An introduction to convex and discrete geometry

Lecture Notes

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Contents

1 Introduction

1.1 Euclidean space

We shall work in d**-dimensional Euclidean space** $\mathbb{R}^d = \{(x_1, \ldots, x_d),\ x_i \in \mathbb{R}, i \le d\}$, the space of all length d real sequences $x = (x_1, \ldots, x_d)$. Its elements are called points or vectors, the origin is $0 = (0, \ldots, 0)$. It is a real vector space. We add vectors coordiante-wise,

$$x + y = (x_1 + y_1, \ldots, x_d + y_d), \qquad x, y \in \mathbb{R}^d,$$

and the same for scalar multiplication

$$\lambda x = (\lambda x_1, \ldots, \lambda x_d), \qquad \lambda \in \mathbb{R}, x \in \mathbb{R}^d.$$

The **standard basis** consits of the vectors $e_1 = (1, 0, 0, \ldots, 0, 0)$, $e_2 = (0, 1, 0, \ldots, 0, 0)$, and so on, $e_d = (0, 0, 0, \ldots, 0, 1)$ and plainly $x = \sum_{j=1}^d x_j e_j$. Addition of vectors suggests a natural way of adding nonempty subsets A, B of $\mathbb{R}^d$, which is defined by

$$A + B = \{a + b,\ a \in A, b \in B\}$$

and called the **Minkowski sum** of A and B. Similarly for multiplication by a scalar λ,

$$\lambda A = \{\lambda a,\ a \in A\},$$

called the **dilation** of A. In particular, $(-1)A$, denoted $-A$ is the symmetric image of A. The set A is called **symmetric** if $A = -A$.

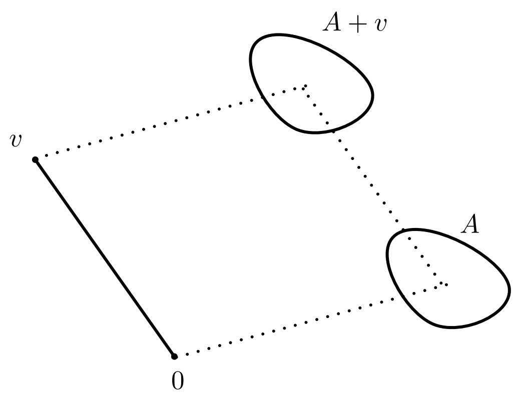

Figure 1.1: $A + v$ is A translated by v.

1.1 Example. The Minkowski sum involving a singleton, say $A + \{v\}$ is the translation of A by v, denoted $A + v$, or $v + A$ (see Figure 1.1).

1.2 Example. The Minkowski sum of a square and a disk is a *rounded* square (see Figure 1.2).

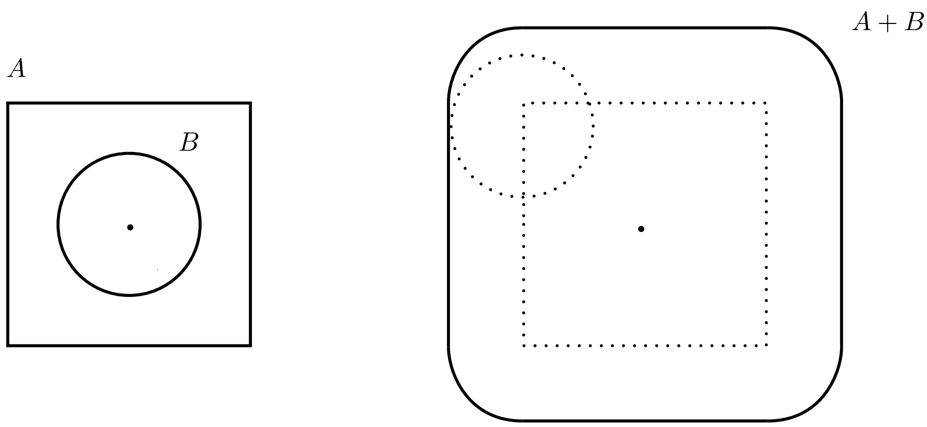

Figure 1.2: The Minkowski sum of a square and a disk.

