infty} \int_{\Theta_m([\alpha_2,\beta_2])} \varepsilon^m_{\alpha_2,\beta_2}(x)w(x)dx + C\delta.$$

Iterating this process, we finally obtain a map $v_{n-1} \in \mathcal{C}^1\left(\overline{\Omega} \setminus \{\alpha_n, \beta_n\}, S^2\right) \cap H^1(\Omega)$ (recall that $\beta_n = N$) verifying $v_{n-1} = u$ outside a δ-neighborhood of $\cup_{k=1}^{n-1}[\alpha_k, \beta_k]$ included in Ω, $\deg(v_{n-1}, \alpha_n) = +1$, $\deg(v_{n-1}, \beta_n) = -1$ and

$$\int_\Omega |\nabla v_{n-1}(x)|^2 w(x)dx \le \int_\Omega |\nabla u(x)|^2 w(x)dx + 8 \sum_{k=1}^{n-1} \liminf_{m\to+\infty} \int_{\Theta_m([\alpha_k,\beta_k])} \varepsilon^m_{\alpha_k,\beta_k}(x)w(x)dx + C\delta.$$

As in Step 1, we consider $0 < r_0 < \delta$ such that $B_{r_0}(\alpha_n) \cap B_{r_0}(\beta_n) = \emptyset$ and $B_{r_0}(\alpha_n) \cup B_{r_0}(\beta_n) \subset \Omega$ and we construct, using Lemma A1 in [1], a map $\tilde{v} \in \mathcal{C}^1\left(\overline{\Omega} \setminus \{\alpha_n, \beta_n\}, S^2\right) \cap H^1(\Omega)$ satisfying

$$\tilde{v}(x) = \begin{cases} u(x) & \text{on } \Omega \setminus B_{r_0}(\alpha_n), \\ R_+\left(\dfrac{x-\alpha_n}{|x-\alpha_n|}\right) & \text{on } B_{r_0}(\alpha_n), \\ -R_-\left(\dfrac{x-\beta_n}{|x-\beta_n|}\right) & \text{on } B_{r_0}(\beta_n), \end{cases}$$

for some rotations R_+ and R_- and

$$\int_\Omega |\nabla \tilde{v}(x)|^2 w(x)dx \le \int_\Omega |\nabla v_{n-1}(x)|^2 w(x)dx + \delta.$$

Applying the construction in Step 2 starting from $\tilde{v}$, we obtain a new map $\tilde{v}_n^{m_n}$ (for some large $m_n \in \mathbb{N}$) defined on δ-neighborhood W' of $[\alpha_n, \beta_n]$ included in Ω, which coincide with $\tilde{v}$ near $\partial W'$, which then has only point singularities of degree zero (since $\deg(\tilde{v}, \beta_n) = -1$) and satisfying

$$\int_{W'} |\nabla v_n^{m_n}(x)|^2 w(x)dx \le \int_{W'} |\nabla \tilde{v}(x)|^2 w(x)dx + 8 \liminf_{m\to+\infty} \int_{\Theta_m([\alpha_n,\beta_n])} \varepsilon^m_{\alpha_n,\beta_n}(x) w(x)dx + C\delta.$$

Since the degree of each singularities of $v_n^{m_n}$ is zero, we can construct a map $v_n \in \mathcal{C}^1(\overline{W}', S^2)$ (see [2, 3]) verifying $v_n = \tilde{v}$ in a neighborhood of $\partial W'$ and

$$\int_{W'} |\nabla v_n(x)|^2 w(x)dx \le \int_{W'} |\nabla v_n^{m_n}(x)|^2 w(x)dx + \delta.$$

Then we define $u_\delta : \overline{\Omega} \to S^2$ by

$$u_\delta(x) = \begin{cases} v_{n-1}(x) & \text{if } x \in \overline{\Omega} \setminus W', \\ v_n(x) & \text{if } x \in \overline{W}'. \end{cases}$$

Since $v_{n-1} = \tilde{v}$ and $\tilde{v} = v_{n-1}$ near $\partial W'$, we deduce that $u_\delta \in \mathcal{C}^1(\overline{\Omega}, S^2)$. Moreover it follows by construction that $u_\delta = u$ outside a δ-neighborhood of $\cup_{k=1}^n [\alpha_k, \beta_k]$ included in Ω and

$$\int_\Omega |\nabla u_\delta(x)|^2 w(x)dx \le \int_\Omega |\nabla u(x)|^2 w(x)dx + 8\pi \overline{\ell}\,(\mathcal{F}) + C\delta,$$

which ends the proof since δ is arbitrary small. ■

Lemma 2.3. *Let $(P_i, N_i)_{i=1}^K$ be $2K$ distinct points in Ω and consider $u \in H^1(\Omega, S^2) \cap \mathcal{C}^1(\overline{\Omega} \setminus \cup_{i=1}^K \{P_i, N_i\})$ such that $\deg(u, P_i) = +1$ and $\deg(u, N_i) = -1$ for $i = 1, \dots, K$. Then There exists a sequence of maps $(u_n)_{n\in\mathbb{N}} \subset \mathcal{C}^1(\overline{\Omega}, S^2)$ satisfying $u_{n|\partial\Omega} = u_{|\partial\Omega}$,*

$$\int_\Omega |\nabla u_n(x)|^2 w(x)dx \le \int_\Omega |\nabla u(x)|^2 w(x)dx + 8\pi L_w(u) + 2^{-n},$$

and

$$\text{meas}\left(\{x \in \Omega,\ u_n(x) \ne u(x)\}\right) \le 2^{-n}.$$

Proof. Without loss of generality we may assume that $\sum_i d_w(P_i, N_i)$ is equal to the length of a minimal connection relative to d_w between the points (P_i) and (N_i). As in [1], we are going to "remove" each dipole (P_i, N_i). More precisely, for each $n \in \mathbb{N}$, we construct successively K maps $\left(u_n^i\right)_{i=1}^K$ satisfying

