Energy with Weight for S^2-Valued Maps with Prescribed Singularities

Vincent Millot

Laboratoire J.L. Lions, Université Pierre et Marie Curie, B.C. 187
4 Place Jussieu, 75252 Paris Cedex 05, France
E-mail adress: millot@ann.jussieu.fr

Abstract

We generalize a result of H. Brezis, J.M. Coron and E. Lieb concerning the infimum of the Dirichlet energy over classes of S^2-valued maps with prescribed singularities to an energy with measurable weight and we prove some geometric properties of such quantity. We also give some stability and approximation results.

1 Introduction and Main Results

Let Ω be a smooth bounded and connected open set of $\mathbb{R}^3$ or $\Omega = \mathbb{R}^3$ and let $w : \Omega \to \mathbb{R}$ be a measurable function such that

$$0 < \lambda \leq \; w \; \leq \Lambda \quad \text{a.e. in } \Omega \tag{1.1}$$

for some constant λ and Λ. We consider N distinct points $a_1, \ldots, a_N$ in Ω and we define the following class of S^2-valued maps

$$\mathcal{E} = \Big\{ u \in \mathcal{C}^1_{\text{loc}}\left(\overline{\Omega} \setminus \cup_i \{a_i\}, S^2\right), \, u = \text{const on } \partial\Omega, \\ \int_\Omega |\nabla u(x)|^2 dx < +\infty, \, \deg(u, a_i) = d_i \quad \text{for } i = 1, \ldots, N \Big\}$$

(without boundary condition if $\Omega = \mathbb{R}^3$) where the d_i's are given in $\mathbb{Z} \setminus \{0\}$ and such that $\sum d_i = 0$ (which is a necessary and sufficient condition for $\mathcal{E}$ to be non-empty, see [9]). Our goal is to establish a formula for

$$E_w\left((a_i, d_i)_{i=1}^N\right) = \operatorname*{Inf}_{u \in \mathcal{E}} \int_\Omega |\nabla u(x)|^2 w(x) dx. \tag{1.2}$$

In [9], H. Brezis, J.M. Coron and E. Lieb have proved that for $w \equiv 1$ this quantity is equal to $8\pi L$ where L is the *length of a minimal connection* associated to the configuration $(a_i, d_i)_{i=1}^N$ and the Euclidean geodesic distance d_Ω on $\overline{\Omega}$ (see also [1, 6, 7, 17]). The first motivation for studying such a problem comes from the theory of liquid crystals (see [14, 15]). Later F. Bethuel, H. Brezis and J.M. Coron have shown that the notion of minimal connection is very useful when dealing with questions of approximation of S^2-maps by smooth S^2-maps in the strong H^1-topology (see [2, 3]). We also refer to the results of J. Bourgain, H. Brezis, P. Mironescu [4] and H. Brezis, P. Mironescu, A.C. Ponce [10] for some similar problems involving S^1-valued maps. In the *dipole case*, namely when we have two prescribed points P and N of degree $+1$ and -1 respectively, the value of L is equal to $d_\Omega(P, N)$. When w is continuous, we prove that $E_w(P, N) = 8\pi\delta_w(P, N)$ where δ_w denotes the Riemannian distance on $\overline{\Omega}$ defined by

$$\delta_w(P, N) = \operatorname{Inf} \int_0^1 w\left(\gamma(t)\right) |\dot{\gamma}(t)| dt, \tag{1.3}$$

where the infimum is taken over all curves $\gamma \in \mathrm{Lip}_{P,N}\left([0,1], \overline{\Omega}\right)$. Here $\mathrm{Lip}_{P,N}\left([0,1], \overline{\Omega}\right)$ denotes the set of all Lipschitz maps γ from $[0,1]$ with values into $\overline{\Omega}$ such that $\gamma(0) = P$ and $\gamma(1) = N$. For a general measurable function w, we prove that $E_w(P, N)$ induces a geodesic distance on $\overline{\Omega}$ (in the sense defined in Section 2.1). We call the attention of the reader to the fact that, in the measurable case, there is no way to define a distance by a formula like (1.3) since w is not well defined on curves which are sets of null Lebesgue measure. To overcome this difficulty, we construct a kind of "length structure" in which the general idea is to thicken the curves. We proceed as follows. For two points x and y in Ω, we consider the class $\mathcal{P}(x, y)$ of all finite collections of segments $\mathcal{F} = ([\alpha_k, \beta_k])_{k=1}^{n(\mathcal{F})}$ such that $\beta_k = \alpha_{k+1}$, $\alpha_1 = x$, $\beta_{n(\mathcal{F})} = y$ and $[\alpha_k, \beta_k] \subset \Omega$. We define "the length" of an element $\mathcal{F} \in \mathcal{P}(x, y)$ by

$$\ell_w(\mathcal{F}) = \sum_{k=1}^{n(\mathcal{F})} \liminf_{\varepsilon \to 0^+} \frac{1}{\pi\varepsilon^2} \int_{\Xi([\alpha_k,\beta_k],\varepsilon)\cap\Omega} w(\xi) d\xi.$$

where $\Xi\left([\alpha_k, \beta_k], \varepsilon\right) = \left\{ \xi \in \mathbb{R}^3, \operatorname{dist}\left(\xi, [\alpha_k, \beta_k]\right) \leq \varepsilon \right\}$ and then we consider the function $d_w : \Omega \times \Omega \to \mathbb{R}_+$ defined by

$$d_w(x, y) = \operatorname*{Inf}_{\mathcal{F} \in \mathcal{P}(x,y)} \ell_w(\mathcal{F}).$$

In Section 2, we extend d_w to $\overline{\Omega}\times\overline{\Omega}$ and we prove the metric and geodesic character of d_w. We also show that d_w agrees with δ_w whenever w is continuous. In the third section, we give the proof of the following result.

Theorem 1.1. *We have*

$$E_w\left((a_i,d_i)_{i=1}^N\right) = 8\pi L_w$$

where L_w is the length of a minimal connection associated to the configuration $(a_i,d_i)_{i=1}^N$ and the distance d_w on $\overline{\Omega}$.

The geodesic character of the distance d_w implies that d_w coincides with the distance induced by the length functional associated to the Finsler metric φ_w obtained by differentiation of d_w (cf. Section 2.2). More precisely, for every P and N in $\overline{\Omega}$, we prove that

$$d_w(P,N) = \mathrm{Min}\ \left\{\int_0^1 \varphi_w\left(\gamma(t),\dot\gamma(t)\right)dt,\ \gamma\in \mathrm{Lip}_{P,N}\left([0,1],\overline{\Omega}\right)\right\}. \tag{1.4}$$

Formula (1.4) shows that, for a non-smooth w, the quantity $E_w\left((a_i,d_i)_{i=1}^N\right)$ is still given in terms of shortest paths between the a_i's but the metric we compute the lengths with might be non-isotropic (a metric φ is said to be isotropic if $\varphi(x,\nu)=p(x)|\nu|$ for some positive function p).

We recall that the length L_w of a minimal connection is computed as follows (see [9]). We relabel the points a_i, taking into account their multiplicity $|d_i|$, as two lists of positive and negative points say $(p_1,\dots,p_K)$ and $(n_1,\dots,n_K)$ (note that this two lists have the same number of elements since $\sum d_i = 0$). Then we have

$$L_w = \underset{\sigma\in\mathcal{S}_K}{\mathrm{Min}} \sum_{j=1}^K d_w(p_j,n_{\sigma(j)}) \tag{1.5}$$

where $\mathcal{S}_K$ denotes the set of all permutations of K indices. Another way to compute L_w is to use the following formula (see [9]),

$$L_w = \mathrm{Max}\sum_{j=1}^K \zeta(p_j)-\zeta(n_j), \tag{1.6}$$

where the supremum is taken over all functions $\zeta:\overline{\Omega}\to\mathbb{R}$ which are 1-Lipschitz with respect to d_w i.e., $|\zeta(x)-\zeta(y)|\le d_w(x,y)$ for all $x,y\in\overline{\Omega}$. In Section 2.3, we give a characterization of 1-Lipschitz functions for the distance d_w. Combining this characterization with formula (1.6), we obtain the lower bound of the energy following the approach in [9]. The upper bound is obtained using explicit test functions based on a *dipole construction*.

Section 4.1 concerns a stability property of problem (1.2). We investigate the following question. Given an arbitrary sequence $(w_n)_{n\in\mathbb{N}}$ of real measurable functions, under which condition on $(w_n)_{n\in\mathbb{N}}$, can we conclude that the sequence $\left\{E_{w_n}\left((a_i,d_i)_{i=1}^N\right)\right\}_{n\in\mathbb{N}}$ converges to $E_w\left((a_i,d_i)_{i=1}^N\right)$? From Theorem 1, we infer that the convergence of $\left\{E_{w_n}\left((a_i,d_i)_{i=1}^N\right)\right\}_{n\in\mathbb{N}}$ is strictly related to the convergence of the variational problems

$$\text{Min}\left\{\int_0^1 \varphi_{w_n}\left(\gamma(t),\dot\gamma(t)\right)dt,\ \gamma\in \text{Lip}_{P,N}\left([0,1],\overline{\Omega}\right)\right\}$$

where $P,N\in\Omega$ and φ_{w_n} denotes the Finsler metric derived from w_n. The same question involving the class $\text{Lip}_{P,N}\left([0,1],\Omega\right)$ instead of the class $\text{Lip}_{P,N}\left([0,1],\overline{\Omega}\right)$ has been studied in [5] by G. Buttazzo, L. De Pascale and I. Fragalà in the Γ-convergence framework. Adapting their result to our setting, we give a necessary and sufficient condition on $(w_n)_{n\in\mathbb{N}}$ under which $\left\{E_{w_n}\left((a_i,d_i)_{i=1}^N\right)\right\}_{n\in\mathbb{N}}$ converges to $E_w\left((a_i,d_i)_{i=1}^N\right)$. In Section 4.2, we concentrate on the approximation procedure by smooth weights. If one requires that w_n is continuous and converges to w uniformly in $\overline{\Omega}$ then we get easily the convergence using formula (1.3) but such an assumption implies that w is continuous and this is quite restrictive in our setting. On the other hand if one assumes that $w_n\to w$ almost everywhere in Ω, we show that the convergence of the problems does not hold in general (c.f. Remark 4.1). However, we prove that $E_w\left((a_i,d_i)_{i=1}^N\right)$ is the limit of a sequence $\left\{E_{w_n}\left((a_i,d_i)_{i=1}^N\right)\right\}_{n\in\mathbb{N}}$ where w_n obtained from w by regularization.

In the last section, we present a partial result on a similar problem involving a matrix field $M=(m_{kl})_{k,l=1}^3$ instead of a weight:

$$E_M\left((a_i,d_i)_{i=1}^N\right)=\underset{u\in\mathcal{E}}{\text{Inf}}\int_\Omega\sum_{k,l=1}^3 m_{kl}(x)\frac{\partial u}{\partial x_k}\cdot\frac{\partial u}{\partial x_l}\,dx.$$

Throughout the paper, a sequence of smooth mollifiers means any sequence $(\rho_n)_{n\in\mathbb{N}}$ satisfying

$$\rho_n\in C^\infty(\mathbb{R}^3,\mathbb{R}),\quad \text{Supp}\,\rho_n\subset B_{1/n}(0),\quad \int_{\mathbb{R}^3}\rho_n=1,\quad \rho_n\geq 0\ \text{ on }\ \mathbb{R}^3.$$

2 Preliminary Results: Metric Properties of d_w

2.1 Metric and Geodesic Character of d_w

First of all we recall that for any metric space (M, d), we may associate the length functional $\mathbb{L}_d$ defined by

$$\mathbb{L}_d(\gamma) = \text{Sup}\left\{\sum_{k=0}^{m-1} d\left(\gamma(t_k), \gamma(t_{k+1})\right),\, 0 = t_0 < t_1 < \ldots < t_m = 1,\, m \in \mathbb{N}\right\}$$

where $\gamma : [0,1] \to M$ is any continuous curve. Note that $\mathbb{L}_d$ is lower semi-continuous on $\mathcal{C}^0([0,1], M)$ endowed with the topology of the uniform convergence on $[0,1]$.

Definition 2.1. A distance d is said to be *geodesic* on M if for all $x, y \in M$,

$$d(x,y) = \text{Inf } \mathbb{L}_d(\gamma)$$

where the infimum is taken over all continuous curves $\gamma : [0,1] \to M$ such that $\gamma(0) = x$ and $\gamma(1) = y$.