1.3 Example. The Minkowski sum of two intervals is a a parallelogram. The Minkowski sum of several intervals is a polygon (see Figure 1.3).

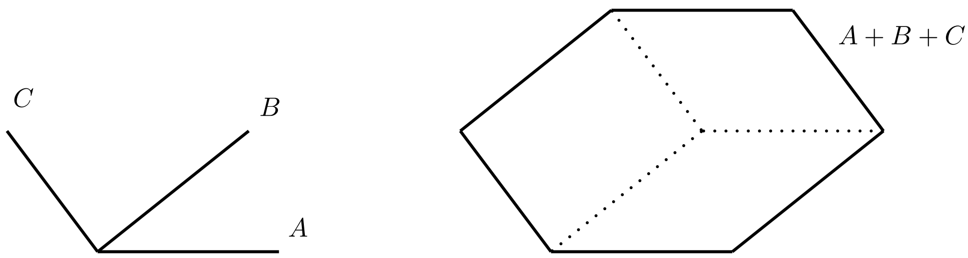

Figure 1.3: The Minkowski sum of several intervals.

The Euclidean structure of $\mathbb{R}^d$ is given by the **standard scalar product**

$$\langle x, y\rangle = \sum_{j=1}^{d} x_j y_j, \qquad x, y \in \mathbb{R}^d.$$

Recall the defining properties of a scalar product: $\langle\cdot,\cdot\rangle\colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is symmetric and bilinear and satisfies $\langle x, x\rangle \geq 0$, for every $x \in \mathbb{R}^d$, with equality if and only if $x = 0$. The **Euclidean norm** of x is

$$|x| = \sqrt{\langle x, x\rangle} = \sqrt{\sum_j x_j^2}$$

which is its distance to the origin. Recall that a function $\|\cdot\|\colon \mathbb{R}^d \to \mathbb{R}$ is called a norm if: 1) $\|x\| \geq 0$, for every $x \in \mathbb{R}^d$ with equality if and only if $x = 0$, 2) $\|\lambda x\| = |\lambda|\|x\|$, for every $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}^d$, 3) $\|x + y\| \leq \|x\| + \|y\|$, for every $x, y \in \mathbb{R}^d$. The **(Euclidean) distance** between x and y in $\mathbb{R}^d$ is

$$d(x, y) = |x - y|$$

which defines the **Euclidean metric** on $\mathbb{R}^d$. In general, every scalar product gives rise to a norm by $\|x\| = \sqrt{\langle x, x\rangle}$. The triangle inequality is a consequence of the Cauchy-Schwarz inequality.

1.4 Theorem. *If $x, y \in \mathbb{R}^d$, then $|\langle x, y\rangle| \le |x| \cdot |y|$. Equality holds if and only if $x = \lambda y$ or $y = \lambda x$ for some $\lambda \in \mathbb{R}$.*

Proof. If $y = 0$, the statement is clear. Assume $y \neq 0$ and consider the following polynomial of degree 2

$$P(t) = \langle x - ty, x - ty\rangle = |x|^2 + t^2|y|^2 - 2t\langle x, y\rangle.$$

Since P is nonnegative and the leading coefficient $|y|^2$ is positive, the discriminant of P is nonpositive, hence $4\langle x, y\rangle^2 - 4|x|^2|y|^2 \le 0$. If we have equality, P has a zero at say λ, that is $0 = P(\lambda) = \langle x - \lambda y, x - \lambda y\rangle$ which by the axioms of a scalar product implies that $x - \lambda y = 0$. $\square$

1.2 Linear and affine hulls

A **linear combination** of points $x_1, \ldots, x_k$ in $\mathbb{R}^d$ is $\sum_{j=1}^k \lambda_j x_j$, where $\lambda_j \in \mathbb{R}$. We call it an **affine combination** if additionally the scalars (weights) λ_j add up to 1, $\sum_{j=1}^k \lambda_j = 1$. We define the **linear hull** (or simply the span) of points $x_1, \ldots, x_k$ as the set of all linear combinations of $x_1, \ldots, x_k$. Similarly for the **affine hull**.