(a) $u_n^i \in H^1(\Omega, S^2) \cap \mathcal{C}^1\left(\overline{\Omega} \setminus \bigcup_{i+1\le j\le K} \{P_j, N_j\}\right)$ for $i = 1, \dots, K$,

(b) $u_n^1 = u$ on $\overline{\Omega} \setminus W_n^1$ and $u_n^i = u_n^{i-1}$ on $\overline{\Omega} \setminus W_n^i$ for $i = 2, \ldots, K$ where W_n^i is is strictly included in $\Omega \setminus \bigcup_{i+1 \leq j \leq K} \{P_j, N_j\}$ and $|W_n^i| \leq 2^{-n}/K$,

(c) $\displaystyle\int_\Omega |\nabla u_n^1(x)|^2 w(x)dx \leq \int_\Omega |\nabla u(x)|^2 w(x)dx + 8\pi d_w(P_1, N_1) + \frac{2^{-n}}{K}$ and

$$\int_\Omega |\nabla u_n^i(x)|^2 w(x)dx \leq \int_\Omega |\nabla u_n^{i-1}(x)|^2 w(x)dx + 8\pi d_w(P_i, N_i) + \frac{2^{-n}}{K} \quad \text{for } i = 2, \ldots, K.$$

We easily check that the sequence $(u_n^K)_{n \in \mathbb{N}}$ then satisfies the requirement since we have $L_w(u) = \sum_i d_w(P_i, N_i)$. We start with the construction of u_n^1.

Construction of u_n^1. By Lemma 2.1, we can find $\mathcal{F}_1 = ([\alpha_1, \beta_1], \ldots, [\alpha_l, \beta_l]) \in \mathcal{Q}(P_1, N_1)$ satisfying

$$\big(\cup_{i=2}^K \{P_i, N_i\} \cup \{N_1\} \big) \cap \Big(\cup_{k=2}^l [\alpha_k, \beta_k] \cup [\alpha_1, \beta_1[\Big) = \emptyset, \tag{2.13}$$

and

$$\overline{\ell}_w\left(\mathcal{F}_1\right) \leq d_w(P_1, N_1) + \frac{2^{-(n+1)}}{8K\pi}.$$

From (2.13), we infer that we can find $\delta > 0$ small enough such that

$$W_\delta^1 = \{x \in \mathbb{R}^3, \, \mathrm{dist}(x, \cup_{k=1}^l [\alpha_k, \beta_k]) \leq \delta\} \subset \Omega \setminus \cup_{i=2}^K \{P_i, N_i\} \quad \text{and} \quad |W_\delta^1| \leq \frac{2^{-n}}{K}.$$

By the method described in the proof of Lemma 2.2, we construct a map $u_n^1 \in H^1(\Omega, S^2) \cap \mathcal{C}^1(\overline{\Omega} \setminus \cup_{i=2}^K \{P_i, N_i\})$ verifying $u_n^1 = u$ outside W_δ^1 and

$$\begin{aligned} \int_\Omega |\nabla u_n^1(x)|^2 w(x)dx amp; &\leq \int_\Omega |\nabla u(x)|^2 w(x)dx + 8\pi \overline{\ell}_w\left(\mathcal{F}_1\right) + \frac{2^{-(n+1)}}{K} \\ amp; &\leq \int_\Omega |\nabla u(x)|^2 w(x)dx + 8\pi d_w(P_1, N_1) + \frac{2^{-n}}{K}. \end{aligned}$$

Construction of u_n^i, $i = 2, \ldots, K$. We iterate the previous process i.e., we proceed as for the construction of u_n^1 but starting from u_n^{i-1} instead of u. ■

3 Proof of Theorem 1.2

3.1 Lower Bound of the Energy

In this section, we denote by F_w the functional defined for maps $u \in H_g^1(\Omega, S^2)$ by

$$F_w(u) = \int_\Omega |\nabla u(x)|^2 w(x)dx + 8\pi L_w(u).$$

Proposition 3.1. *The functional F_w is sequentially lower semi-continuous on $H_g^1(\Omega, S^2)$ for the weak H^1-topology.*

Proof. We follow the method in [4]. Since the supremum of a family of sequentially lower semi-continuous functionals is sequentially lower semi-continuous, it suffices to show that for any function $\zeta : \overline{\Omega} \to \mathbb{R}$ which is 1-Lipschitz with respect to d_w, the functional

$$u \in H^1_g \mapsto \int_\Omega |\nabla u(x)|^2 w(x) dx + 2 \int_\Omega D(u) \cdot \nabla \zeta \, dx$$

is sequentially lower semi-continuous for the weak H^1-topology (the term $\int_{\partial\Omega}(D(u)\cdot\nu)\zeta$ only depends on g and ζ). Consider $(u_n)_{n\in\mathbb{N}} \subset H^1_g(\Omega, S^2)$ and $u \in H^1_g(\Omega, S^2)$ such that $u_n \rightharpoonup u$ weakly in H^1. Setting $v_n = u_n - u$, we have

$$\int_\Omega |\nabla u_n(x)|^2 w(x) dx = \int_\Omega |\nabla u(x)|^2 w(x) dx + \int_\Omega |\nabla v_n(x)|^2 w(x) dx + o(1),$$

and writing

$$2 \int_\Omega D(u_n) \cdot \nabla \zeta \, dx = A_n + B_n + C_n$$

with

$$A_n = 2 \int_\Omega u_n \cdot \left(\frac{\partial u}{\partial x_2} \wedge \frac{\partial u}{\partial x_3} \frac{\partial \zeta}{\partial x_1} + \frac{\partial u}{\partial x_3} \wedge \frac{\partial u}{\partial x_1} \frac{\partial \zeta}{\partial x_3} + \frac{\partial u}{\partial x_1} \wedge \frac{\partial u}{\partial x_2} \frac{\partial \zeta}{\partial x_3} \right),$$