Proposition 2.1. *d_w defines a geodesic distance on $\overline{\Omega}$ which is equivalent to the Euclidean geodesic distance d_Ω and d_w agrees with δ_w whenever w is continuous.*

Proof. Step 1. Let $x, y \in \Omega$ and let $\mathcal{F} = ([\alpha_1, \beta_1], \ldots, [\alpha_n, \beta_n])$ be an element of $\mathcal{P}(x,y)$. From assumption (1.1), we get that

$$\ell_w(\mathcal{F}) \geq \sum_{k=1}^{n} \lim_{\varepsilon \to 0^+} \frac{\lambda}{\pi\varepsilon^2} \int_{\Xi([\alpha_k,\beta_k],\varepsilon)\cap\Omega} d\xi = \lambda \sum_{k=1}^{n} |\alpha_k - \beta_k| \geq \lambda\, d_\Omega(x,y). \tag{2.1}$$

By the definition of d_w and (1.1), for any $\mathcal{F} = ([\alpha_1, \beta_1], \ldots, [\alpha_n, \beta_n])$ in $\mathcal{P}(x,y)$, we have

$$d_w(x,y) \leq \Lambda \sum_{k=1}^{n} \lim_{\varepsilon \to 0^+} \frac{1}{\pi\varepsilon^2} \int_{\Xi([\alpha_k,\beta_k],\varepsilon)\cap\Omega} d\xi = \Lambda \sum_{k=1}^{n} |\alpha_k - \beta_k|.$$

Taking the infimum over all $\mathcal{F} \in \mathcal{P}(x,y)$, we infer that

$$d_w(x,y) \leq \Lambda\, d_\Omega(x,y). \tag{2.2}$$

From (2.1) and (2.2), we deduce that $d_w(x,y) = 0$ if and only if $x = y$. Now let us now prove that d_w is symmetric. Let $x, y \in \Omega$ and $\delta > 0$ arbitrary small. We can find $\mathcal{F}_\delta = ([\alpha_1, \beta_2], \ldots, [\alpha_n, \beta_n])$ in $\mathcal{P}(x,y)$ satisfying

$$\ell_w(\mathcal{F}_\delta) \leq d_w(x,y) + \delta.$$

Then for $\mathcal{F}'_\delta = ([\beta_n, \alpha_n], \ldots, [\beta_1, \alpha_1]) \in \mathcal{P}(y, x)$, we have

$$d_w(y, x) \leq \ell_w\left(\mathcal{F}'_\delta\right) = \ell_w\left(\mathcal{F}_\delta\right) \leq d_w(x, y) + \delta.$$

Since δ is arbitrary, we obtain $d_w(y, x) \leq d_w(x, y)$ and we conclude that $d_w(y, x) = d_w(x, y)$ inverting the roles of x and y. The triangle inequality is immediate since the juxtaposition of $\mathcal{F}_1 \in \mathcal{P}(x, z)$ with $\mathcal{F}_2 \in \mathcal{P}(z, y)$ is an element of $\mathcal{P}(x, y)$. Hence d_w defines a distance on Ω verifying

$$\lambda d_\Omega(x, y) \leq d_w(x, y) \leq \Lambda d_\Omega(x, y) \quad \text{for all } x, y \in \Omega. \tag{2.3}$$

Therefore distance d_w extends uniquely to $\overline{\Omega} \times \overline{\Omega}$ into a distance function that we still denote by d_w. By continuity, d_w satisfies (2.3) on $\overline{\Omega}$.

If w is continuous, it is easy to see that for a segment $[\alpha, \beta] \subset \Omega$ we have

$$\lim_{\varepsilon \to 0^+} \frac{1}{\pi \varepsilon^2} \int_{\Xi([\alpha,\beta],\varepsilon)\cap\Omega} w(\xi) d\xi = \int_{[\alpha,\beta]} w(s) ds,$$

and we obtain for $\mathcal{F} = ([\alpha_1, \beta_1], \ldots, [\alpha_n, \beta_n]) \in \mathcal{P}(x, y)$ and $x, y \in \Omega$,

$$\ell_w\left(\mathcal{F}\right) = \int_{\cup_{k=1}^n [\alpha_k, \beta_k]} w(s) ds. \tag{2.4}$$

Since w is continuous, the infimum in (1.3) can be taken over all piecewise affine curves $\gamma : [0, 1] \to \Omega$ such that $\gamma(0) = x$ and $\gamma(1) = y$ and we infer from (2.4) that $d_w(x, y) = \delta_w(x, y)$. Then $d_w \equiv \delta_w$ on $\Omega \times \Omega$ which implies that the equality holds on $\overline{\Omega} \times \overline{\Omega}$ by continuity.

Step 2. We prove the geodesic character of d_w on $\overline{\Omega}$. Since d_w is equivalent to d_Ω, $\overline{\Omega}$ endowed with d_w remains complete. By Theorem 1.8 in [16], it suffices to prove that for any $x, y \in \overline{\Omega}$ and any $\delta > 0$, we can find a point $z \in \overline{\Omega}$ verifying

$$\max(d_w(x, z), d_w(z, y)) \leq \frac{1}{2}\, d_w(x, y) + \delta.$$

Fix $x, y \in \overline{\Omega}$ and then $\tilde{x}, \tilde{y} \in \Omega$ such that $d_w(x, \tilde{x}) + d_w(y, \tilde{y}) \leq \delta/2$ and let $\mathcal{F} = ([\alpha_1, \beta_1], \ldots, [\alpha_n, \beta_n])$ in $\mathcal{P}(x, y)$ satisfying $\ell_w(\mathcal{F}) \leq d_w(\tilde{x}, \tilde{y}) + \delta/2$. For every $1 \leq m \leq n$, we set $\mathcal{F}_m = ([\alpha_1, \beta_1], \ldots, [\alpha_m, \beta_m])$. We consider $n_\star \in \mathbb{N}$ defined by

$$n_\star = \begin{cases} \text{Max}\left\{ m,\ 2 \leq m \leq n,\ \ell_w\left(\mathcal{F}_{m-1}\right) < \frac{1}{2}\,\ell_w(\mathcal{F}) \right\} & \text{if } \ell_w\left(\mathcal{F}_1\right) < \frac{1}{2}\,\ell_w(\mathcal{F}), \\ 1 & \text{otherwise,} \end{cases}$$

and $s \in (0,1)$ defined by

$$s = \begin{cases} \dfrac{\ell_w(\mathcal{F}) - 2\,\ell_w([\alpha_1,\beta_1],\ldots,[\alpha_{n_\star-1},\beta_{n_\star-1}])}{2\,\ell_w([\alpha_{n_\star},\beta_{n_\star}])} & \text{if } n_\star > 1, \\ \dfrac{\ell_w(\mathcal{F})}{2\ell_w([\alpha_{n_\star},\beta_{n_\star}])} & \text{if } n_\star = 1. \end{cases}$$

Let $\varepsilon_k \to 0^+$ as $k \to +\infty$ such that

$$\ell_w([\alpha_{n_\star},\beta_{n_\star}]) = \lim_{k\to+\infty} \frac{1}{\pi\varepsilon_k^2} \int_{\Xi([\alpha_{n_\star},\beta_{n_\star}],\varepsilon_k)\cap\Omega} w(\xi)d\xi.$$

For each $k \in \mathbb{N}$, we choose $z_k \in [\alpha_{n_\star},\beta_{n_\star}]$ verifying

$$\begin{aligned} \frac{1}{\pi\varepsilon_k^2} \int_{\Xi([\alpha_{n_\star},z_k],\varepsilon_k)\cap\Omega} w(\xi)d\xi =& \frac{s}{\pi\varepsilon_k^2} \int_{\Xi([\alpha_{n_\star},\beta_{n_\star}],\varepsilon_k)\cap\Omega} w(\xi)d\xi + \mathcal{O}(\varepsilon_k), \\ \frac{1}{\pi\varepsilon_k^2} \int_{\Xi([z_k,\beta_{n_\star}],\varepsilon_k)\cap\Omega} w(\xi)d\xi =& \frac{1-s}{2\pi\varepsilon_k^2} \int_{\Xi([\alpha_{n_\star},\beta_{n_\star}],\varepsilon_k)\cap\Omega} w(\xi)d\xi + \mathcal{O}(\varepsilon_k). \end{aligned}$$

Extracting a subsequence if necessary, we may assume that $z_k \underset{k\to+\infty}{\to} z$ with $z \in [\alpha_{n_\star},\beta_{n_\star}]$. Then we have

$$\begin{aligned} \frac{1}{\pi\varepsilon_k^2} \int_{\Xi([\alpha_{n_\star},z],\varepsilon_k)\cap\Omega} w(\xi)d\xi =& \frac{s}{\pi\varepsilon_k^2} \int_{\Xi([\alpha_{n_\star},\beta_{n_\star}],\varepsilon_k)\cap\Omega} w(\xi)d\xi \\ & + \mathcal{O}(\varepsilon_k) + \mathcal{O}(|z-z_k|), \\ \frac{1}{\pi\varepsilon_k^2} \int_{\Xi([z,\beta_{n_\star}],\varepsilon_k)\cap\Omega} w(\xi)d\xi =& \frac{1-s}{2\pi\varepsilon_k^2} \int_{\Xi([\alpha_{n_\star},\beta_{n_\star}],\varepsilon_k)\cap\Omega} w(\xi)d\xi \\ & + \mathcal{O}(\varepsilon_k) + \mathcal{O}(|z-z_k|). \end{aligned}$$

Taking the $\liminf$ in k, we derive

$$\ell_w([\alpha_{n_\star},z]) \le s\ell_w([\alpha_{n_\star},\beta_{n_\star}]) \quad \text{and} \quad \ell_w([z,\beta_{n_\star}]) \le (1-s)\ell_w([\alpha_{n_\star},\beta_{n_\star}]).$$

Therefore we obtain that the elements $\mathcal{F}_{\tilde{x}} = ([\alpha_1,\beta_1],\ldots,[\alpha_{n_\star},z]) \in \mathcal{P}(\tilde{x},z)$ and $\mathcal{F}_{\tilde{y}} = ([z,\beta_{n_\star}],\ldots,[\alpha_n,\beta_n]) \in \mathcal{P}(z,\tilde{y})$ verify

$$\begin{aligned} d_w(\tilde{x},z) \le \ell_w(\mathcal{F}_{\tilde{x}}) \le \frac{1}{2}\,\ell_w(\mathcal{F}) \le \frac{1}{2}\,d_w(\tilde{x},\tilde{y}) + \delta/4, \\ d_w(\tilde{y},z) \le \ell_w(\mathcal{F}_{\tilde{y}}) \le \frac{1}{2}\,\ell_w(\mathcal{F}) \le \frac{1}{2}\,d_w(\tilde{x},\tilde{y}) + \delta/4, \end{aligned}$$

and we conclude that

$$\max(d_w(x,z), d_w(y,z)) \leq \max(d_w(\tilde{x},z), d_w(\tilde{y},z)) + \frac{\delta}{2} \leq \frac{1}{2}\, d_w(\tilde{x},\tilde{y}) + \frac{3\delta}{4}$$
$$\leq \frac{1}{2}\, d_w(x,y) + \delta$$

i.e. the point z meets the requirement. ■

Remark 2.1. The geodesic character of d_w implies that two arbitrary points of $\left(\overline{\Omega}, d_w\right)$ can be linked by a minimizing geodesic. We mean by a minimizing geodesic any curve $\gamma : I \to \overline{\Omega}$ such that

$$d_w(\gamma(t), \gamma(t')) = |t - t'| \quad \text{for all } t, t' \in I,$$

where I is some interval of $\mathbb{R}$. In particular we obtain the existence for all $x, y \in \overline{\Omega}$ of a curve $\gamma_{xy} \in \mathrm{Lip}_{x,y}\left([0,1], \overline{\Omega}\right)$ satisfying

$$d_w\left(\gamma_{xy}(t), \gamma_{xy}(t')\right) = \mathbb{L}_{d_w}(\gamma_{xy})|t - t'| \quad \text{for all } t, t' \in [0,1]$$

(and then $d_w(x,y) = \mathbb{L}_{d_w}(\gamma_{xy})$). Indeed, $\left(\overline{\Omega}, d_w\right)$ defines a complete and locally compact metric space and since d_w is of geodesic type, the existence of a minimizing geodesic is ensured by the Hopf-Rinow Theorem (see [16], Chapter 1). Moreover we deduce from (2.3) that any minimizing geodesic for the distance d_w is a λ^{-1}-Lipschitz curve for the Euclidean geodesic distance.

2.2 Integral Representation of the Length Functional

In this section, we show that d_w is actually induced by a Finsler metric in the sense defined below.

Definition 2.2. A Borel measurable function $\varphi : \overline{\Omega} \times \mathbb{R}^3 \to [0, +\infty)$ is said to be a *Finsler metric* if $\varphi(x, \cdot)$ is positively 1-homogeneous for every $x \in \overline{\Omega}$ and convex for almost every $x \in \overline{\Omega}$.