1.5 Example. The linear hull of two points a, b is a line if they are colinear with the origin and otherwise, it is a plane. The affine hull of two different points a, b is the line passing through a and b. The linear hull of e_1, e_2, e_3 in $\mathbb{R}^3$ is $\mathbb{R}^3$ and their affine hull is the plane $\{x \in \mathbb{R}^3,\ x_1 + x_2 + x_3 = 1\}$.

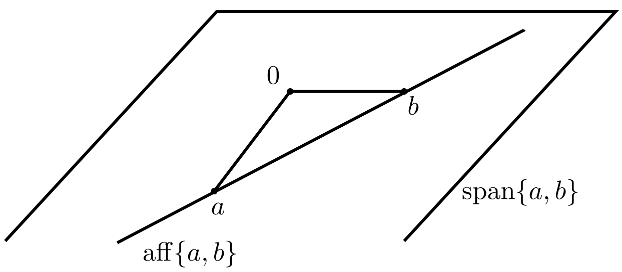

Figure 1.4: The affine and linear hull of two points.

The definitions of linear and affine hulls extend from finite sets to arbitrary ones. That is, given a subset A of $\mathbb{R}^d$, its affine hull is defined as

$$\begin{aligned}\mathrm{aff}(A) &= \{\text{all affine combinations of points from } A\} \\ &= \left\{ \sum_{i=1}^{k} \lambda_i x_i,\ k \geq 1, a_i \in A, \lambda_i \in \mathbb{R}, \sum_{i=1}^{k} \lambda_i = 1 \right\}.\end{aligned}$$

Similarly for the linear hull of A, denoted $\mathrm{lin}(A)$ or $\mathrm{span}(A)$.

A set A in $\mathbb{R}^d$ is called a **flat** (or an affine subspace) if it is a translate of a subspace F, that is $A = F + x_0$ for some $x_0 \in \mathbb{R}^d$. The **dimension** of A is the dimension of F. We have a basic result relating flats and affine hulls.

1.6 Theorem. *If $x_1, \ldots, x_k$ are points in $\mathbb{R}^d$, then their affine hull is*

(a) a flat,

(b) the smallest flat containing them, that is if $x_1, \ldots, x_k \in F$ for some flat F, then $\mathrm{aff}(x_1, \ldots, x_k) \subset F$,

(c) $\mathrm{aff}(x_1, \ldots, x_k) = \bigcap \{F : F$ is a flat containing $x_1, \ldots, x_k\}$.

Proof. Exercise. □

Some affine subspaces have spacial names: 1-dimensional ones are called **lines**, 1-codimensional ones are called **hyperplanes** and in general k-dimensional ones are called k-flats. A hyperplane H in $\mathbb{R}^d$ can be specified by a single linear equation $a_1 x_1 + \cdots + a_d x_d = b$, that is $H = \{x \in \mathbb{R}^d,\ \langle a, x \rangle = b\}$ and H is perpendicular to a. It gives rise to two closed half-spaces: $H^- = \{x \in \mathbb{R}^d,\ \langle a, x \rangle \leq b\}$ and $H^+ = \{x \in \mathbb{R}^d,\ \langle a, x \rangle \geq b\}$.

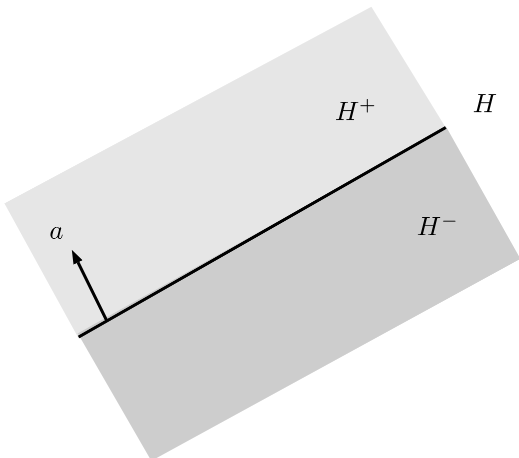

Figure 1.5: Convex sets are intersections of halfspaces.