$$B_n = 2 \int_\Omega u_n \cdot \left(\frac{\partial v_n}{\partial x_2} \wedge \frac{\partial u}{\partial x_3} + \frac{\partial u}{\partial x_2} \wedge \frac{\partial v_n}{\partial x_3} \right) \frac{\partial \zeta}{\partial x_1} + 2 \int_\Omega u_n \cdot \left(\frac{\partial v_n}{\partial x_3} \wedge \frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_3} \wedge \frac{\partial v_n}{\partial x_1} \right) \frac{\partial \zeta}{\partial x_2}$$

$$+ 2 \int_\Omega u_n \cdot \left(\frac{\partial v_n}{\partial x_1} \wedge \frac{\partial u}{\partial x_2} + \frac{\partial u}{\partial x_1} \wedge \frac{\partial v_n}{\partial x_2} \right) \frac{\partial \zeta}{\partial x_3},$$

$$C_n = 2 \int_\Omega u_n \cdot \left(\frac{\partial v_n}{\partial x_2} \wedge \frac{\partial v_n}{\partial x_3} \frac{\partial \zeta}{\partial x_1} + \frac{\partial v_n}{\partial x_3} \wedge \frac{\partial v_n}{\partial x_1} \frac{\partial \zeta}{\partial x_3} + \frac{\partial v_n}{\partial x_1} \wedge \frac{\partial v_n}{\partial x_2} \frac{\partial \zeta}{\partial x_3} \right).$$

We easily obtain that $A_n \to 2 \int_\Omega D(u) \cdot \nabla\zeta$ as $n \to +\infty$ since $u_n \rightharpoonup u$ weak$\star$ in L^∞ and that $B_n \to 0$ since $v_n \rightharpoonup 0$ weakly in L^2 and $u_n \to u$ strongly in L^2. Now we set

$$V_n = \left(u_n \cdot \frac{\partial v_n}{\partial x_2} \wedge \frac{\partial v_n}{\partial x_3}, \, u_n \cdot \frac{\partial v_n}{\partial x_3} \wedge \frac{\partial v_n}{\partial x_1}, \, u_n \cdot \frac{\partial v_n}{\partial x_1} \wedge \frac{\partial v_n}{\partial x_2} \right).$$

We have

$$|C_n| = 2 \left| \int_\Omega V_n \cdot \nabla \zeta \right| \leq 2 \int_\Omega |V_n| |\nabla \zeta|.$$

By Lemma 1 in [4], we know that $2|V_n| \leq |\nabla v_n|^2$ and by Proposition 2.3 in [14], any $\zeta : \overline{\Omega} \to \mathbb{R}$ which 1-Lipschitz with respect to d_w satisfies $|\nabla \zeta| \leq w$ a.e. on Ω. Then we obtain

$$|C_n| \leq \int_\Omega |\nabla v_n(x)|^2 w(x) dx$$

and we conclude that

$$\int_\Omega |\nabla u_n(x)|^2 w(x) dx + 2 \int_\Omega D(u_n) \cdot \nabla \zeta \, dx \geq \int_\Omega |\nabla u(x)|^2 w(x) dx + 2 \int_\Omega D(u) \cdot \nabla \zeta \, dx + o(1)$$

which clearly implies the result. ■

Proof of "$\geq$" in Theorem 1.2. Let $u \in H^1_g(\Omega, S^2)$ and consider an arbitrary sequence $(u_n)_{n\in\mathbb{N}} \subset H^1_g(\Omega, S^2) \cap \mathcal{C}^1(\overline{\Omega})$ such that $u_n \rightharpoonup u$ weakly in H^1. Since u_n is smooth in Ω, we have $T(u_n) \equiv 0$ and then $L_w(u_n) = 0$. We conclude by Proposition 3.1 that

$$\liminf_{n\to+\infty} \int_\Omega |\nabla u_n(x)|^2 w(x)dx = \liminf_{n\to+\infty} F_w(u_n) \geq F_w(u) = \int_\Omega |\nabla u(x)|^2 w(x)dx + 8\pi L_w(u).$$

Since the sequence $(u_n)_{n\in\mathbb{N}}$ is arbitrary, we get the announced result. ■

3.2 Upper Bound of the Energy

Proposition 3.2. *Let $u \in H^1_g(\Omega, S^2)$. Then there exists a sequence of maps $(u_n)_{n\in\mathbb{N}} \subset H^1_g(\Omega, S^2) \cap \mathcal{C}^1(\overline{\Omega})$ such that $u_n \rightharpoonup u$ weakly in H^1 and*

$$\limsup_{n\to+\infty} \int_\Omega |\nabla u_n(x)|^2 w(x)dx \leq \int_\Omega |\nabla u(x)|^2 w(x)dx + 8\pi L_w(u).$$

End of the proof of Theorem 1.2. Let $u \in H^1_g(\Omega, S^2)$ and let $(u_n)_{n\in\mathbb{N}}$ be the sequence of maps given by Proposition 3.2. By definition of $E_w(u)$ and Proposition 3.2, we have

$$E_w(u) \leq \liminf_{n\to+\infty} \int_\Omega |\nabla u_n(x)|^2 w(x)dx \leq \int_\Omega |\nabla u(x)|^2 w(x)dx + 8\pi L_w(u),$$

which ends the proof of Theorem 1.2. ■

To prove Proposition 3.2, we need the following Lemma. We postpone its proof at the end of this section.

Lemma 3.1. *For any $u, v \in H^1_g(\Omega, S^2)$, we have*

$$|L_w(u) - L_w(v)| \leq C\Lambda \left(\|\nabla u\|_{L^2(\Omega)} + \|\nabla v\|_{L^2(\Omega)}\right) \|\nabla u - \nabla v\|_{L^2(\Omega)}, \tag{3.1}$$

for a constant C independent of w.