Proposition 2.2. *There exists a Finsler metric $\varphi_w : \overline{\Omega} \times \mathbb{R}^3 \to [0, +\infty)$ such that for every Lipschitz curve $\gamma : [0,1] \to \overline{\Omega}$,*

$$\mathbb{L}_{d_w}(\gamma) = \int_0^1 \varphi_w\left(\gamma(t), \dot{\gamma}(t)\right) dt. \tag{2.5}$$

Moreover, for every $x, y \in \overline{\Omega}$, we have

$$d_w(x,y) = \mathrm{Min}\left\{\int_0^1 \varphi_w\left(\gamma(t), \dot{\gamma}(t)\right) dt,\ \gamma \in \mathrm{Lip}_{x,y}\left([0,1], \overline{\Omega}\right)\right\}. \tag{2.6}$$

Proof. *Step 1.* Assume that $\Omega = \mathbb{R}^3$. To distance d_w we associate the function $\varphi_w : \mathbb{R}^3 \times \mathbb{R}^3 \to [0, +\infty)$ defined by

$$\varphi_w(x, \nu) = \limsup_{t \to 0^+} \frac{d_w(x, x + t\nu)}{t}.$$

In [19], it is proved that φ_w defines a Finsler metric and the proof of (2.5) is given in [13], Theorem 2.5. Then (2.6) directly follows from Remark 2.1.
Step 2. Assume that Ω is a smooth bounded and connected open set of $\mathbb{R}^3$. For $\delta > 0$, we consider $\Omega_\delta = \{x \in \mathbb{R}^3, \operatorname{dist}(x, \Omega) < \delta\}$ where "dist" denotes the usual Euclidean distance on $\mathbb{R}^3$. We choose δ sufficiently small for the projection Πx of $x \in \Omega_\delta$ on $\overline{\Omega}$ to be well defined and smooth. Setting $x_\perp = x - \Pi x$ for $x \in \Omega_\delta$, we define the function $d_{w,\delta} : \Omega_\delta \times \Omega_\delta \to [0, +\infty)$ by

$$d_{w,\delta}(x, y) = d_w(\Pi x, \Pi y) + |x_\perp - y_\perp|.$$

We easily check that $d_{w,\delta}$ defines a distance on Ω_δ. Then we consider for $x, y \in \Omega_\delta$,

$$\overline{d}_{w,\delta}(x, y) = \operatorname{Inf}\, \mathbb{L}_{d_{w,\delta}}(\gamma),$$

where the infimum is taken over all $\gamma \in \mathcal{C}^0([0,1], \Omega_\delta)$ satisfying $\gamma(0) = x$ and $\gamma(1) = y$. We also easily verify that $\overline{d}_{w,\delta}$ defines a distance on Ω_δ and it follows from Proposition 1.6 in [16] that

$$\mathbb{L}_{\overline{d}_{w,\delta}} = \mathbb{L}_{d_{w,\delta}} \quad \text{on } \mathcal{C}^0([0,1], \Omega_\delta). \tag{2.7}$$

Therefore $\overline{d}_{w,\delta}(x, y)$ is a geodesic distance on Ω_δ. Moreover we infer from (2.3) that $\overline{d}_{w,\delta}$ is equivalent to the Euclidean geodesic distance on Ω_δ. Now we consider $\varphi_{w,\delta} : \Omega_\delta \times \mathbb{R}^3 \to [0, +\infty)$ defined by

$$\varphi_{w,\delta}(x, \nu) = \limsup_{t \to 0^+} \frac{\overline{d}_{w,\delta}(x, x + t\nu)}{t}.$$

By the results in [19], $\varphi_{w,\delta}$ is Borel measurable, positively 1-homogeneous in ν for every $x \in \Omega_\delta$ and convex in ν for almost every $x \in \Omega_\delta$. By Theorem 2.5 in [13], we have for every Lipschitz curve $\gamma : [0,1] \to \Omega_\delta$,

$$\mathbb{L}_{\overline{d}_{w,\delta}}(\gamma) = \int_0^1 \varphi_{w,\delta}(\gamma(t), \dot\gamma(t))\, dt. \tag{2.8}$$

Since $d_{w,\delta} = d_w$ on $\overline{\Omega}$, we deduce that

$$\mathbb{L}_{d_{w,\delta}} = \mathbb{L}_{d_w} \quad \text{on } \mathcal{C}^0\big([0,1], \overline{\Omega}\big). \tag{2.9}$$

If we denote by φ_w the restriction of $\varphi_{w,\delta}$ to $\overline{\Omega} \times \mathbb{R}^3$, we obtain (2.5) combining (2.7-2.9). Then (2.6) follows from Remark 2.1. ■

Remark 2.2. If we assume that w is continuous in Ω, we have

$$\varphi_w(x,\nu) = w(x)|\nu| \quad \text{for every } (x,\nu) \in \Omega \times \mathbb{R}^3.$$

Indeed, fix $(x,\nu) \in \Omega \times \mathbb{R}^3 \setminus \{0\}$, $t > 0$ such that $B(x, 2t\lambda^{-1}|\nu|) \subset \Omega$ and consider a sequence $\gamma_n \in \mathrm{Lip}([0,1], \overline{\Omega})$ verifying

$$\int_0^1 w\left(\gamma_n(s)\right) |\dot{\gamma}_n(s)| ds \to d_w(x, x + t\nu) \quad \text{as } n \to +\infty.$$

Since $d_w \geq \lambda d_\Omega$, we infer that $\gamma_n([0,1]) \subset B(x, 2t\lambda^{-1}|\nu|)$ and therefore

$$\int_0^1 w\left(\gamma_n(s)\right) |\dot{\gamma}_n(s)| ds \geq w(x) \int_0^1 |\dot{\gamma}_n(s)| ds - o(t) \geq w(x)t|\nu| - o(t).$$

Letting $n \to +\infty$, we obtain

$$\frac{d_w(x, x + t\nu)}{t} \geq w(x)|\nu| - o(1).$$

But we trivially have

$$\frac{d_w(x, x + t\nu)}{t} \leq \frac{1}{t} \int_0^t w(x + s\nu)|\nu| ds = w(x)|\nu| + o(1).$$

We derive the result from these two last inequalities letting $t \to 0$.

2.3 Characterization of 1-Lipschitz Functions

Proposition 2.3. *Assume that (1.1) holds. Then for all* $\zeta : \overline{\Omega} \to \mathbb{R}$*, the following properties are equivalent:*

i) $|\zeta(x) - \zeta(y)| \leq d_w(x,y)$ *for all* $x, y \in \overline{\Omega}$.

ii) ζ *is Lipschitz continuous and* $|\nabla\zeta(x)| \leq w(x)$ *for a.e.* $x \in \Omega$.

Proof. i) $\Rightarrow$ *ii).* Let $\zeta : \overline{\Omega} \to \mathbb{R}$ satisfying *i)*. From Proposition 2.1, we infer that ζ is Lipschitz continuous. Fix $x_0 \in \Omega$ and $R > 0$ such that $B_{3R}(x_0) \subset \Omega$. Let $(\rho_n)_{n \in \mathbb{N}}$ be a sequence of smooth mollifiers and consider, for $n > 1/R$, the smooth function $\zeta_n = \rho_n * \zeta : B_R(x_0) \to \mathbb{R}$. We write

$$\zeta_n(x) = \int_{B_{1/n}} \rho_n(-z)\zeta(x+z) dz$$

and therefore for all $x, y \in B_R(x_0)$,

$$\begin{aligned} |\zeta_n(x) - \zeta_n(y)| &\leq \int_{B_{1/n}} \rho_n(-z) \, |\zeta(x+z) - \zeta(y+z)| \, dz \\ &\leq \int_{B_{1/n}} \rho_n(-z) \, d_w(x+z, y+z) dz \\ &\leq \int_{B_{1/n}} \rho_n(-z) \, \ell_w \left([x+z, y+z] \right) dz. \end{aligned}$$

Taking an arbitrary sequence of positive numbers $\varepsilon_k \to 0$ as $k \to +\infty$ and using Fatou's lemma, we get that

$$\begin{aligned} |\zeta_n(x) - \zeta_n(y)| \leq \int_{B_{1/n}} \rho_n(-z) \left(\liminf_{k \to +\infty} \frac{1}{\pi \varepsilon_k^2} \int_{\Xi([x+z, y+z], \varepsilon_k) \cap \Omega} w(\xi) d\xi \right) dz \\ \leq \liminf_{k \to +\infty} \frac{1}{\pi \varepsilon_k^2} \int_{B_{1/n}} \int_{\Xi([x+z, y+z], \varepsilon_k) \cap \Omega} \rho_n(-z) w(\xi) d\xi dz. \end{aligned}$$

For $k \in \mathbb{N}$ sufficiently large, we have $\Xi\left([x+z, y+z], \varepsilon_k\right) \subset B_{3R}(x_0)$ and accordingly

$$\begin{aligned} \int_{B_{1/n}} \int_{\Xi([x+z, y+z], \varepsilon_k)} \rho_n(-z) w(\xi) d\xi dz &= \int_{\Xi([x,y], \varepsilon_k)} \int_{B_{1/n}} \rho_n(-z) w(\xi + z) dz d\xi \\ &= \int_{\Xi([x,y], \varepsilon_k)} \rho_n * w(\xi) d\xi. \end{aligned}$$

Since $\rho_n * w$ is smooth, we obtain as in the proof of Proposition 2.1,

$$\frac{1}{\pi \varepsilon_k^2} \int_{\Xi([x,y], \varepsilon_k)} \rho_n * w(\xi) d\xi \to \int_{[x,y]} \rho_n * w(s) ds \quad \text{as } k \to +\infty.$$

Thus for each $x, y \in B_R(x_0)$ we have

$$|\zeta_n(x) - \zeta_n(y)| \leq \int_{[x,y]} \rho_n * w(s) ds.$$

Then for $x \in B_R(x_0)$, $h \in S^2$ fixed and $\delta > 0$ small, we derive

$$\frac{|\zeta_n(x + \delta h) - \zeta_n(x)|}{\delta} \leq \frac{1}{\delta} \int_{[x, x+\delta h]} \rho_n * w(s) ds \underset{\delta \to 0^+}{\to} \rho_n * w(x)$$

and we conclude, letting $\delta \to 0$, that $|\nabla \zeta_n(x) \cdot h| \leq \rho_n * w(x)$ for each $x \in B_R(x_0)$ and $h \in S^2$ which implies that $|\nabla \zeta_n| \leq \rho_n * w$ on $B_R(x_0)$. Since

$\nabla\zeta_n \to \nabla\zeta$ and $\rho_n * w \to w$ a.e. on $B_R(x_0)$ as $n \to +\infty$, we deduce that $|\nabla\zeta| \leq w$ a.e. on $B_R(x_0)$. Since x_0 is arbitrary in Ω, we get the result.

ii) $\Rightarrow$ *i)* The reverse implication follows from the lemma below.

Lemma 2.1. *Let $\zeta : \overline{\Omega} \to \mathbb{R}$ be a Lipschitz continuous function. For all $a, b \in \Omega$ with $[a,b] \subset \Omega$ and all $\varepsilon > 0$ sufficiently small, we have*

$$|\zeta(a) - \zeta(b)| \leq \frac{1}{\pi\varepsilon^2} \int_{\Xi([a,b],\varepsilon)\cap\Omega} |\nabla\zeta(z)| dz + 2\varepsilon \|\nabla\zeta\|_\infty .$$

Indeed, let ζ be a Lipschitz continuous function satisfying *ii)*. We deduce from Lemma 2.1 and (1.1) that for all $\mathcal{F} = ([\alpha_1, \beta_1], \ldots, [\alpha_n, \beta_n]) \in \mathcal{P}(x,y)$ and all parameters $\varepsilon_1, \ldots, \varepsilon_n > 0$ sufficiently small, we have

$$|\zeta(x) - \zeta(y)| \leq \sum_{k=1}^{n} |\zeta(\beta_k) - \zeta(\alpha_k)| \leq \sum_{k=1}^{n} \left(\frac{1}{\pi\varepsilon_k^2} \int_{\Xi([\alpha_k,\beta_k],\varepsilon_k)\cap\Omega} w(z)dz + 2\Lambda\varepsilon_k \right).$$

Taking successively the lim inf in $\varepsilon_k \to 0^+$ for each parameter ε_k, we get that $|\zeta(x) - \zeta(y)| \leq \ell_w(\mathcal{F})$. We obtain the result for $x, y \in \Omega$ taking the infimum over all $\mathcal{F} \in \mathcal{P}(x,y)$. We conclude that *i)* holds in all $\overline{\Omega}$ by continuity. ■

Proof of Lemma 2.1. First note that we just have to prove the inequality for smooth functions ζ, the general case follows by a density argument. Let ζ be a smooth real valued function. Without loss of generality, we may assume that $a = (0,0,0)$ and $b = (0,0,R)$. Then for all $\varepsilon > 0$ such that the 3D-cylinder $B_\varepsilon^{(2)}(0) \times [0,R]$ is included in Ω, and all $(x_1, x_2) \in B_\varepsilon^{(2)}(0)$, we have

$$\begin{aligned} |\zeta(b) - \zeta(a)| &\leq |\zeta(0,0,R) - \zeta(x_1,x_2,R)| + |\zeta(x_1,x_2,R) - \zeta(x_1,x_2,0)| \\ &\quad + |\zeta(x_1,x_2,0) - \zeta(0,0,0)| \\ &\leq \int_0^R |\nabla\zeta(x_1,x_2,x_3)|\, dx_3 + 2\varepsilon \|\nabla\zeta\|_\infty . \end{aligned}$$

Integrating the last inequality in $(x_1, x_2) \in B_\varepsilon^{(2)}(0)$ yields

$$\pi\varepsilon^2 |\zeta(b) - \zeta(a)| \leq \int_{B_\varepsilon^{(2)}(0)\times[0,R]} |\nabla\zeta(x_1,x_2,x_3)|\, dx_1 dx_2 dx_3 + 2\pi\varepsilon^3 \|\nabla\zeta\|_\infty .$$

Dividing by $\pi\varepsilon^2$, we get the result since $B_\varepsilon^{(2)}(0) \times [0,R] \subset \Xi([a,b],\varepsilon) \cap \Omega$. ■

Remark 2.3. In [11], F. Camilli and A. Siconolfi study the Hamilton-Jacobi equation

$$H(x, \nabla u) = 0 \quad \text{a.e. in } \Omega$$

where the Hamiltonian $H(x, \nu)$ is measurable in x, continuous and quasi-convexe in ν. They construct the *optical length function* $L^\Omega : \overline{\Omega} \times \overline{\Omega}$ giving a class of "fundamental solutions". They show that for every $y_0 \in \overline{\Omega}$, $L^\Omega(y_0, \cdot)$ is the maximal element of the set

$$\mathcal{C}(y_0) = \left\{ v \in W^{1,\infty}(\Omega, \mathbb{R}),\ H(x, \nabla v) \leq 0 \text{ a.e in } \Omega,\ v(y_0) = 0 \right\}.$$

In the case $H(x, \nu) = |\nu| - w(x)$, Proposition 2.3 shows that d_w and the optical length function L^Ω coincide i.e., $d_w(x, y) = L^\Omega(x, y)$ for all $x, y \in \overline{\Omega}$.