Affine and linear notions are obviously connected. Recall that points $x_1, \ldots, x_k$ in $\mathbb{R}^d$ are affinely dependent if one of them can be written as an affine combination of the others. Equivalently, there are reals $\alpha_1, \ldots, \alpha_k$ not all zero such that $\sum_{j=1}^{k} \alpha_j = 0$ and $\sum_{j=1}^{k} \alpha_j x_j = 0$. We have the following convenient criterion.

1.7 Theorem. *Given points $x_1, \ldots, x_k$ in $\mathbb{R}^d$, define $y_j = \left[\begin{smallmatrix} x_j \\ 1 \end{smallmatrix}\right]$ in $\mathbb{R}^{d+1}$ by appending* 1 *as the last coordinate. Then $x_1, \ldots, x_k$ are affinely independent if and only if $y_1, \ldots, y_k$ are linearly independent.*

Proof. Exercise. $\square$

This correspondence of affine notions in $\mathbb{R}^d$ and linear notions in $\mathbb{R}^{d+1}$ is quite general. For example, d dimensional nonhorizontal subspaces in $\mathbb{R}^{d+1}$ are bijective to $d-1$ dimensional affine subspaces in $\mathbb{R}^d \times \{1\}$ (see Figure 1.6).

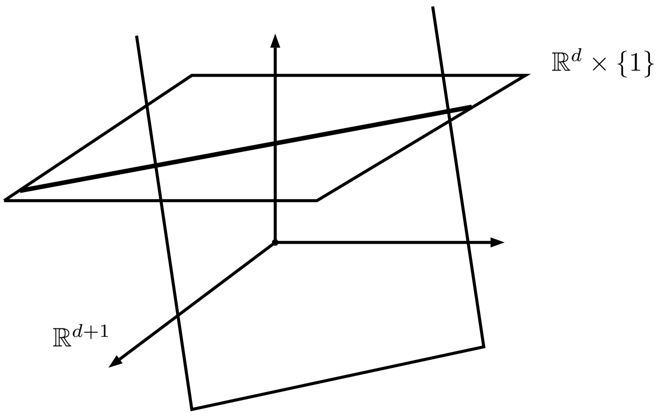

Figure 1.6: Subspaces in $\mathbb{R}^{d+1}$ are bijective to flats in $\mathbb{R}^d \times \{1\}$.

1.3 Exercises

1. Show that for any set A in $\mathbb{R}^d$, the set $A - A$ is symmetric.

2. Give an example of a subset A of $\mathbb{R}$ such that $A + A \neq 2A$.

3. Consider the unit disk $D = \{x \in \mathbb{R}^2,\ x_1^2 + x_2^2 \leq 1\}$. Give an example of a set A different than $\frac{1}{2}D$ such that $A - A = D$.

4. Let $f\colon [0,\infty) \to [0,\infty)$ be a continuous strictly increasing function with $f(0) = 0$. Then for every nonnegative a, b, we have
$$ab \leq \int_0^a f + \int_0^b f^{-1}$$
(called Young's inquality). Here f^{-1} denotes the inverse function.

   Hint: Draw a plot of f and interpret the integrals as areas.

5. Let $p, q \in (1,\infty)$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Show that for positive a, b, we have
$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$
Show that for vectors x, y in $\mathbb{R}^d$, we have
$$|\langle x, y\rangle| \leq \left(\sum_{j=1}^d |x_j|^p\right)^{1/p} \left(\sum_{j=1}^d |y_j|^q\right)^{1/q}$$
(called Hölder's inequality).

6. Show that for vectors x, y in $\mathbb{R}^d$, we have
$$\left(\sum_{j=1}^d |x_j + y_j|^p\right)^{1/p} \leq \left(\sum_{j=1}^d |x_j|^p\right)^{1/p} + \left(\sum_{j=1}^d |y_j|^p\right)^{1/p}$$
(called Minkowski's inequality).