Proof of Proposition 3.2. Let $u \in H^1_g(\Omega, S^2)$. By the result in [1, 3], we can find a sequence of maps $(v_n)_{n\in\mathbb{N}} \subset H^1_g(\Omega, S^2)$ such that $v_n \in \mathcal{C}^1(\overline{\Omega} \setminus \cup_{i=1}^{K_n}\{P_i, N_i\})$ for some $2K_n$ distinct points (P_i, N_i) in Ω, $\deg(v_n, P_i) = +1$ and $\deg(v_n, N_i) = -1$ for $i = 1, \ldots, K_n$ and such that

$$\|\nabla(v_n - u)\|_{L^2(\Omega)} \leq 2^{-n}. \tag{3.2}$$

From this inequality we infer that

$$\text{meas}\left(\left\{x \in \Omega,\ |v_n(x) - u(x)| < 2^{-n/2}\right\}\right) \leq C\, 2^{-n}. \tag{3.3}$$

Applying Lemma 2.3 to v_n, we find a map $u_n \in \mathcal{C}^1(\overline{\Omega}, S^2)$ satisfying $u_{n|\partial\Omega} = g$,

$$\int_\Omega |\nabla u_n(x)|^2 w(x)dx \leq \int_\Omega |\nabla v_n(x)|^2 w(x)dx + 8\pi L_w(v_n) + 2^{-n} \tag{3.4}$$

and

$$\text{meas}\left(\{x \in \Omega,\, u_n(x) \neq v_n(x)\}\right) \leq 2^{-n}. \tag{3.5}$$

From (3.2) and Lemma 3.1 we deduce that $L_w(v_n) \to L_w(u)$ as $n \to +\infty$ and then it follows that $(u_n)_{n\in\mathbb{N}}$ is bounded in H^1. Moreover we obtain from (3.3) and (3.5) that $u_n \to u$ a.e. in Ω and we conclude that $u_n \rightharpoonup u$ weakly in H^1. Letting $n \to +\infty$ in (3.4) leads to

$$\limsup_{n\to+\infty} \int_\Omega |\nabla u_n(x)|^2 w(x)dx \leq \int_\Omega |\nabla u(x)|^2 w(x)dx + 8\pi L_w(u),$$

which completes the proof. ■

Proof of Lemma 3.1. To prove Lemma 3.1, we follow the method in [4]. For $u, v \in H^1_g(\Omega, S^2)$, we set

$$L_w(u,v) = \text{Sup}\left\{\int_\Omega (D(u) - D(v)) \cdot \nabla\zeta\,,\ \zeta : \overline{\Omega} \to \mathbb{R} \text{ 1-Lipschitz with respect to } d_w\right\}.$$

Since $D(u) \cdot \nu = D(v) \cdot \nu$ on $\partial\Omega$ (it only depends on g), we have

$$\int_\Omega D(u)\cdot\nabla\zeta - \int_{\partial\Omega} (D(u)\cdot\nu)\,\zeta = \int_\Omega D(v)\cdot\nabla\zeta - \int_{\partial\Omega} (D(v)\cdot\nu)\,\zeta + \int_\Omega (D(u) - D(v))\cdot\nabla\zeta\,,$$

and we easily derive that

$$|L_w(u) - L_w(v)| \leq L_w(u,v).$$

Similar computations to those in [4], proof of Theorem 1, lead to

$$\left|\int_\Omega (D(u) - D(v))\cdot\nabla\zeta\right| \leq C\left(\|\nabla u\|_{L^2(\Omega)} + \|\nabla v\|_{L^2(\Omega)}\right)\|\nabla u - \nabla v\|_{L^2(\Omega)}\|\nabla\zeta\|_{L^\infty(\Omega)}.$$

By Proposition 2.3 in [14], any real function ζ which is 1-Lipschitz with respect to d_w satisfies $|\nabla\zeta| \leq w$ a.e. on Ω. We deduce that (3.1) holds since $w \leq \Lambda$ a.e. on Ω. ■

4 Stability and Approximation Properties

4.1 A Stability Property

Before stating the result, we need to recall some previous ones obtained in [14]. For any real measurable function w satisfying assumption (1.1), we may associate to distance d_w the length functional $\mathbb{L}_{d_w}$ defined by

$$\mathbb{L}_{d_w}(\gamma) = \text{Sup}\left\{\sum_{k=0}^{m-1} d_w\left(\gamma(t_k), \gamma(t_{k+1})\right),\ 0 = t_0 < t_1 < \ldots < t_m = 1,\ m \in \mathbb{N}^\star\right\},$$

where $\gamma : [0,1] \to \overline{\Omega}$ is any continuous curve. In [14], we have proved that for any $x, y \in \overline{\Omega}$,

$$d_w(x,y) = \text{Inf}\left\{\mathbb{L}_{d_w}(\gamma),\ \gamma \in \text{Lip}([0,1], \overline{\Omega}),\ \gamma(0) = x \text{ and } \gamma(1) = y\right\} \tag{4.1}$$

where $\mathrm{Lip}([0,1],\overline{\Omega})$ denotes the class of all Lipschitz maps from $[0,1]$ into $\overline{\Omega}$. We have also shown that the infimum in (4.1) is in fact achieved.

The following stability result relies on the Γ-convergence of the length functionals (we refer to [11] for the notion of Γ-convergence). In the sequel, we endow $\mathrm{Lip}([0,1],\overline{\Omega})$ with the topology of the uniform convergence on $[0,1]$.

Theorem 4.1. *Let $(w_n)_{n\in\mathbb{N}}$ be a sequence of measurable real functions such that*

$$0 < c_0 \leq w_n \leq C_0 \quad a.e.\ in\ \Omega \tag{4.2}$$

for some constants c_0 and C_0 independent of $n \in \mathbb{N}$. Then the following properties are equivalent:

(i) the functionals $\mathbb{L}_{d_{w_n}}$ Γ-converge to $\mathbb{L}_{d_w}$ in $\mathrm{Lip}([0,1],\overline{\Omega})$ and

$$\int_\Omega |\nabla\varphi(x)|^2 w_n(x)dx \underset{n\to+\infty}{\to} \int_\Omega |\nabla\varphi(x)|^2 w(x)dx \quad \text{for any } \varphi \in H^1(\Omega,\mathbb{R}), \tag{4.3}$$

(ii) for every smooth boundary data $g:\partial\Omega\to S^2$ such that $\deg(g)=0$,

$$E_{w_n}(u) \underset{n\to+\infty}{\to} E_w(u) \quad \text{for any } u \in H^1_g(\Omega,S^2).$$