3 Energy Estimates - Proof of Theorem 1

Theorem 1.1 follows from the combination of Lemma 3.1 and Lemma 3.4 below. In Section 3.2, we give an explicit *dipole construction*.

3.1 Lower Bound for the Energy

Lemma 3.1. *For all $u \in \mathcal{E}$, we have*

$$\int_\Omega |\nabla u|^2 w(x) dx \geq 8\pi L_w.$$

Proof. The proof is essentially the same as in [9] once we have the results of Section 2. We introduce for each $u \in \mathcal{E}$ the vector field D defined by

$$D = \left(u \cdot \frac{\partial u}{\partial x_2} \wedge \frac{\partial u}{\partial x_3},\ u \cdot \frac{\partial u}{\partial x_3} \wedge \frac{\partial u}{\partial x_1},\ u \cdot \frac{\partial u}{\partial x_1} \wedge \frac{\partial u}{\partial x_2} \right). \tag{3.1}$$

As in [9], we have $2|D| \leq |\nabla u|^2$ and $D \in L^1(\Omega)$ defines a distribution which satisfies

$$\operatorname{div} D = 4\pi \sum_{i=1}^N d_i \delta_{a_i} \quad \text{in } \mathcal{D}'(\Omega). \tag{3.2}$$

Relabelling the points (a_i) as positive and negative points taking into account their multiplicity $|d_i|$, we get a list (p_j) of positive points and a list (n_j) of negative points. Since $\sum d_i = 0$, we have as many positive points as negative points. Then we write (3.2) as

$$\operatorname{div} D = 4\pi \sum_{j=1}^K \delta_{p_j} - \delta_{n_j}. \tag{3.3}$$

From Proposition 2.3 and the properties of D, we deduce that for all functions $\zeta : \overline{\Omega} \to \mathbb{R}$ which is 1-Lipschitz with respect to d_w,

$$\int_\Omega |\nabla u|^2 w(x)dx \geq 2\int_\Omega |D|w(x)dx \geq -2\int_\Omega D\cdot\nabla\zeta. \tag{3.4}$$

Using (3.3), we get that

$$\int_\Omega |\nabla u|^2 w(x)dx \geq 8\pi\left(\sum_{j=1}^K \zeta(p_j)-\zeta(n_j)\right) - 8\pi\int_{\partial\Omega}(D\cdot\eta)\zeta\, d\sigma$$

without the boundary term if $\Omega = \mathbb{R}^3$. On $\partial\Omega$, we have $D\cdot\eta = \mathrm{Jac}_2(u_{/\partial\Omega})$ where η denotes the outward normal and $\mathrm{Jac}_2(u_{/\partial\Omega})$ denotes the 2×2 Jacobian determinant of u restricted to $\partial\Omega$. Since each $u \in \mathcal{E}$ is constant on $\partial\Omega$, we have $D\cdot\eta \equiv 0$ on $\partial\Omega$ and therefore we derive

$$\int_\Omega |\nabla u|^2 w(x)dx \geq 8\pi\,\mathrm{Max}\sum_{j=1}^K \zeta(p_j)-\zeta(n_j)$$

where the maximum is taken over all functions ζ which 1-Lipschitz with respect to d_w. By (1.6) we conclude that

$$\int_\Omega |\nabla u|^2 w(x)dx \geq 8\pi L_w$$

for all maps $u \in \mathcal{E}$ which completes the proof of the lower bound. ■

3.2 The Dipole Construction

Lemma 3.2. *Let P, N be two distinct points in Ω. For all $\delta > 0$, there exists $u_\delta \in \mathcal{C}^1_{\mathrm{loc}}\left(\overline{\Omega}\setminus\{P,N\}, S^2\right)$ such that $\deg(u_\delta, P) = +1$, $\deg(u_\delta, N) = -1$ and*

$$\int_\Omega |\nabla u_\delta|^2 w(x)dx \leq 8\pi d_w(P,N) + \delta.$$

Moreover u_δ is constant outside a small neighborhood of a polygonal curve running between P and N.

Proof. For $\varepsilon > 0$, we consider the map $\omega_\varepsilon : \mathbb{R}^2 \to S^2$ defined by

$$\omega_\varepsilon(x,y) = \begin{cases} \dfrac{2\varepsilon^2}{\varepsilon^4+r^2}(x,-y,-\varepsilon^2)+(0,0,1) & \text{if } r\leq\varepsilon \\ (A(r)\cos\theta, -A(r)\sin\theta, C(r)) & \text{if } \varepsilon\leq r\leq 2\varepsilon \\ (0,0,1) & \text{if } 2\varepsilon\leq r \end{cases} \tag{3.5}$$

where $(x,y)=(r\cos\theta, r\sin\theta)$ and

$$A(r)=\frac{-2\varepsilon^2}{\varepsilon^4+\varepsilon^2}\,r+\frac{4\varepsilon^3}{\varepsilon^4+\varepsilon^2}\ ,\ C(r)=\sqrt{1-(A(r))^2}\,.$$

According to the results in [8], ω_ε is Lipschitz continuous and $\deg\omega_\varepsilon=+1$ when one identifies $\mathbb{R}^2\cup\{\infty\}$ with S^2. As in [9], the map ω_ε will be the main ingredient in our construction. First we define the following objects. For two distinct points $\alpha,\beta\in\Omega$ with $[\alpha,\beta]\subset\Omega$, we denote by $p_{\alpha,\beta}(x)$ the projection of $x\in\mathbb{R}^3$ on the straight line passing by α and β and

$$r_{\alpha,\beta}(x)=\operatorname{dist}(x,[\alpha,\beta])\,,\quad h_{\alpha,\beta}(x)=\operatorname{dist}(p_{\alpha,\beta}(x),\{\alpha,\beta\})\,,$$

where "dist" denotes the Euclidean distance in $\mathbb{R}^3$. For some small $\sigma>0$, we consider the following sets:

$$C^\sigma_\varepsilon(\alpha,\beta)=\big\{x\in\mathbb{R}^3,\ p_{\alpha,\beta}(x)\in]\alpha,\beta[,\ \sigma r_{\alpha,\beta}(x)\le h_{\alpha,\beta}(x),\ 0\le h_{\alpha,\beta}(x)\le\sigma\varepsilon\big\}$$
$$T^\sigma_\varepsilon(\alpha,\beta)=\big\{x\in\mathbb{R}^3,\ p_{\alpha,\beta}(x)\in[\alpha,\beta],\ r_{\alpha,\beta}(x)\le\varepsilon,\ h_{\alpha,\beta}(x)\ge\sigma\varepsilon\big\}$$
$$V_\varepsilon(\alpha,\beta)=\big\{x\in\mathbb{R}^3,\ p_{\alpha,\beta}(x)\in[\alpha,\beta],\ r_{\alpha,\beta}(x)\le\varepsilon\big\}\,.$$

We choose ε small enough such that $C^\sigma_{2\varepsilon}(\alpha,\beta)\cup T^\sigma_{2\varepsilon}(\alpha,\beta)\cup V_{2\varepsilon}(\alpha,\beta)\subset\Omega$. We fix $\delta>0$ and we consider $\mathcal{F}=([\alpha_1,\beta_1],\ldots,[\alpha_n,\beta_n])\in\mathcal{P}(P,N)$ such that the curve $\gamma=\cup_k[\alpha_k,\beta_k]$ has no self-intersection points. Then for each $k\in\{1,\ldots,n\}$, we fix two unit vectors i_k and j_k in the orthogonal plane to $\beta_k-\alpha_k$ such that $(i_k,j_k,\frac{\beta_k-\alpha_k}{|\beta_k-\alpha_k|})$ defines a direct orthonormal basis of $\mathbb{R}^3$ and we consider $u^{(k)}_\varepsilon:\Omega\to S^2$ defined by

$$u^{(k)}_\varepsilon(x)=\begin{cases}\omega_\varepsilon\left(X_k(x),Y_k(x)\right) & \text{if } x\in C^\sigma_{2\varepsilon}(\alpha_k,\beta_k),\\ \omega_\varepsilon\left((x-p_{\alpha_k,\beta_k}(x))\cdot i_k\,,(x-p_{\alpha_k,\beta_k}(x))\cdot j_k\right) & \text{if } x\in T^\sigma_{2\varepsilon}(\alpha_k,\beta_k),\\ (0,0,1) & \text{otherwise}\end{cases}$$

with

$$X_k(x)=\frac{2\sigma\varepsilon}{h_{\alpha_k,\beta_k}(x)}(x-p_{\alpha_k,\beta_k}(x))\cdot i_k\ ,\ Y_k(x)=\frac{2\sigma\varepsilon}{h_{\alpha_k,\beta_k}(x)}(x-p_{\alpha_k,\beta_k}(x))\cdot j_k.$$

We easily check that $u^{(k)}_\varepsilon\in W^{1,\infty}_{\rm loc}\left(\overline{\Omega}\setminus\{\alpha_k,\beta_k\},S^2\right)$, $\deg(u^{(k)}_\varepsilon,\alpha_k)=+1$, $\deg(u^{(k)}_\varepsilon,\beta_k)=-1$. Using coordinates in the basis $(i_k,j_k,\frac{\beta_k-\alpha_k}{|\beta_k-\alpha_k|})$, some classical computations (see [6]) lead to

$$|\nabla u^{(k)}_\varepsilon(x)|^2\le(1+C\varepsilon^2)\frac{4\sigma^2\varepsilon^2}{h^2_{\alpha_k,\beta_k}(x)}\,|\nabla\omega_\varepsilon\left(X_k(x),Y_k(x)\right)|^2\ \text{ in } C^\sigma_{2\varepsilon}(\alpha_k,\beta_k).\quad(3.6)$$

By the results in [8], we have

$$\int_{B_{2\varepsilon}(0)\setminus B_\varepsilon(0)} |\nabla\omega_\varepsilon|^2 = \mathcal{O}(\varepsilon)\,, \ \int_{B_\varepsilon(0)} |\nabla\omega_\varepsilon|^2 = 8\pi + \mathcal{O}(\varepsilon) \tag{3.7}$$

and therefore

$$\int_{(T^\sigma_{2\varepsilon}\setminus T^\sigma_\varepsilon)(\alpha_k,\beta_k)} |\nabla\omega_\varepsilon((x-p_{\alpha_k,\beta_k}(x))\cdot i_k\,,(x-p_{\alpha_k,\beta_k}(x))\cdot j_k)|^2 dx = \mathcal{O}(\varepsilon), \tag{3.8}$$

$$\int_{C^\sigma_{2\varepsilon}(\alpha_k,\beta_k)} \frac{4\sigma^2\varepsilon^2}{h^2_{\alpha_k,\beta_k}(x)} \left|\nabla\omega_\varepsilon\left(X_k(x),Y_k(x)\right)\right|^2 dx = \mathcal{O}(\varepsilon). \tag{3.9}$$

We infer from (3.6-3.9) that

$$\int_\Omega |\nabla u_\varepsilon^{(k)}|^2 w(x)dx \leq$$
$$\leq \int_{T^\sigma_\varepsilon(\alpha_k,\beta_k)} |\nabla\omega_\varepsilon\left((x-p_{\alpha_k,\beta_k}(x))\cdot i_k\,,(x-p_{\alpha_k,\beta_k}(x))\cdot j_k\right)|^2 w(x)dx + \mathcal{O}(\varepsilon).$$

Since we have

$$|\nabla\omega_\varepsilon(x,y)|^2 = \frac{8\varepsilon^4}{(\varepsilon^4+x^2+y^2)^2} \quad \text{for } (x,y)\in B_\varepsilon(0),$$

we conclude that

$$\int_\Omega |\nabla u_\varepsilon^{(k)}|^2 w(x)dx \leq 8\int_{V_\varepsilon(\alpha_k,\beta_k)} \frac{\varepsilon^4 w(x)}{\left(\varepsilon^4 + r^2_{\alpha_k,\beta_k}(x)\right)^2}\,dx + \mathcal{O}(\varepsilon). \tag{3.10}$$