7. Using the first part of Exercise 5, show that for $t \geq 0$, $\alpha \geq 1$, we have
$$t(\alpha - t^{\alpha-1}) \leq \alpha - 1.$$
Given nonnegative numbers $a_1, \ldots, a_n$, apply it to $t = \left(\frac{na_j}{\sum_{k=1}^n a_k}\right)^{1/n}$ to obtain the inequality of arithmetic and geometric means (the AM-GM inequality): $\frac{\sum_{j=1}^n a_j}{n} \geq \left(\prod_{j=1}^n a_j\right)^{1/n}$.

8. Prove Theorem 1.6.

9. Prove Theorem 1.7.

10. Points $x_1, \ldots, x_k$ in $\mathbb{R}^d$ are affinely independent if and only if $x_2 - x_1, \ldots, x_k - x_1$ are linearly independent.

11. Show that every $k \geq d+2$ points in $\mathbb{R}^d$ are affinely dependent.

12. Write an equation of a hyperplane in $\mathbb{R}^3$ orthogonal to the vector $(1, 2, 1)$, passing through $(1, -1, 0)$. Given two vectors x_0 and a in $\mathbb{R}^d$, describe geometrically the set $\{x \in \mathbb{R}^d,\ -1 \leq \langle x - x_0, a\rangle \leq 1\}$.

2 Basic convexity

2.1 Convex hulls

A set K in $\mathbb{R}^d$ is **convex** if for every two points x, y in K the whole segments $[x, y] = \{\lambda x + (1-\lambda)y,\ \lambda \in [0,1]\}$ belongs to K. In other words, for every $x, y \in K$ and $\lambda \in [0,1]$, we have $\lambda x + (1-\lambda)y \in K$, or equivalently $\lambda K + (1-\lambda)K \subset K$ (if fact we can write equality instead of the inclusion "$\subset$" because the opposite inclusion always holds). The **dimension** of a convex set is the dimension of its affine hull. Convexity is clearly preserved by taking intersections.

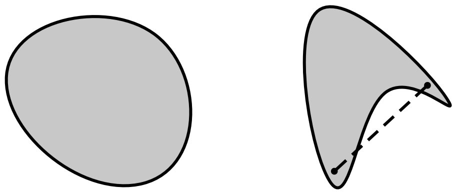

Figure 2.1: An example of a convex and a nonconvex set.

2.1 Theorem. *Intersections of (arbitrarily many) convex sets are convex.*

2.2 Example. Subspaces, flats are convex. Open/closed half-spaces are convex.

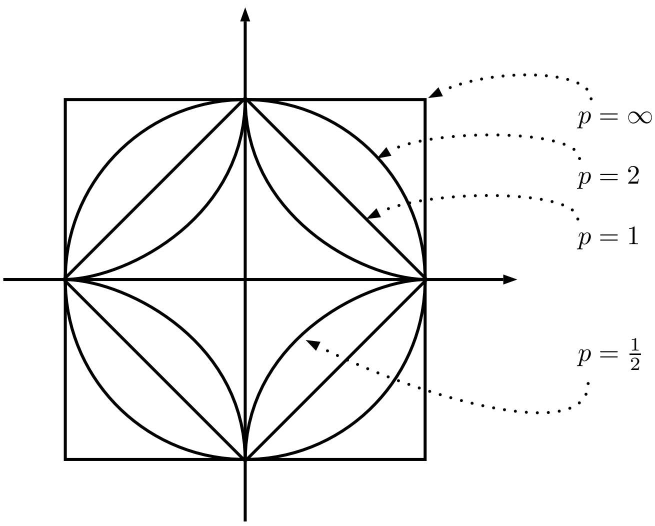

Figure 2.2: B_p^d

2.3 Example. Given $p > 0$ define

$$\|x\|_p = \left(\sum_{j=1}^{d} |x_j|^p\right)^{1/p}, \qquad x \in \mathbb{R}^d$$

and

$$\|x\|_\infty = \max_{j \le d} |x_j|$$

(as suggested by taking the limit $p \to \infty$). Define the set

$$B_p^n = \{x \in \mathbb{R}^d, \ \|x\|_p \le 1\}$$

called the unit ℓ_p-ball in $\mathbb{R}^d$. It is convex if and only if $p \in [1, \infty]$ (see Figure 2.2). In particular, B_1^d is called the cross-polytope, B_2^d is the unit (centred) Euclidean ball, $B_\infty^d = [-1, 1]^d$ is the d-dimensional cube (see Figure 2.3).