Proof. (i)⇒(ii). We fix a smooth boundary data $g:\Omega\to S^2$ such that $\deg(g)=0$. Clearly (4.3) implies that

$$\int_\Omega |\nabla u(x)|^2 w_n(x)dx \underset{n\to+\infty}{\to} \int_\Omega |\nabla u(x)|^2 w(x)dx \quad \text{for any } u \in H^1_g(\Omega,S^2),$$

and by Theorem 1.2, it remains to prove that

$$L_{w_n}(u) \underset{n\to+\infty}{\to} L_w(u) \quad \text{for any } u \in H^1_g(\Omega,S^2). \tag{4.4}$$

Consider $u \in H^1_g(\Omega,S^2)$. By the result in [1, 3], there exits a sequence of maps $(v_k)_{k\in\mathbb{N}} \subset H^1_g(\Omega,S^2)$ such that $v_k \in \mathcal{C}^1(\overline{\Omega}\setminus\cup_{j=1}^{M_k}\{P_j,N_j\},S^2)$ for some $2M_k$ points (P_j,N_j) in Ω, $\deg(v_k,P_j)=+1$ and $\deg(v_k,N_j)=-1$ for $j=1,\dots,M_k$, and $v_k\to u$ strongly in H^1. We have

$$L_{w_n}(v_k) = \underset{\sigma\in\mathcal{S}_{M_k}}{\mathrm{Min}} \sum_{j=1}^{M_k} d_{w_n}(P_j,N_{\sigma(j)}) \quad \text{and} \quad L_w(v_k) = \underset{\sigma\in\mathcal{S}_{M_k}}{\mathrm{Min}} \sum_{j=1}^{M_k} d_w(P_j,N_{\sigma(j)})$$

Since the functionals $\mathbb{L}_{d_{w_n}}$ Γ-converge to $\mathbb{L}_{d_w}$ in $\mathrm{Lip}([0,1],\overline{\Omega})$, we deduce from Theorem 4.1 in [14] that for every $k\in\mathbb{N}$, $L_{w_n}(v_k)\to L_w(v_k)$ as $n\to+\infty$. Now we fix a small $\delta>0$. Since $v_k\to u$ strongly in H^1, we derive from Lemma 3.1 and (4.2) that exists $k_0\in\mathbb{N}$ which only depends on u, δ and C_0 such that

$$L_{w_n}(v_k)-\delta \leq L_{w_n}(u) \leq L_{w_n}(v_k)+\delta \quad \text{for any } n\in\mathbb{N} \text{ and } k\geq k_0.$$

Letting $n \to +\infty$ in this inequality, we get that

$$L_w(v_k) - \delta \leq \liminf_{n\to+\infty} L_{w_n}(u) \leq \limsup_{n\to+\infty} L_{w_n}(u) \leq L_w(v_k) + \delta \quad \text{for } k \geq k_0.$$

Passing to the limit in k and using Lemma 3.1, we obtain

$$L_w(u) - \delta \leq \liminf_{n\to+\infty} L_{w_n}(u) \leq \limsup_{n\to+\infty} L_{w_n}(u) \leq L_w(u) + \delta,$$

which leads to the result since δ is arbitrary small.

(ii)⇒(i). First we prove (4.3) for $\varphi \in \mathcal{C}^\infty(\overline{\Omega}, \mathbb{R})$. Let $\varphi \in \mathcal{C}^\infty(\overline{\Omega}, \mathbb{R})$ and consider the smooth map $g : \partial\Omega \to S^2$ defined by $g(x) = (\cos(\varphi(x)), \sin(\varphi(x)), 0)$. We easily check that $\deg(g) = 0$. Now consider the map u defined for $x \in \overline{\Omega}$ by

$$u(x) = (\cos(\varphi(x)), \sin(\varphi(x)), 0).$$

We have $u \in H^1_g(\Omega, S^2) \cap \mathcal{C}^\infty(\overline{\Omega})$ and then $L_{w_n}(u) = L_w(u) = 0$ for any $n \in \mathbb{N}$. Since $|\nabla u|^2 = |\nabla \varphi|^2$, we derive from assumption *(ii)* and Theorem 1.2 that

$$\int_\Omega |\nabla\varphi(x)|^2 w_n(x)dx \underset{n\to+\infty}{\longrightarrow} \int_\Omega |\nabla\varphi(x)|^2 w(x)dx.$$

Let us now prove (4.3) for any $\varphi \in H^1(\Omega, \mathbb{R})$. Let $\varphi \in H^1(\Omega, \mathbb{R})$ and consider a sequence $(\varphi_k)_{k\in\mathbb{N}} \subset \mathcal{C}^\infty(\overline{\Omega}, \mathbb{R})$ such that $\varphi_k \to \varphi$ strongly in H^1. We fix a small $\delta > 0$. From assumption (4.2), we infer that exists $k_0 \in \mathbb{N}$ which only depends on φ, δ and C_0 such that for any $n \in \mathbb{N}$ and $k \geq k_0$,

$$\int_\Omega |\nabla\varphi_k(x)|^2 w_n(x)dx - \delta \leq \int_\Omega |\nabla\varphi(x)|^2 w_n(x)dx \leq \int_\Omega |\nabla\varphi_k(x)|^2 w_n(x)dx + \delta.$$

Since φ_k is smooth, letting $n \to +\infty$ we obtain for $k \geq k_0$,

$$\begin{aligned}\int_\Omega |\nabla\varphi_k(x)|^2 w(x)dx - \delta \leq\ & \liminf_{n\to+\infty} \int_\Omega |\nabla\varphi(x)|^2 w_n(x)dx \\ \leq\ & \limsup_{n\to+\infty} \int_\Omega |\nabla\varphi(x)|^2 w_n(x)dx \leq \int_\Omega |\nabla\varphi_k(x)|^2 w(x)dx + \delta.\end{aligned}$$