Then we set

$$\tilde{\ell}_w(\mathcal{F}) = \sum_{k=1}^n \liminf_{\varepsilon\to 0^+} \frac{1}{\pi}\int_{V_\varepsilon(\alpha_k,\beta_k)} \frac{\varepsilon^4 w(x)}{\left(\varepsilon^4 + r^2_{\alpha_k,\beta_k}(x)\right)^2}\,dx. \tag{3.11}$$

By (3.10) and (3.11), we can choose $\varepsilon_1,\ldots,\varepsilon_n > 0$ arbitrarily small to have

$$\sum_{k=1}^n \int_\Omega |\nabla u_{\varepsilon_k}^{(k)}|^2 w(x)dx \leq 8\pi\tilde{\ell}_w(\mathcal{F}) + \frac{\delta}{4}. \tag{3.12}$$

We choose σ and then each ε_k for $\left\{C^\sigma_{2\varepsilon_k}(\alpha_k,\beta_k)\cup T^\sigma_{2\varepsilon_k}(\alpha_k,\beta_k)\right\}_{k=1}^n$ to define a family of disjoint sets (which is possible since the curve γ has no self intersection points) and such that (3.12) holds. Then we consider the map $\tilde{u}_\delta : \Omega \to S^2$ defined by

$$\tilde{u}_\delta(x) = \begin{cases} u^{(k)}_{\varepsilon_k} & \text{if } x \in C^\sigma_{2\varepsilon_k}(\alpha_k, \beta_k) \cup T^\sigma_{2\varepsilon_k}(\alpha_k, \beta_k), \\ (0,0,1) & \text{if } x \notin \cup_k C^\sigma_{2\varepsilon_k}(\alpha_k, \beta_k) \cup T^\sigma_{2\varepsilon_k}(\alpha_k, \beta_k). \end{cases}$$

By construction, $\tilde{u}_\delta \in W^{1,\infty}_{\text{loc}}\left(\overline{\Omega} \setminus \{P, \alpha_2, \dots, \alpha_n, N\}, S^2\right)$, $\deg(\tilde{u}_\delta, P) = 1$, $\deg(\tilde{u}_\delta, N) = -1$ and $\deg(\tilde{u}_\delta, \alpha_k) = 0$ for $k = 2, \dots, n$. From (3.12), we derive that

$$\int_\Omega |\nabla \tilde{u}_\delta|^2 w(x) dx \leq 8\pi \tilde{\ell}_w(\mathcal{F}) + \frac{\delta}{4}.$$

Since $\deg(\tilde{u}_\delta, \alpha_k) = 0$ for $k = 2, \dots, n$, we can smoothen $\tilde{u}_\delta$ around γ, using the result in [2], in order to obtain a new map $u_\delta \in \mathcal{C}^1_{\text{loc}}\left(\overline{\Omega} \setminus \{P, N\}, S^2\right)$ verifying $\deg(u_\delta, P) = 1$, $\deg(u_\delta, N) = -1$ and

$$\int_\Omega |\nabla u_\delta|^2 w(x) dx \leq 8\pi \tilde{\ell}_w(\mathcal{F}) + \frac{\delta}{2}. \tag{3.13}$$

Now we recall that the collection $\mathcal{F} = ([\alpha_1, \beta_1], \dots, [\alpha_n, \beta_n]) \in \mathcal{P}(P, N)$ such that the curve $\gamma = \cup_k [\alpha_k, \beta_k]$ has no self-intersection points, can be chosen for the construction of u_δ. From Lemma 3.3 below, we can find $\mathcal{F}$ such that

$$\tilde{\ell}_w(\mathcal{F}) \leq d_w(P, N) + \frac{\delta}{16\pi}$$

and according to (3.13), the map u_δ satisfies the required properties. ∎

Lemma 3.3. *For any $x, y \in \Omega$, let $\mathcal{P}'(x, y)$ be the class of all elements $\mathcal{F} = ([\alpha_1, \beta_1], \dots, [\alpha_n, \beta_n])$ in $\mathcal{P}(x, y)$ such that the curve $\gamma = \cup_k [\alpha_k, \beta_k]$ has no self intersection points. Then*

$$\tilde{d}_w(x, y) = \operatorname*{Inf}_{\mathcal{F} \in \mathcal{P}'(x,y)} \tilde{\ell}_w(\mathcal{F}) \leq d_w(x, y),$$

where $\tilde{\ell}_w(\mathcal{F})$ is defined in (3.11).

Proof. Step 1. First we prove that $\tilde{d}_w$ defines a distance. As for distance d_w, we infer that $\tilde{d}_w(x, y) = 0$ if and only if $x = y$ and $\tilde{d}_w$ is symmetric. Then we just have to check the triangle inequality. We remark that the juxtaposition of $\mathcal{F}_1 \in \mathcal{P}'(x, z)$ with $\mathcal{F}_2 \in \mathcal{P}'(z, y)$ is not an element of $\mathcal{P}'(x, y)$ in general and we can't proceed as for d_w. Let x, y, z be three distinct points in Ω. We consider two arbitrary elements $\mathcal{F}_1 = ([\alpha^1_1, \beta^1_1], \dots, [\alpha^1_{n_1}, \beta^1_{n_1}]) \in \mathcal{P}'(x, z)$,

$\mathcal{F}_2 = ([\alpha_1^2, \beta_1^2], \dots, [\alpha_{n_2}^2, \beta_{n_2}^2]) \in \mathcal{P}'(z, y)$, and the curves $\gamma_1 = \cup_k [\alpha_k^1, \beta_k^1]$ and $\gamma_2 = \cup_k [\alpha_k^2, \beta_k^2]$. We have to prove that we can construct $\mathcal{F}_3 \in \mathcal{P}'(x, y)$ such that $\tilde{\ell}_w(\mathcal{F}_3) \leq \tilde{\ell}_w(\mathcal{F}_1) + \tilde{\ell}_w(\mathcal{F}_2)$.
First Case: If the curve $\gamma_1 \cup \gamma_2$ has no self intersection points then we take $\mathcal{F}_3 = ([\alpha_1^1, \beta_1^1], \dots, [\alpha_{n_1}^1, \beta_{n_1}^1], [\alpha_1^2, \beta_1^2], \dots, [\alpha_{n_2}^2, \beta_{n_2}^2]) \in \mathcal{P}'(x, y)$ and we have

$$\tilde{\ell}_w(\mathcal{F}_3) = \tilde{\ell}_w(\mathcal{F}_1) + \tilde{\ell}_w(\mathcal{F}_2).$$

Second Case: If $\gamma_1 \cup \gamma_2$ has self intersection points then we rewrite the curves γ_1 and γ_2 as $\gamma_1 = \cup_{k=1}^{\tilde{n}_1} [\tilde{\alpha}_k^1, \tilde{\beta}_k^1]$ and $\gamma_2 = \cup_{k=1}^{\tilde{n}_2} [\tilde{\alpha}_k^2, \tilde{\beta}_k^2]$ such that

a) $(\alpha_k^i)_{k=1}^{n_i} \subset (\tilde{\alpha}_k^i)_{k=1}^{\tilde{n}_i}$ for $i = 1, 2$,

b) if S is a connected component of $\gamma_1 \cap \gamma_2$ then one of the following cases holds:

b1) $S \subset \left(\cup_{k=1}^{\tilde{n}_1} \{\tilde{\alpha}_k^1, \tilde{\beta}_k^1\} \right) \cap \left(\cup_{k=1}^{\tilde{n}_1} \{\tilde{\alpha}_k^2, \tilde{\beta}_k^2\} \right)$,

b2) $S \in \left\{ [\tilde{\alpha}_1^1, \tilde{\beta}_1^1], \dots, [\tilde{\alpha}_{\tilde{n}_1}^1, \tilde{\beta}_{\tilde{n}_1}^1] \right\} \cap \left\{ [\tilde{\alpha}_1^2, \tilde{\beta}_1^2], \dots, [\tilde{\alpha}_{\tilde{n}_2}^2, \tilde{\beta}_{\tilde{n}_2}^2] \right\}$,

c) $\tilde{\mathcal{F}}_1 = ([\tilde{\alpha}_1^1, \tilde{\beta}_1^1], \dots, [\tilde{\alpha}_{\tilde{n}_1}^1, \tilde{\beta}_{\tilde{n}_1}^1]) \in \mathcal{P}'(x, z)$,

d) $\tilde{\mathcal{F}}_2 = ([\tilde{\alpha}_1^2, \tilde{\beta}_1^2], \dots, [\tilde{\alpha}_{\tilde{n}_2}^2, \tilde{\beta}_{\tilde{n}_2}^2]) \in \mathcal{P}'(z, y)$.

By construction, we can write $[\alpha_k^i, \beta_k^i] = \cup_{l=1}^{m_k^i} [\tilde{\alpha}_l^i, \tilde{\beta}_l^i]$ for some $m_k^i \in \mathbb{N}$ and for any $k = 1, \dots, n_i$ and $i = 1, 2$. Since we have

$$V_\varepsilon(\alpha_k^i, \beta_k^i) = \cup_{l=1}^{m_k^i} V_\varepsilon(\tilde{\alpha}_l^i, \tilde{\beta}_l^i),$$

we get that

$$\begin{aligned} \liminf_{\varepsilon \to 0^+} \frac{1}{\pi} \int_{V_\varepsilon(\alpha_k^i, \beta_k^i)} \frac{\varepsilon^4 w(x)}{\left(\varepsilon^4 + r^2_{\alpha_k^i, \beta_k^i}(x) \right)^2} \, dx \geq \\ \geq \sum_{l=1}^{m_k^i} \liminf_{\varepsilon \to 0^+} \frac{1}{\pi} \int_{V_\varepsilon(\tilde{\alpha}_l^i, \tilde{\beta}_l^i)} \frac{\varepsilon^4 w(x)}{\left(\varepsilon^4 + r^2_{\tilde{\alpha}_l^i, \tilde{\beta}_l^i}(x) \right)^2} \, dx \end{aligned}$$

and we conclude that $\tilde{\ell}_w(\tilde{\mathcal{F}}_i) \leq \tilde{\ell}_w(\mathcal{F}_i)$ for $i = 1, 2$. In the collection $([\tilde{\alpha}_1^1, \tilde{\beta}_1^1], \dots, [\tilde{\alpha}_{\tilde{n}_1}^1, \tilde{\beta}_{\tilde{n}_1}^1], [\tilde{\alpha}_1^2, \tilde{\beta}_1^2], \dots, [\tilde{\alpha}_{\tilde{n}_2}^2, \tilde{\beta}_{\tilde{n}_2}^2])$, we just have to delete some segments in order to obtain a new element $\mathcal{F}_3 \in \mathcal{P}'(x, y)$ which then satisfies

$\tilde{\ell}_w(\mathcal{F}_3) \leq \tilde{\ell}_w(\tilde{\mathcal{F}}_1) + \tilde{\ell}_w(\tilde{\mathcal{F}}_2) \leq \tilde{\ell}_w(\mathcal{F}_1) + \tilde{\ell}_w(\mathcal{F}_2)$.
From these constructions, we conclude that $\tilde{d}_w(x, y) \leq \tilde{\ell}_w(\mathcal{F}_1) + \tilde{\ell}_w(\mathcal{F}_2)$. Taking the infimum over all $\mathcal{F}_1 \in \mathcal{P}'(x, z)$ and all $\mathcal{F}_2 \in \mathcal{P}'(z, y)$, we derive the triangle inequality.
Step 2. We fix two arbitrary points x_0 and y_0 in Ω and we consider $\zeta : \Omega \to \mathbb{R}$ defined by

$$\zeta(x) = \tilde{d}_w(x, y_0).$$

From the triangle inequality, we get that ζ is 1-Lipschitz with respect to the distance $\tilde{d}_w$. Let $z_0 \in \Omega$ and $R > 0$ such that $B_{3R}(z_0) \subset \Omega$ and let $(\rho_n)_{n\in\mathbb{N}}$ be a sequence of smooth mollifiers. For $n > 1/R$, we consider $\zeta_n = \rho_n * \zeta : B_R(z_0) \to \mathbb{R}$. We have for all $x, y \in B_R(z_0)$,

$$\begin{aligned}
|\zeta_n(x) - \zeta_n(y)| &\leq \int_{B_{1/n}} \rho_n(-z)|\zeta(x+z) - \zeta(y+z)|dz \\
&\leq \int_{B_{1/n}} \rho_n(-z)\tilde{d}_w(x+z, y+z)dz \\
&\leq \int_{B_{1/n}} \rho_n(-z)\tilde{\ell}_w\left([x+z, y+z]\right)dz.
\end{aligned}$$

We remark that $V_\varepsilon(x+z, y+z) = z + V_\varepsilon(x, y)$ and that for all $\xi \in V_\varepsilon(x, y)$, we have $r_{x,y}(\xi) = r_{x+z,y+z}(\xi + z)$. Then we obtain for all $z \in B_{1/n}(0)$,

$$\tilde{\ell}_w\left([x+z, y+z]\right) = \liminf_{\varepsilon\to 0^+} \frac{1}{\pi}\int_{V_\varepsilon(x,y)} \frac{\varepsilon^4 w(\xi+z)}{\left(\varepsilon^4 + r_{x,y}^2(\xi)\right)^2}\,d\xi.$$