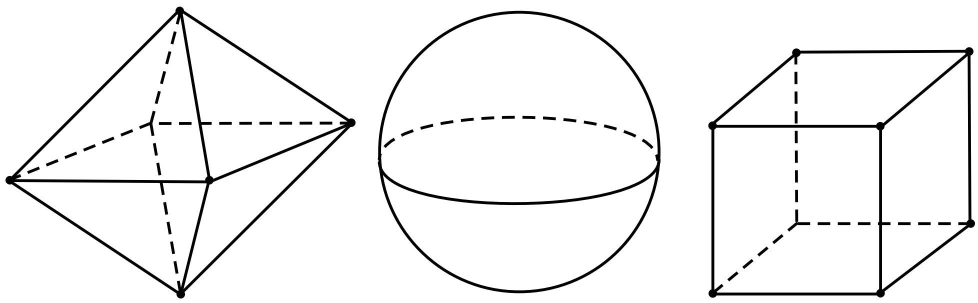

Figure 2.3: The cross-polytope, Euclidean ball and cube.

A **convex combination** of points $x_1, \ldots, x_k$ in $\mathbb{R}^d$ is $\sum_{j=1}^k \lambda_j x_j$ for some nonnegative $\lambda_1, \ldots, \lambda_k$ with $\sum_{j=1}^k \lambda_j = 1$ (often called weights). The **convex hull** of a subset A of $\mathbb{R}^d$ is the set of all convex combinations of points from A,

$$\mathrm{conv}(A) = \left\{ \sum_{j=1}^k \lambda_j x_j, \ k \ge 1, x_1, \ldots, x_k \in A, \lambda_1, \ldots, \lambda_k \ge 0, \sum_{j=1}^k \lambda_j = 1 \right\}.$$

Similarly to affine hulls, convex hulls are smallest convex sets containing given ones.

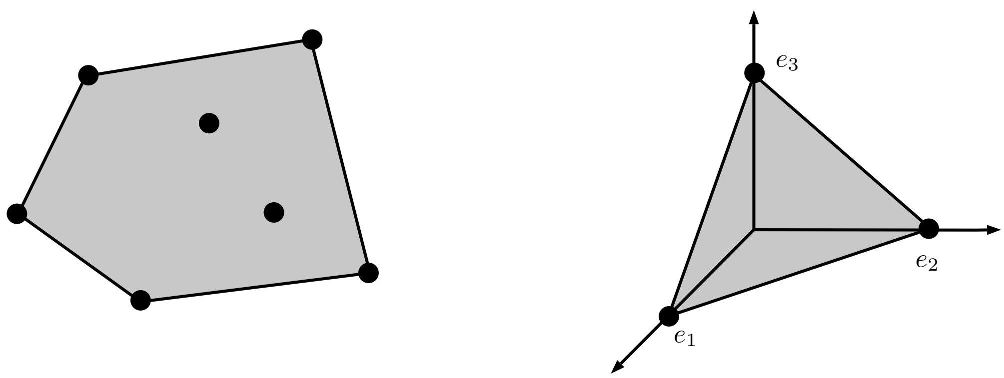

Figure 2.4: The convex hull of several points in the plane and the convex hull of the standard basis in $\mathbb{R}^3$.

2.4 Theorem. *Let A be a subset of $\mathbb{R}^d$. Then*

(a) $\mathrm{conv}A$ *is convex,*

(b) $\mathrm{conv}A$ *is the smallest convex set containing A, that is if $A \subset K$ for a convex set K, then* $\mathrm{conv}A \subset K$*,*

(c) $\mathrm{conv}A = \bigcap\{K : \ K \text{ is convex and } K \supset A\}$.