Passing to the limit in k and then $\delta \to 0$, we conclude

$$\lim_{n\to+\infty} \int_\Omega |\nabla\varphi(x)|^2 w_n(x)dx = \int_\Omega |\nabla\varphi(x)|^2 w(x)dx.$$

It remains to prove that the functionals $\mathbb{L}_{d_{w_n}}$ Γ-converge to $\mathbb{L}_{d_w}$ in $\text{Lip}([0,1], \overline{\Omega})$. Let P and N be two distinct points in Ω. We take $g \equiv (0,0,1)$ and consider $u \in H^1_g(\Omega, S^2) \cap \mathcal{C}^1(\overline{\Omega} \setminus \{P, N\})$ (such a map is constructed for instance in [6, 9]). By Theorem 1.2, we have

$$E_{w_n}(u) = \int_\Omega |\nabla u(x)|^2 w_n(x)dx + 8\pi d_{w_n}(P, N)$$

and

$$E_w(u) = \int_\Omega |\nabla u(x)|^2 w(x)dx + 8\pi d_w(P, N).$$

From (4.3) we get that $\int_\Omega |\nabla u(x)|^2 w_n(x)dx \to \int_\Omega |\nabla u(x)|^2 w(x)dx$ and from assumption *(ii)* we deduce that

$$d_{w_n}(P,N) \to d_w(P,N) \quad \text{as } n \to +\infty.$$

Since the points P and N are arbitrary in Ω, we derive that d_{w_n} converges to d_w pointwise on $\Omega \times \Omega$ and the conclusion follows by the results in [14] Section 4. ■

In the next proposition, we give some sufficient condition on a sequence $(w_n)_{n\in\mathbb{N}}$ converging pointwise a.e. to w for property *(ii)* in Theorem 4.1 to hold.

Proposition 4.1. *Let $(w_n)_{n\in\mathbb{N}}$ be a sequence of measurable real functions satisfying (4.2) and assume that one of the following conditions holds:*

(a) $w_n \geq w$ *and* $w_n \to w$ *a.e. in* Ω,

(b) $w_n \to w$ *in* $L^\infty(\Omega)$.

Then property (ii) in Theorem 4.1 holds.

Proof. By Proposition 4.1 and Theorem 4.1 in [14], *(a)* or *(b)* implies that the functionals $\mathbb{L}_{d_{w_n}}$ Γ-converge to $\mathbb{L}_{d_w}$ in $\mathrm{Lip}([0,1],\overline{\Omega})$. We also check that *(a)* or *(b)* implies (4.3) by dominated convergence. Then the conclusion follows from Theorem 4.1. ■

Remark 4.1. The conclusion of Proposition 4.1 may fails if one only assumes that $w_n \to w$ a.e. in Ω (see Remark 4.1 in [14]).

4.2 Approximation Property

In this section, we show that the functional E_w can be obtain as pointwise limit of a sequence $(E_{w_n})_{n\in\mathbb{N}}$ in which the weight function w_n is smooth.

Proposition 4.2. *Let $(\rho_n)_{n\in\mathbb{N}}$ be a sequence of smooth mollifiers. Extending w by a sufficiently large constant and setting $w_n = \rho_n * w$, we have*

$$E_{w_n}(u) \underset{n\to+\infty}{\to} E_w(u) \quad \textit{for any } u \in H^1_g(\Omega, S^2).$$

Proof. By construction, (4.3) clearly holds. Then property *(i)* in Theorem 4.1 follows from Theorem 4.1 in [14] and Theorem 4.2 in [14] which leads to the result by Theorem 4.1. ■

5 The Relaxed Energy without Prescribed Boundary Data

In this section, we consider the relaxed type functional

$$\tilde{E}_w(u) = \mathrm{Inf}\left\{ \liminf_{n\to+\infty} \int_\Omega |\nabla u_n(x)|^2 w(x)dx,\ u_n \in \mathcal{C}^1(\overline{\Omega}, S^2),\ u_n \rightharpoonup u \text{ weakly in } H^1 \right\}$$

defined for $u \in H^1(\Omega, S^2)$. We recall that F. Bethuel has also proved (see [1]) that $\mathcal{C}^1(\overline{\Omega}, S^2)$ is sequentially dense in $H^1(\Omega, S^2)$ for the weak H^1 topology and then $\tilde{E}_w$ is well defined.

As in [4], there is also a notion of length of a minimal connection relative to d_w defined for any $u \in H^1(\Omega, S^2)$:

$$\tilde{L}_w(u) = \frac{1}{4\pi} \operatorname{Sup} \big\{ \langle T(u), \zeta \rangle,\, \zeta : \overline{\Omega} \to \mathbb{R} \text{ 1-Lipschitz with respect to } d_w \text{ and } \zeta = 0 \text{ on } \partial\Omega \big\}.$$

Since no assumptions are made on $u_{|\partial\Omega}$, it may happen that $\deg(u_{|\partial\Omega}) \neq 0$ or that $\deg(u_{|\partial\Omega})$ is not well defined. But clearly $\tilde{L}_w(u)$ always makes sense. When u is smooth except at a finite number of point in Ω, $\tilde{L}_w(u)$ is equal to the length of a minimal connection relative to d_w between the singularities of u and some virtual singularities on the boundary (see [9]). More precisely, one adds some virtual singularities on the boundary in such a way that the new configuration has the same number of positive and negative points and one consider the length of a minimal connection relative to d_w for this configuration. Then $\tilde{L}_w(u)$ corresponds to the infimum of these quantities when one varies the position and the number of the boundary points. There is the variant of Theorem 1.2 for $\tilde{E}_w$.

Theorem 5.1. *For any $u \in H^1(\Omega, S^2)$, we have*

$$\tilde{E}_w(u) = \int_\Omega |\nabla u(x)|^2 w(x) dx + 8\pi \tilde{L}_w(u).$$

5.1 Proof of Theorem 5.1

The inequality "$\geq$" in Theorem 5.1 can be proved using a method similar to the one used in Section 3.1 and we omit it. We obtain "$\leq$" as in Section 3.2 using Proposition 5.1 and Lemma 5.1 below instead of Proposition 3.2 and Lemma 3.1. The proof of Lemma 5.1 is almost identical to the proof of Lemma 3.1 and we also omit it (note that all the boundary integrals vanish since $\zeta = 0$ on $\partial\Omega$).