Taking an arbitrary sequence $\varepsilon_k \to 0^+$ and using Fatou's lemma, we get that

$$\begin{aligned}
|\zeta_n(x) - \zeta_n(y)| &\leq \liminf_{k\to+\infty} \frac{1}{\pi}\int_{B_{1/n}}\int_{V_{\varepsilon_k}(x,y)} \frac{\varepsilon_k^4\,\rho_n(-z)w(\xi+z)}{\left(\varepsilon_k^4 + r_{x,y}^2(\xi)\right)^2}\,d\xi dz \\
&\leq \liminf_{k\to+\infty} \frac{1}{\pi}\int_{V_{\varepsilon_k}(x,y)} \frac{\varepsilon_k^4}{\left(\varepsilon_k^4 + r_{x,y}^2(\xi)\right)^2}\,\rho_n * w(\xi)\,d\xi.
\end{aligned}$$

Without loss of generality we may assume that $[x, y] = \{(0,0)\} \times [-R, R]$. Then we have $V_\varepsilon(x, y) = \left\{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3,\, |\xi_3| \leq R,\, \sqrt{\xi_1^2 + \xi_2^2} \leq \varepsilon\right\}$ and

$r_{x,y}(\xi) = \sqrt{\xi_1^2 + \xi_2^2}$ for $\xi \in V_\varepsilon(x,y)$. Therefore we can write

$$\int_{V_{\varepsilon_k}(x,y)} \frac{\varepsilon_k^4\, \rho_n * w(\xi)}{\left(\varepsilon_k^4 + r_{x,y}^2(\xi)\right)^2}\, d\xi = \int_{B_{\varepsilon_k}(0)\times[-R,R]} \frac{\varepsilon_k^4\, \rho_n * w\,(\xi)}{\left(\varepsilon_k^4 + \xi_1^2 + \xi_2^2\right)^2}\, d\xi$$
$$= \int_{B_{\varepsilon_k}(0)\times[-R,R]} \frac{\varepsilon_k^4\, \left(\rho_n * w\,(0,0,\xi_3) + \mathcal{O}_n(\varepsilon_k)\right)}{\left(\varepsilon_k^4 + \xi_1^2 + \xi_2^2\right)^2}\, d\xi,$$

where $\mathcal{O}_n(\varepsilon_k)$ denotes a quantity which tends to 0 as $\varepsilon_k \to 0$ for n fixed. Since we have

$$\int_{B_{\varepsilon_k}(0)} \frac{\varepsilon_k^4}{\left(\varepsilon_k^4 + \xi_1^2 + \xi_2^2\right)^2}\, d\xi = \pi + \mathcal{O}(\varepsilon_k),$$

it follows that

$$|\zeta_n(x) - \zeta_n(y)| \leq \int_{-R}^{R} \rho_n * w(0,0,\xi_3) d\xi_3 = \int_{[x,y]} \rho_n * w(s) ds.$$

As in the proof of Proposition 2.3, we conclude that $|\nabla\zeta| \leq w$ a.e. in $B_R(z_0)$ and since z_0 is arbitrary in Ω, we get that $|\nabla\zeta| \leq w$ a.e. in Ω. According to Proposition 2.3, it implies that for all $x, y \in \Omega$,

$$|\zeta(x) - \zeta(y)| \leq d_w(x,y)$$

which leads to $\tilde{d}_w(x_0,y_0) \leq d_w(x_0,y_0)$ taking $x = x_0$ and $y = y_0$. ∎

3.3 Upper Bound for the Energy

Lemma 3.4. *For all $\delta > 0$, there exists a map $u_\delta \in \mathcal{E}$ such that*

$$\int_\Omega |\nabla u_\delta|^2 w(x) dx \leq 8\pi L_w + \delta.$$

Proof. We relabel the list $(a_i)_{i=1}^N$ as a list of positive points $(p_j)_{j=1}^K$ and a list of negative points $(n_j)_{j=1}^K$ and we may assume that $\sum_j d_w(p_j, n_j) = L_w$. We will construct dipoles between each pair (p_j, n_j) which do not intersect each other. We claim that we can find $\mathcal{F}_1 = ([\alpha_1^1, \beta_1^1], \ldots, [\alpha_{m_1}^1, \beta_{m_1}^1]) \in \mathcal{P}'(p_1, n_1)$ such that

(A.1) $\gamma_1 = \cup_k [\alpha_k^1, \beta_k^1]$ does not contain any $p_j \neq p_1$ and any $n_j \neq n_1$,

(A.2) $\tilde{\ell}_w(\mathcal{F}_1) \leq d_w(p_1, n_1) + \frac{\delta}{8K\pi}$.

Indeed if we define for $x, y \in \Omega_A = \Omega \setminus \{p_j, n_j | p_j \neq p_1, n_j \neq n_1\}$,

$$D_w^A(x,y) = \operatorname{Inf} \tilde{\ell}_w(\mathcal{F})$$

where the infimum is taken over all $\mathcal{F} = ([\alpha_1, \beta_1], \ldots, [\alpha_m, \beta_m]) \in \mathcal{P}'(x,y)$ such that $\cup_k [\alpha_k, \beta_k] \subset \Omega_A$ then we prove, using the arguments in the proof of Lemma 3.3 that $D_w^A(x,y) \leq d_w(x,y)$ for all $x, y \in \Omega_A$. Since $p_1, n_1 \in \Omega_A$, we obtain $D_w^A(p_1, n_1) \leq d_w(p_1, n_1)$ and by the definition of D_w^A, we draw the existence of $\mathcal{F}_1 \in \mathcal{P}'(p_1, n_1)$ satisfying *(A.1)* and *(A.2)*.

Now we will show that we can find $\mathcal{F}_2 = ([\alpha_1^2, \beta_1^2], \ldots, [\alpha_{m_2}^2, \beta_{m_2}^2])$ in $\mathcal{P}'(p_2, n_2)$ such that

(B.1) $\gamma_2 = \cup_k [\alpha_k^2, \beta_k^2]$ does not contain any $p_j \neq p_2$ and any $n_j \neq n_2$ and does not intersect $\gamma_1 \setminus \{p_1, n_1\}$,

(B.2) $\tilde{\ell}_w(\mathcal{F}_2) \leq d_w(p_2, n_2) + \frac{\delta}{8K\pi}$.

As previously we define

$$\Omega_B = \Omega \setminus (\{p_j, n_j | p_j \neq p_2, n_j \neq n_2\} \cup \gamma_1 \setminus \{p_1, n_1\})$$

and

$$D_w^B(x,y) = \operatorname{Inf} \tilde{\ell}_w(\mathcal{F}) \quad \text{for } x, y \in \Omega_B$$

where the infimum is taken over all $\mathcal{F} = ([\alpha_1, \beta_1], \ldots, [\alpha_m, \beta_m]) \in \mathcal{P}'(x,y)$ such that $\cup_k [\alpha_k, \beta_k] \subset \Omega_B$. In the same way we infer that for all $x, y \in \Omega_2$, $D_w^B(x,y) \leq d_w(x,y)$ and the existence of $\mathcal{F}_2 \in \mathcal{P}'(p_2, n_2)$ satisfying *(B.1)* and *(B.2)* follows.

Iterating this process, we finally reach the existence of K elements $\mathcal{F}_j = ([\alpha_1^j, \beta_1^j], \ldots, [\alpha_{m_j}^j, \beta_{m_j}^j])$ in $\mathcal{P}'(p_j, n_j)$ such that $\tilde{\ell}_w(\mathcal{F}_j) \leq d_w(p_j, n_j) + \frac{\delta}{8K\pi}$, $\gamma_j = \cup_k [\alpha_k^j, \beta_k^j]$ and $\gamma_i = \cup_k [\alpha_k^i, \beta_k^i]$ do not intersect except maybe at their extremities for $i \neq j$. From the dipole construction in Lemma 3.2, we find K maps $u_\delta^j \in \mathcal{C}^1_{\text{loc}}(\overline{\Omega} \setminus \{p_j, n_j\}, S^2)$ constant outside an arbitrary small open neighborhood $\mathcal{N}_j$ of γ_j and such that $\deg(u_\delta^j, p_j) = +1$, $\deg(u_\delta^j, n_j) = -1$ and

$$\int_\Omega |\nabla u_\delta^j|^2 w(x) dx \leq 8\pi d_w(p_j, n_j) + \frac{\delta}{K}.$$

By construction of the $\mathcal{F}_j$'s, we can choose the $\mathcal{N}_j$ sufficiently small for $\mathcal{N}_j$ and $\mathcal{N}_i$ to not intersect whenever $j \neq i$. Then the map

$$u_\delta(x) = \begin{cases} u_\delta^j(x) & \text{if } x \in \mathcal{N}_j, \\ (0,0,1) & \text{if } x \notin \cup_j \mathcal{N}_j, \end{cases}$$

is well defined and satisfies the required properties. ■

Remark 3.1. In a forthcoming paper (see [18]), we study, in the case of a smooth bounded open set $\Omega \subset \mathbb{R}^3$, the *relaxed energy* defined for $u \in H^1_g(\Omega, S^2)$ by

$$E_w(u) = \text{Inf} \left\{ \liminf_{n \to +\infty} \int_\Omega |\nabla u_n(x)|^2 w(x) dx \right\}$$

where the infimum is taken over all sequences $(u_n)_{n\in\mathbb{N}} \subset \mathcal{C}^1\left(\overline{\Omega}, S^2\right)$ satisfying $u_{n/\partial\Omega} = g$, $u_n \to u$ weakly in H^1 and $g : \partial\Omega \to S^2$ is a given smooth map such that $\deg(g, \partial\Omega) = 0$. In the case $w \equiv 1$, F. Bethuel, H. Brezis and J.M. Coron have proved (see [3]) that

$$E_1(u) = \int_\Omega |\nabla u(x)|^2 dx + 8\pi L(u)$$

where $L(u)$ denotes the length of a minimal connection (relative to the Euclidean geodesic distance d_Ω in Ω) between the singularities of u. We believe that a similar result holds for any function w satisfying (1.1), computing minimal connections with d_w instead of d_Ω.

4 Some Stability and Approximation Results

4.1 Stability Results

The stability result below is based on Theorem 3.1 in [5]. It relies on the Γ-convergence of the length functionals (we refer to [12] for the notion of Γ-convergence). In the sequel, we denote by $\text{Lip}\left([0,1], \overline{\Omega}\right)$ the class of all Lipschitz map from $[0,1]$ into $\overline{\Omega}$ and we endow $\text{Lip}\left([0,1], \overline{\Omega}\right)$ with the topology of the uniform convergence on $[0,1]$.

Theorem 4.1. *Let $(w_n)_{n\in\mathbb{N}}$ be a sequence of measurable real functions such that*

$$0 < c_0 \leq w_n \leq C_0 \quad a.e \text{ in } \Omega$$

for some constants c_0 and C_0 independent of $n \in \mathbb{N}$. Then the following properties are equivalent:

(i) $E_{w_n}\left((a_i, d_i)_{i=1}^N\right) \underset{n\to+\infty}{\longrightarrow} E_w\left((a_i, d_i)_{i=1}^N\right)$ *for any configuration* $(a_i, d_i)_{i=1}^N$,

(ii) the functionals $\mathbb{L}_{d_{w_n}}$ Γ*-converge to* $\mathbb{L}_{d_w}$ *in* $\text{Lip}\left([0,1], \overline{\Omega}\right)$.

In the proof of Theorem 4.1, we will make use of the following lemma.

Lemma 4.1. *Let $(d_n)_{n\in\mathbb{N}}$ be a sequence of geodesic distances on $\overline{\Omega}$ such that*

$$c_0 d_\Omega \le d_n \le C_0 d_\Omega \tag{4.1}$$

for some positive constants c_0 and C_0 independent of $n \in \mathbb{N}$. Then there exits a subsequence $(n_k)_{k\in\mathbb{N}}$ and a geodesic distance d' on $\overline{\Omega}$ such that $d_{n_k} \to d'$ as $k \to +\infty$ uniformly on every compact subset of $\overline{\Omega} \times \overline{\Omega}$.