Proposition 5.1. *Let $u \in H^1(\Omega, S^2)$. Then there exists a sequence of maps $(u_n)_{n\in\mathbb{N}} \subset \mathcal{C}^1(\overline{\Omega}, S^2)$ such that*

$$u_n \rightharpoonup u \quad \textit{weakly in } H^1$$

and

$$\limsup_{n\to+\infty} \int_\Omega |\nabla u_n(x)|^2 w(x) dx \leq \int_\Omega |\nabla u(x)|^2 w(x) dx + 8\pi \tilde{L}_w(u).$$

Lemma 5.1. *For any $u, v \in H^1(\Omega, S^2)$, we have*

$$\left| \tilde{L}_w(u) - \tilde{L}_w(v) \right| \leq C\Lambda \left(\|\nabla u\|_{L^2(\Omega)} + \|\nabla v\|_{L^2(\Omega)} \right) \|\nabla u - \nabla v\|_{L^2(\Omega)}, \tag{5.1}$$

for a constant C independent of w.

Proof of Proposition 5.1. Let $u \in H^1(\Omega, S^2)$. By the result in [1, 3], we can find a sequence $(v_n)_{n\in\mathbb{N}} \subset H^1(\Omega, S^2)$ such that $v_n \in \mathcal{C}^1(\overline{\Omega} \setminus \{(a_i)_{i=1}^{N_n}\})$ for some N_n distinct points $a_1, \ldots, a_{N_n}$ in Ω and

$$\|u - v_n\|_{H^1(\Omega)} \leq 2^{-n}. \tag{5.2}$$

Since we are working with an approximating sequence, we may assume that $|\deg(v_n, a_i)| = 1$ for $i = 1, \ldots, N_n$ (see [1]). Since v_n is smooth except at a finite number of point in Ω, the length of a minimal connection $\tilde{L}_w(v_n)$ is computed as follows (see [9], part II). We pair each singularity a_i either to another singularity in Ω of opposite degree or to a virtual singularity on the boundary with opposite degree. In other words, we allow connections to the boundary of Ω. Pairing all the singularities in this way, we take a configuration that minimizes the sum of the distances between the paired singularities, computing the distances with d_w. We relabel all the singularities (the a_i's and the virtual singularities on the boundary), according to their multiplicity for those on the boundary, as a list of positive and negative points say $(P_1, \ldots, P_{K_n})$ and $(N_1, \ldots, N_{K_n})$ such that

$$\tilde{L}_w(v_n) = \sum_{j=1}^{K_n} d_w(P_j, N_j).$$

Using Lemma 2 bis in [1], we can find $\tilde{v}_n \in H^1(\Omega, S^2) \cap \mathcal{C}^1(\overline{\Omega} \setminus \cup_{j=1}^{K_n} \{\tilde{P}_j, \tilde{N}_j\})$ for some $2K_n$ distinct points $(\tilde{P}_j, \tilde{N}_j)$ in Ω such that $\tilde{v}_n = v_n$ outside a small neighborhood of $\partial\Omega$, $\deg(\tilde{v}_n, \tilde{P}_j) = +1$ and $\deg(\tilde{v}_n, \tilde{N}_j) = -1$ for $j = 1, \ldots, K_n$, $\tilde{P}_j = P_j$ (respectively $\tilde{N}_j = N_j$) if $P_j \in \Omega$ (respectively if $N_j \in \Omega$) and $|\tilde{P}_j - P_j| \leq \frac{2^{-n}}{K_n}$ otherwise (respectively $|\tilde{N}_j - N_j| \leq \frac{2^{-n}}{K_n}$), and

$$\|\tilde{v}_n - v_n\|_{H^1(\Omega)} \leq 2^{-n}. \tag{5.3}$$

Note that, for each pair (P_j, N_j), we necessarily have $\tilde{P}_j = P_j$ or $\tilde{N}_j = N_j$ and then

$$\left| \sum_{j=1}^{K_n} d_w(P_j, N_j) - \sum_{j=1}^{K_n} d_w(\tilde{P}_j, \tilde{N}_j) \right| \leq C\, 2^{-n}, \tag{5.4}$$

and from (5.2) and (5.3), we infer that

$$\text{meas}\left(\{x \in \Omega,\, |u(x) - \tilde{v}_n(x)| < 2^{-n/2}\}\right) \leq C\, 2^{-n}. \tag{5.5}$$

Applying Lemma 2.3 to $\tilde{v}_n$, we find a map $u_n \in \mathcal{C}^1(\overline{\Omega}, S^2)$ satisfying

$$\int_\Omega |\nabla u_n(x)|^2 w(x) dx \leq \int_\Omega |\nabla \tilde{v}_n(x)|^2 w(x) dx + 8\pi \sum_{j=1}^{K_n} d_w(\tilde{P}_j, \tilde{N}_j) + 2^{-n} \tag{5.6}$$

and

$$\text{meas}\left(\{x \in \Omega,\, u_n(x) \neq \tilde{v}_n(x)\}\right) \leq 2^{-n}. \tag{5.7}$$

From (5.4) and (5.6), we derive that

$$\int_\Omega |\nabla u_n(x)|^2 w(x) dx \leq \int_\Omega |\nabla v_n(x)|^2 w(x) dx + 8\pi \tilde{L}_w(v_n) + C\, 2^{-n}. \tag{5.8}$$

Since $v_n \to u$ strongly in H^1, we deduce from Lemma 5.1 that $\tilde{L}_w(v_n) \to \tilde{L}_w(u)$ as $n \to +\infty$ which implies that $(u_n)_{n\in\mathbb{N}}$ is bounded in H^1. From (5.3) and (5.7) we obtain $u_n \to u$ a.e. in Ω and then we conclude that $u_n \rightharpoonup u$ weakly in H^1. Passing to the limit in (5.8) leads to

$$\limsup_{n\to+\infty} \int_\Omega |\nabla u_n(x)|^2 w(x) dx \leq \int_\Omega |\nabla u(x)|^2 w(x) dx + 8\pi \tilde{L}_w(u)$$

and the proof is complete. ■

5.2 Stability and Approximation Properties for $\tilde{E}_w$

We present in this section the variants for $\tilde{E}_w$ of the results in Section 4.

Theorem 5.2. *Let $(w_n)_{n\in\mathbb{N}}$ be a sequence of measurable real functions satisfying (4.2) and assume that (i) in Theorem 4.1 holds. Then we have*

$$\tilde{E}_{w_n}(u) \underset{n\to+\infty}{\to} \tilde{E}_w(u) \quad \textit{for any } u \in H^1(\Omega, S^2). \tag{5.9}$$

Proof. Assumption (4.3) clearly implies that

$$\int_\Omega |\nabla u(x)|^2 w_n(x)dx \underset{n\to+\infty}{\to} \int_\Omega |\nabla u(x)|^2 w(x)dx \quad \text{for any } u \in H^1(\Omega, S^2),$$

and by Theorem 5.1, we just have to prove that

$$\tilde{L}_{w_n}(u) \underset{n\to+\infty}{\to} \tilde{L}_w(u) \quad \text{for any } u \in H^1(\Omega, S^2). \tag{5.10}$$

Consider $u \in H^1(\Omega, S^2)$. By the result in [1, 3], we can find a sequence $(v_k)_{k\in\mathbb{N}} \subset H^1(\Omega, S^2)$ such that $v_k \in \mathcal{C}^1(\overline{\Omega} \setminus \cup_{i=1}^{M_k}\{a_j\}, S^2)$ for some M_k points (a_i) in Ω and $v_k \to u$ strongly in H^1. We easily check that a minimal connection for v_k relative to distance d_{w_n} does not allow more than $\sum_{i=1}^{M_k} |\deg(v_k, a_i)|$ connections to the boundary. Therefore, extracting a subsequence $(n_l)_{l\in\mathbb{N}}$, we can relabel the singularities of v_k and the virtual singularities on the boundary given by a minimal connection relative to $d_{w_{n_l}}$, as a list of positive points $(P_1^l, \dots, P_{K_k}^l)$ and a list of negative points $(N_1^l, \dots, N_{K_k}^l)$ with K_k independent of l and such that

$$\tilde{L}_{w_{n_l}}(v_k) = \underset{\sigma\in\mathcal{S}_{K_k}}{\operatorname{Min}} \sum_{j=1}^{K_k} d_{w_{n_l}}(P_j^l, N_{\sigma(j)}^l) = \sum_{j=1}^{K_k} d_{w_{n_l}}(P_j^l, N_{\sigma_l(j)}^l)$$

for some permutation $\sigma_l \in \mathcal{S}_{K_k}$. Extracting another subsequence if necessary, we may assume that $\sigma_l = \sigma_\star$ is independent of $l \in \mathbb{N}$ and that $P_j^l \underset{l\to+\infty}{\to} P_j$ and $N_j^l \underset{l\to+\infty}{\to} N_j$ for $j = 1, \dots, K_k$. From the results in [14] Section 4.1, we know that assumption *(i)* implies that d_{w_n} converges to d_w uniformly on $\overline{\Omega} \times \overline{\Omega}$ and then we have

$$\tilde{L}_{w_{n_l}}(v_k) = \sum_{j=1}^{K_k} d_{w_{n_l}}(P_j^l, N_{\sigma_\star(j)}^l) \underset{l\to+\infty}{\to} \sum_{j=1}^{K_k} d_w(P_j, N_{\sigma_\star(j)})$$

By definition of $\tilde{L}_w(v_k)$, we obtain that

$$\tilde{L}_w(v_k) \leq \lim_{l\to+\infty} \tilde{L}_{w_{n_l}}(v_k).$$

On the other hand, we can also relabel the singularities of v_k and the virtual singularities on the boundary given by a minimal connection relative to d_w, as a list of positive points $(\overline{P}_1, \dots, \overline{P}_{\overline{K}})$ and a list of negative points $(\overline{N}_1, \dots, \overline{N}_{\overline{K}})$ such that

$$\tilde{L}_w(v_k) = \sum_{j=1}^{\overline{K}} d_w(\overline{P}_j, \overline{N}_j).$$

As previously, we have for any $l \in \mathbb{N}$,

$$\tilde{L}_{w_{n_l}}(v_k) \leq \sum_{j=1}^{\overline{K}} d_{w_{n_l}}(\overline{P}_j, \overline{N}_j).$$

Letting $l \to +\infty$, we obtain

$$\lim_{l \to +\infty} \tilde{L}_{w_{n_l}}(v_k) \leq \sum_{j=1}^{\overline{K}} d_w(\overline{P}_j, \overline{N}_j)$$

and then we conclude that $\lim\limits_{l \to +\infty} \tilde{L}_{w_{n_l}}(v_k) = \tilde{L}_w(v_k)$. By uniqueness of the limit, we get that the convergence holds for the full sequence i.e.,

$$\tilde{L}_{w_n}(v_k) \underset{n \to +\infty}{\to} \tilde{L}_w(v_k).$$

At this stage, we can proceed as in the proof of Theorem 4.2 *(i)⇒(ii)* using Lemma 5.1 instead of Lemma 3.1. ■

We obtain the following variants of Proposition 4.1 and Proposition 4.2 using Theorem 5.2 instead of Theorem 4.1.

Proposition 5.2. *Let $(w_n)_{n \in \mathbb{N}}$ be a sequence of measurable real functions satisfying (4.2) and assume that (a) or (b) in Proposition 4.1 holds. Then (5.9) holds.*

Proposition 5.3. *Let $(\rho_n)_{n \in \mathbb{N}}$ be a sequence of smooth mollifiers. Extending w by a sufficiently large constant and setting $w_n = \rho_n * w$, then (5.9) holds.*

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