Proof. For $(x_1, y_1), (x_2, y_2) \in \overline{\Omega} \times \overline{\Omega}$ we have

$$\begin{aligned} d_{w_n}(x_1, y_1) - d_{w_n}(x_2, y_2) &\le d_{w_n}(x_1, x_2) + d_{w_n}(x_2, y_1) - d_{w_n}(x_2, y_2) \\ &\le d_{w_n}(x_1, x_2) + d_{w_n}(y_1, y_2) \\ &\le C_0 \left(d_\Omega(x_1, x_2) + d_\Omega(y_1, y_2) \right). \end{aligned}$$

Inverting the roles of (x_1, y_1) and (x_2, y_2) we infer that

$$|d_{w_n}(x_1, y_1) - d_{w_n}(x_2, y_2)| \le C_0 \left(d_\Omega(x_1, x_2) + d_\Omega(y_1, y_2) \right).$$

Thus d_{w_n} is C_0-Lipschitz on $\overline{\Omega} \times \overline{\Omega}$ for every $n \in \mathbb{N}$ and we conclude by Ascoli's theorem that we can find a subsequence $(n_k)_{k\in\mathbb{N}}$ and a Lipschitz function d' on $\overline{\Omega} \times \overline{\Omega}$ such that $d_{n_k} \to d'$ as $k \to +\infty$ uniformly on every compact subset of $\overline{\Omega} \times \overline{\Omega}$. We easily check that d' defines a distance on $\overline{\Omega}$ and it remains to prove that d' is geodesic. Since d' satisfies (4.1) as the pointwise limit of $(d_{n_k})_{k\in\mathbb{N}}$, $\overline{\Omega}$ endowed with d' is a complete metric space. By Theorem 1.8 in [16], it suffices to prove that for any $x, y \in \overline{\Omega}$ and $\delta > 0$ there exists $z \in \overline{\Omega}$ such that $\max\left(d'(x, z), d'(z, y)\right) \le \frac{1}{2} d'(x, y) + \delta$. We fix $x, y \in \overline{\Omega}$ and $\delta > 0$. Since d_{n_k} is of geodesic type, we can find $z_k \in \overline{\Omega}$ such that $\max\left(d_{n_k}(x, z), d_{n_k}(z, y)\right) \le \frac{1}{2} d_{n_k}(x, y) + \delta$. Then the sequence (z_k) is bounded and we may assume that $z_k \to z \in \overline{\Omega}$. Since $d_{n_k} \to d'$ uniformly on every compact subset of $\overline{\Omega} \times \overline{\Omega}$, we deduce that $d_{n_k}(x, z_k) \to d'(x, z)$ and $d_{n_k}(z_k, y) \to d'(z, y)$. Letting $k \to +\infty$ in the last inequality we draw that z satisfies the requirement. ∎

Proof of Theorem 4.1. Step 1. We prove $(i) \Rightarrow (ii)$. From *(i)* we derive that $E_{w_n}(P, N) \to E_w(P, N)$ in the dipole case for any distinct points $P, N \in \Omega$. By Theorem 1.1 we conclude that $d_{w_n} \to d_w$ pointwise on Ω. As in the proof of Proposition 2.1 we have $c_0 d_\Omega \le d_{w_n} \le C_0 d_\Omega$ in $\overline{\Omega}$. By Lemma 4.1 and the uniqueness of the limit we get that $d_{w_n} \to d_w$ uniformly on every compact subset of $\overline{\Omega} \times \overline{\Omega}$. Using the arguments of the proof of *i) ⇒ ii)* Theorem 3.1 in [5], we infer that $\mathbb{L}_{d_{w_n}} \xrightarrow{\Gamma} \mathbb{L}_{d_w}$ in $\mathrm{Lip}\left([0, 1], \overline{\Omega}\right)$.
Step 2. We prove *(ii) ⇒ (i)*. Since we have $c_0 d_\Omega \le d_{w_n} \le C_0 d_{w_n}$ in $\overline{\Omega}$

we draw from Lemma 4.1 that we can find a subsequence $(n_k)_{k\in\mathbb{N}}$ and a geodesic distance d' on $\overline{\Omega}$ such that $d_{w_{n_k}} \to d'$ uniformly on every compact subset of $\overline{\Omega}\times\overline{\Omega}$. As in the previous step, we obtain using the method in [5] that $\mathbb{L}_{d_{w_{n_k}}} \xrightarrow{\Gamma} \mathbb{L}_{d'}$ in Lip $\left([0,1],\overline{\Omega}\right)$. Then we conclude by assumption *(ii)* that $\mathbb{L}_{d'} \equiv \mathbb{L}_{d_w}$ on Lip $\left([0,1],\overline{\Omega}\right)$. Since $c_0 d_\Omega \le d' \le C_0 d_\Omega$ as the pointwise limit of $(d_{w_{n_k}})_{k\in\mathbb{N}}$, we can proceed as in Remark 2.1 to prove that for any $x, y \in \overline{\Omega}$ there exists a curve $\gamma \in$ Lip $\left([0,1],\overline{\Omega}\right)$ such that $d'(x,y) = \mathbb{L}_{d'}(\gamma)$. Since the same property holds for d_w we finally get that $d' \equiv d_w$. The uniqueness of the limit implies the convergence of the full sequence. Then *(i)* follows by Theorem 1.1. ■

In the next proposition, we give some sufficient conditions on a sequence $(w_n)_{n\in\mathbb{N}}$ converging pointwise to w for Property *(i)* in Theorem 4.1 to hold.

Proposition 4.1. *Let $(w_n)_{n\in\mathbb{N}}$ be a sequence of measurable functions such that*

$$0 < c_0 \le w_n \le C_0 \quad a.e \text{ in } \Omega$$

for some constants c_0 and C_0 independent of $n \in \mathbb{N}$. Assume that one of the following conditions holds:

(a) $w_n \ge w$ and $w_n \to w$ a.e. in Ω,

(b) $w_n \to w$ in $L^\infty(\Omega)$.

Then Property (i) in Theorem 4.1 holds.

Proof. Step 1. Assume that *(a)* holds. Since $w \le w_n$ a.e. in Ω we infer that $E_w\left((a_i,d_i)_{i=1}^N\right) \le E_{w_n}\left((a_i,d_i)_{i=1}^N\right)$ for any $n\in\mathbb{N}$ and therefore

$$E_w\left((a_i,d_i)_{i=1}^N\right) \le \liminf_{n\to+\infty} E_{w_n}\left((a_i,d_i)_{i=1}^N\right). \tag{4.2}$$

Fix some $u \in \mathcal{E}$. Since $w_n \le C_0$ and $w_n \to w$ a.e. on Ω, we obtain by dominated convergence that

$$\int_\Omega |\nabla u|^2 w_n(x)dx \underset{n\to+\infty}{\to} \int_\Omega |\nabla u|^2 w(x)dx.$$

Then we derive

$$\limsup_{n\to+\infty} E_{w_n}\left((a_i,d_i)_{i=1}^N\right) \le \int_\Omega |\nabla u|^2 w(x)dx,$$

and since u is arbitrary we conclude

$$\limsup_{n\to+\infty} E_{w_n}\left((a_i,d_i)_{i=1}^N\right) \leq E_w\left((a_i,d_i)_{i=1}^N\right). \tag{4.3}$$

Finally the announced result follows from (4.2) and (4.3).
Step 2. Assume that *(b)* holds. We consider $\delta_n = \| w_n - w \|_{L^\infty(\Omega)}$ and

$$\tilde{w}_n = (1 + c_0^{-1}\delta_n)w_n.$$

By construction we have $\tilde{w}_n \geq w$ and $\tilde{w}_n \to w$ a.e. in Ω. From the previous case we deduce that

$$\lim_{n\to+\infty} E_{\tilde{w}_n}\left((a_i,d_i)_{i=1}^N\right) = E_w\left((a_i,d_i)_{i=1}^N\right),$$

which leads to the result since $E_{\tilde{w}_n}\left((a_i,d_i)_{i=1}^N\right) = (1+c_0^{-1}\delta_n)E_{w_n}\left((a_i,d_i)_{i=1}^N\right)$ and $1 + c_0^{-1}\delta_n \to 1$. ■

Remark 4.1. The conclusion of Proposition 4.1 case *(b)* may fail if the sequence $\{w_n\}$ converges to w almost everywhere in Ω. Indeed, if one considers a sequence $(w_n)_{n\in\mathbb{N}}$ of smooth functions on $\Omega = B_1(0)$ satisfying

$$w_n(x) = \begin{cases} 1 & \text{if } |x_3| \geq 1/n\,, \\ 1/2 & \text{if } |x_3| = 0\,, \end{cases}$$

and $1/2 \leq w_n \leq 1$ in Ω, one can easily check that $w_n \to 1$ in $L^p(\Omega)$ for any $1 \leq p < +\infty$. Now if we choose two distinct points $P, N \in \{(x_1,x_2,0)\in\Omega\}$, we obtain in the dipole case $E_{w_n}(P,N) = 1/2|P-N|$ for any $n \in \mathbb{N}$ and $E_1(P,N) = |P-N|$. Note that if we consider the sequence of variational problems

$$P_n = \text{Min}\left\{\int_\Omega |\nabla u(x)|^2 w_n(x)dx,\ u \in H_g^1(\Omega,\mathbb{R})\right\},$$

where g denotes some given function in $H^{1/2}(\partial\Omega,\mathbb{R})$, then it follows by classical results (see [12] for instance) that

$$P_n \underset{n\to+\infty}{\longrightarrow} \text{Min}\left\{\int_\Omega |\nabla u(x)|^2 dx,\ u \in H_g^1(\Omega,\mathbb{R})\right\}.$$

4.2 Approximation Result

In this section, we give an approximation procedure by smooth weights.

Theorem 4.2. *Let $(\rho_n)_{n\in\mathbb{N}}$ be a sequence of smooth mollifiers. Extending w outside Ω by a sufficiently large positive constant and taking $w_n = \rho_n * w$, we have*

$$E_{w_n}\left((a_i,d_i)_{i=1}^N\right) \to E_w\left((a_i,d_i)_{i=1}^N\right) \quad \text{as } n \to +\infty.$$

Proof. Step 1. Assume that $\Omega = \mathbb{R}^3$. Let $(\rho_n)_{n\in\mathbb{N}}$ be a sequence of smooth mollifiers. Fix any function ζ which is 1-Lipschitz with respect to d_w. Using the arguments in the proof of Proposition 2.3, we obtain that the function $\zeta_n = \rho_n * \zeta$ satisfies $|\nabla \zeta_n| \le \rho_n * w$ on $\mathbb{R}^3$. Then we conclude that ζ_n is 1-Lipschitz with respect to the distance $\delta_{\rho_n * w}$. Relabelling the a_i's as a list of positive and negative points $(p_j, n_j)_{j=1}^K$, we get from formula (1.6) and Theorem 1.1,

$$8\pi \sum_{j=1}^K \zeta_n(p_j) - \zeta_n(n_j) \le E_{\rho_n * w}\left((a_i,d_i)_{i=1}^N\right).$$

Taking the lim inf as $n \to +\infty$, we obtain

$$8\pi \sum_{j=1}^K \zeta(p_j) - \zeta(n_j) \le \liminf_{n\to+\infty} E_{\rho_n * w}\left((a_i,d_i)_{i=1}^N\right).$$

Since ζ is arbitrary, we deduce from (1.6) and Theorem 1.1 that

$$E_w\left((a_i,d_i)_{i=1}^N\right) \le \liminf_{n\to+\infty} E_{\rho_n * w}\left((a_i,d_i)_{i=1}^N\right). \tag{4.4}$$

Since $\rho_n * w \le \Lambda$, we obtain by dominated convergence that for any $u \in \mathcal{E}$,

$$\int_\Omega |\nabla u|^2 \rho_n * w(x)dx \underset{n\to+\infty}{\to} \int_\Omega |\nabla u|^2 w(x)dx$$

and therefore

$$\limsup_{n\to+\infty} E_{\rho_n * w}\left((a_i,d_i)_{i=1}^N\right) \le \int_\Omega |\nabla u|^2 w(x)dx.$$

Since u is arbitrary, we infer that

$$\limsup_{n\to+\infty} E_{\rho_n * w}\left((a_i,d_i)_{i=1}^N\right) \le E_w\left((a_i,d_i)_{i=1}^N\right), \tag{4.5}$$

and the result follows from (4.4) and (4.5).
Step 2: Assume that Ω is a smooth bounded and connected open set. We extend w by setting $w = M$ in $\mathbb{R}^3 \setminus \Omega$ for a large positive constant M that we will choose later. We fix some $\delta > 0$ small enough and consider

$$\Omega_\delta = \left\{x \in \mathbb{R}^3,\, \mathrm{dist}(x, \Omega) < \delta\right\}.$$

We extend to Ω_δ any function ζ which is 1-Lipschitz with respect to d_w by setting $\zeta(x) = \zeta(\Pi x)$ for $x \in \Omega_\delta$ where Πx denotes the projection of $x \in \Omega_\delta$ on $\overline{\Omega}$. By construction, such a ζ is Lipschitz continuous on Ω_δ and $|\nabla\zeta| \leq C(\Omega, \delta, \Lambda)$ a.e. on $\Omega_\delta \setminus \Omega$ and $|\nabla\zeta| \leq w$ a.e. on Ω. Then we choose $M \geq C(\Omega, \delta, \Lambda)$. Setting $\zeta_n : x \in \Omega \to \rho_n * \zeta(x)$ for $n \geq 1/\delta$, we have $|\nabla\zeta_n| \leq \rho_n * w$ on Ω. Then ζ_n is 1-Lipschitz with respect to the distance $\delta_{\rho_n * w}$ and we can proceed as in Step 1. ■

Remark 4.2. If $(w_n)_{n\in\mathbb{N}}$ denotes the sequence constructed in Theorem 4.2, the previous results show that $d_{w_n} \to d_w$ uniformly on every compact subset of $\overline{\Omega} \times \overline{\Omega}$ and the functionals $\mathbb{L}_{d_{w_n}}$ Γ-converge to $\mathbb{L}_{d_w}$ in $\mathrm{Lip}\left([0,1], \overline{\Omega}\right)$.

5 Energy involving a Matrix Field

In this section, we consider $M = (m_{kl})_{k,l=1}^3$ a continuous map from $\overline{\Omega}$ onto the set of real symmetric 3×3 matrices such that

$$\lambda|\xi|^2 \leq M(x)\xi \cdot \xi \leq \Lambda|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^3 \text{ and } x \in \overline{\Omega}$$

(here "$\cdot$" denotes the Euclidean scalar product on $\mathbb{R}^3$) and we investigate on the problem

$$E_M\left((a_i, d_i)_{i=1}^N\right) = \operatorname*{Inf}_{u\in\mathcal{E}} \int_\Omega \sum_{k,l=1}^3 m_{kl}(x) \frac{\partial u}{\partial x_k} \cdot \frac{\partial u}{\partial x_l}\, dx.$$

Under the continuity assumption above, we show that $E_M\left((a_i, d_i)_{i=1}^N\right)$ can also be computed in terms of minimal connections relative to some geodesic distance on $\overline{\Omega}$.

In order to state the result we introduce the following objects. For $x \in \overline{\Omega}$, we denote by $\mathrm{cof}(M(x))$ the cofactor matrix of $M(x)$. For any Lipschitz curve $\gamma : [0,1] \to \overline{\Omega}$, we define the length $\mathbb{L}_M(\gamma)$ by

$$\mathbb{L}_M(\gamma) = \int_0^1 \sqrt{\mathrm{cof}\left(M(\gamma(t))\right)\dot{\gamma}(t) \cdot \dot{\gamma}(t)}\; dt$$

and we construct from $\mathbb{L}_M$ the Riemannian distance d_M on $\overline{\Omega}$ defined by

$$d_M(x,y) = \text{Inf } \mathbb{L}_M(\gamma)$$

where the infimum is taken over all curves $\gamma \in \text{Lip}_{x,y}\left([0,1],\overline{\Omega}\right)$.

Theorem 5.1. *We have*

$$E_M\left((a_i,d_i)_{i=1}^N\right) = 8\pi L_M$$

where L_M is the length of a minimal connection associated to the configuration $(a_i,d_i)_{i=1}^N$ and the distance d_M on $\overline{\Omega}$.

Remark 5.1. One can slightly relax the continuity assumption on M. For example, we can assume that

$$M(x) = \begin{cases} M_1(x) & \textit{if } x \in \Omega_1, \\ M_2(x) & \textit{if } x \in \Omega_2, \end{cases}$$

where Ω_1 and Ω_2 are two open sets of Ω with piecewise smooth boundaries such that $\overline{\Omega_1 \cup \Omega_2} = \overline{\Omega}$, and $x \to M_j(x)$ is continuous on $\overline{\Omega}_j$ for $j = 1, 2$. Hence M is possibly discontinuous on the surface $\Sigma = \overline{\Omega}_1 \cap \overline{\Omega}_2$. Then the conclusion of Theorem 5.1 holds with the geodesic distance d_M constructed from the length $\mathbb{L}_M$ defined by

$$\mathbb{L}_M(\gamma) = \int_0^1 \varphi\left(\gamma(t), \dot{\gamma}(t)\right) dt \quad \text{for } \gamma \in \text{Lip}\left([0,1],\overline{\Omega}\right),$$

where

$$\varphi(x,\nu) = \begin{cases} \sqrt{\text{cof}\left(M(x)\right)\nu\cdot\nu} & \text{if } x \in \overline{\Omega}\setminus\Sigma, \\ \min\left\{\sqrt{\text{cof}\left(M_1(x)\right)\nu\cdot\nu},\ \sqrt{\text{cof}\left(M_2(x)\right)\nu\cdot\nu}\right\} & \text{if } x \in \Sigma. \end{cases}$$

Open Problem . Assuming that the coefficients of M are only in $L^\infty(\Omega)$, is the conclusion of Theorem 5.1 still valid for a certain distance?

Sketch of the Proof of Theorem 3. The Lower Bound. We follow the strategy in Section 3. For any $u \in \mathcal{E}$, we have

$$2[\text{cof}(M)D\cdot D]^{1/2} \leq \sum_{k,l=1}^{3} m_{kl}(x)\frac{\partial u}{\partial x_k}\cdot\frac{\partial u}{\partial x_l} \quad \text{a.e. on } \Omega \tag{5.1}$$

where D is the vector field defined by (3.1). Next we infer that

$$\int_\Omega \sum_{k,l=1}^{3} m_{kl}(x)\frac{\partial u}{\partial x_k}\cdot\frac{\partial u}{\partial x_l}\,dx \geq -2\int_\Omega D\cdot\nabla\zeta = 8\pi\sum_{j=1}^{K}\zeta(p_j)-\zeta(n_j) \quad (5.2)$$

for any Lipschitz function $\zeta : \overline{\Omega}\to\mathbb{R}$ such that

$$\left[\operatorname{cof}(M)^{-1}\,\nabla\zeta\cdot\nabla\zeta\right]^{1/2}\leq 1 \quad \text{a.e. in } \Omega. \quad (5.3)$$

Since a function ζ satisfies (5.3) if and only if ζ is 1-Lipschitz with respect to the distance d_M, we conclude from (5.2) that

$$E_M\left((a_i,d_i)_{i=1}^N\right)\geq 8\pi\ \mathrm{Max}\sum_{j=1}^{K}\zeta(p_j)-\zeta(n_j) = 8\pi L_M$$

where the maximum is taken over all functions ζ which is 1-Lipschitz with respect to the distance d_M.

The Upper Bound. The proof relies on the dipole construction.

Lemma 5.1. *For any distinct points $P,N\in\Omega$, any smooth simple curve $\gamma\subset\Omega$ running between P and N and $\delta>0$, there exists a map u_δ in $\mathcal{C}^1_{\mathrm{loc}}\left(\overline{\Omega}\setminus\{P,N\},S^2\right)$ such that* $\deg(u_\delta,P)=+1$, $\deg(u_\delta,N)=-1$ *and*

$$\int_\Omega \sum_{k,l=1}^{3} m_{kl}(x)\frac{\partial u_\delta}{\partial x_k}\cdot\frac{\partial u_\delta}{\partial x_l}\,dx \leq 8\pi\mathbb{L}_M(P,N)+\delta. \quad (5.4)$$

Moreover u_δ is constant outside an arbitrary small neighborhood of γ.

We may assume that $\sum_j d_M(p_j,n_j)=L_M$. Then we choose K smooth simple curves γ_j running between p_j and n_j which do not intersect except at their endpoints and such that $\mathbb{L}_M(p_j,n_j)\leq d_M(p_j,n_j)+\delta$. By Lemma 5.1, we construct K maps u_j constant outside a small neighborhood $\mathcal{N}_j$ of γ_j and $\mathcal{N}_j\cap\mathcal{N}_i=\emptyset$ if $j\neq i$. Letting $u_\delta=u_j$ on $\mathcal{N}_j$ for $j=1,\ldots,K$ and $u_\delta=(0,0,1)$ outside $\cup_j\mathcal{N}_j$, we have $u_\delta\in\mathcal{E}$ and

$$E_M\left((a_i,d_i)_{i=1}^N\right)\leq\int_\Omega \sum_{k,l=1}^{3} m_{kl}(x)\frac{\partial u_\delta}{\partial x_k}\cdot\frac{\partial u_\delta}{\partial x_l}\,dx\leq 8\pi L_M+C\delta.$$

Since δ is arbitrary, we obtain that $E_M\left((a_i,d_i)_{i=1}^N\right)\leq 8\pi L_M$. ■

Sketch of the Proof of Lemma 5.1. Since we can approximate the coefficients of M locally uniformly by smooth coefficients, we just have to prove Lemma

5.1 for M with smooth entries. We construct as in [1] a smooth diffeomorphism Φ from a small neighborhood $\mathcal{V}$ of γ into a small neighborhood of $\{(0,0)\} \times [-|\gamma|/2, |\gamma|/2]$ such that $\Phi(\gamma) = \{(0,0)\} \times [-|\gamma|/2, |\gamma|/2]$ (here $|\gamma|$ denotes the Euclidean length of γ) and $\Phi^{-1}(0,0,\cdot) : [-|\gamma|/2, |\gamma|/2] \to \mathbb{R}^3$ defines a normal parametrization of γ orientating γ from N to P. Then we set for $y_3 \in [-|\gamma|/2, |\gamma|/2]$,

$$B(y_3) = (b_{k,l}(y_3))_{k,l=1}^3 = [\nabla\Phi^{-1}(0,0,y_3)]^{-1} M(\Phi^{-1}(0,0,y_3)) \nabla\Phi^{-1}(0,0,y_3),$$

and

$$\hat{B}(y_3) = (b_{k,l}(y_3))_{k,l=1}^2 \,.$$

For small $\varepsilon > 0$ and $n \in \mathbb{N}$ large, we consider the map $\tilde{u}_n : \Phi(\mathcal{V}) \to S^2$ defined by

$$\tilde{u}_n(y_1, y_2, y_3) = \omega_\varepsilon \left(\frac{n}{\frac{|\gamma|^2}{4} - y_3^2} \, \hat{B}^{-1/2}(y_3) \cdot (y_1, y_2) \right)$$

where ω_ε is given by (3.5). Then we take

$$u_n(x) = \begin{cases} \tilde{u}_n(\Phi(x)) & \text{if } x \in \mathcal{V}, \\ (0,0,1) & \text{if } x \notin \mathcal{V}. \end{cases}$$

Following the computations in [6] and using the properties of Φ, we check that $u_n \in W^{1,\infty}_{\text{loc}}\left(\overline{\Omega} \setminus \{P,N\}, S^2\right)$, $\deg(u_n, P) = +1$, $\deg(u_n, N) = -1$. Choosing n sufficiently large and smoothening u_n around γ by the procedure in [2], we get a new map $u_\delta \in \mathcal{E}$ which satisfies (5.4). ∎

Acknowledgements. The author is deeply grateful to H. Brezis, who proposed this problem, for his hearty encouragement and to I. Shafrir for very helpful comments and suggestions during the preparation of this work. The research of the author was partially supported by the project RTN of European Commission HPRN-CT-2002-00274.

References

[1] F. Almgrem, W. Browder, E.H. Lieb, Co-area, liquid crystals and minimal surfaces, in *Partial differential equations* (Tianjin 1986), Lecture Notes in Math. **1306**, Springer (1988).

[2] F. Bethuel, A characterization of maps in $H^1(B^3, S^2)$ which can be approximated by smooth maps. *Ann. Inst. Henri Poincaré, Analyse Non linéaire* **7** (1990), 269-286.

[3] F. Bethuel, H. Brezis, J.M. Coron, Relaxed energies for harmonic maps, in *Variational Problems* (H. Berestycki , J.M. Coron, I. Ekeland, eds), Birkhäuser (1990), 37-52.

[4] J. Bourgain, H. Brezis, P. Mironescu, $H^{1/2}-$maps with values into the circle: Minimal connections, lifting, and the Ginzburg-Landau equation, to appear in Publications Math. Inst. Hautes Etudes Sci.

[5] G. Buttazzo, L. De Pascale, I. Fragalà, Topological equivalence of some variational problems involving distances. *Discrete Cont. Dynam. Systems* **7** (2) (2001), 247-258.

[6] H. Brezis, Liquid crystals and energy estimates for S^2-valued maps, in [15].

[7] H. Brezis, S^k-valued maps with singularities, in *Topics in calculus of variations* (Montecatini Terme 1987), Lecture Notes in Math. **1365**, Springer (1989).

[8] H. Brezis, J.M. Coron, Large Solutions for Harmonic Maps in Two Dimensions. *Comm. Math. Phys.* **92** (1983), 203-215.

[9] H. Brezis, J.M. Coron, E. Lieb, Harmonics Maps with Defects. *Comm. Math. Phys.* **107** (1986), 649-705.

[10] H. Brezis, P.M. Mironescu, A.C. Ponce, $W^{1,1}$-Maps with values into S^1, in *Geometric Analysis of PDE and Several Complex Variables*, S. Chanillo, P. Cordaro, N. Hanges, J. Hounie and A. Meziani (eds.), Contemporary Mathematics series, AMS, to appear.

[11] F. Camilli, A. Siconolfi, Hamilton-Jacobi equation with measurable dependence on the state variable. *Adv. Differential Equations* **8**, no. 6 (2003), 733-768.

[12] G. Dal Maso, *Introduction to* Γ*-convergence*, Progress in Nonlinear Differential Equations and their Applications **8**, Birkhäuser (1993).

[13] G. De Cecco, G. Palmieri, Lip manifolds: from metric to Finslerian structure. *Math. Z.* **218** (1995), 223-237.

[14] P.G. De Gennes, *The Physics of Liquid Crystals*, Clarendon Press, Oxford (1974).

[15] J. Ericksen, D. Kinderlehrer ed., *Theory and Applications of liquid crystals*, IMA Series **5**, Springer (1987).

[16] M. Gromov, *Metric structures for Riemannian and non-Riemannian spaces*, Progress in Mathematics **152**, Birkhäuser (1999).

[17] M. Giaquinta, G. Modica, J. Souček, *Cartesian Currents in the Calculus of Variations, vol. 2*, Springer (1998).

[18] V. Millot, On a Relaxed Type Energy for S^2-valued Maps, in preparation.

[19] S. Venturini, Derivation of distance functions in $\mathbb{R}^N$, Preprint (1991).