Proof. Exercise. □

A d-dimensional **simplex** (or just d-simplex) is a convex hull of $d+1$ affinely independent points in $\mathbb{R}^d$. A **convex polytope** in $\mathbb{R}^d$ is the convex hull of finitely many points in $\mathbb{R}^d$ (we often say just "polytope" meaning "convex polytope").

2.2 Carathéodory's theorem

A basic theorem in combinatorial geometry due to Carathéodory asserts that points from convex hulls can in fact be expresses as convex combinations of only dimension plus one many points.

2.5 Theorem (Carathéodory)**.** *Let A be a subset of $\mathbb{R}^d$ and let x belong to* $\mathrm{conv}\, A$*. Then*

$$x = \lambda_1 a_1 + \ldots + \lambda_{d+1} a_{d+1}$$

for some points $a_1, \ldots, a_{d+1}$ from A and nonnegative weights $\lambda_1, \ldots, \lambda_{d+1}$ adding up to 1*.*

Proof. For $y \in \mathbb{R}^d$ and $t \in \mathbb{R}$ by $\left[\begin{smallmatrix} y \\ t \end{smallmatrix}\right]$ we mean the vector in $\mathbb{R}^{d+1}$ whose last component is t and the first d are given by y. Since x belongs to $\mathrm{conv}\, A$, we can write for some $a_1, \ldots, a_k$ from A and nonnegative $\lambda_1, \ldots, \lambda_k$,

$$\left[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right] = \sum_{i=1}^{k} \lambda_i \left[\begin{smallmatrix} a_i \\ 1 \end{smallmatrix}\right]$$

(the last equation taking care of $\sum \lambda_i = 1$). Let k be the smallest possible for which this is possible. We can assume that the λ_i used for that are positive. We want to show that $k \leq d+1$. If not, $k \geq d+2$, the vectors $\left[\begin{smallmatrix} a_1 \\ 1 \end{smallmatrix}\right], \ldots, \left[\begin{smallmatrix} a_k \\ 1 \end{smallmatrix}\right]$ are not linearly independent, thus there are reals $\mu_1, \ldots, \mu_k$, not all zero, such that

$$\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right] = \sum_{i=1}^{k} \mu_i \left[\begin{smallmatrix} a_i \\ 1 \end{smallmatrix}\right].$$

Therefore, for every $t \in \mathbb{R}$ we get

$$\left[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right] = \sum_{i=1}^{k} (\lambda_i + t\mu_i) \left[\begin{smallmatrix} a_i \\ 1 \end{smallmatrix}\right].$$

Notice that the weights $\lambda_i + t\mu_i$ are all positive for $t = 0$, so they all remain positive for small t and there is a choice for t so that (at least) one of the weights becomes zero with the rest remaining positive. This contradicts the minimality of k. $\square$

In particular, Carathéodory theorem says that convex sets can be covered with d-simplices.

2.3 Exercises

1. Prove that subspaces, flats, half-spaces are convex.

2. Prove that a centred ellipsoid $\left\{x \in \mathbb{R}^d,\ \sum_{i=1}^d \frac{x_i^2}{\alpha_i^2} \le 1\right\}$ is convex (the α_i are positive numbers – the lengths of the principal axes).

3. Prove that a set K in $\mathbb{R}^d$ is convex if and only if for every $n \ge 2$ points from K, their convex combination is in K.

4. Prove Theorem 2.4.

5. Describe all convex subsets of $\mathbb{R}$.

6. If sets $K_1, \dots, K_n$ in $\mathbb{R}^d$ are convex, then so is their Minkowski sum $K_1 + \cdots + K_n$.

7. Show that $B_\infty^d = [-e_1, e_1] + \cdots + [-e_d, e_d]$.

8. Let $p \in [1, \infty]$. Prove that the function $x \mapsto \|x\|_p$ is a norm on $\mathbb{R}^d$.

9. Prove that B_p^d is convex if and only if $p \in [1, \infty]$.

10. Show that $B_1^d = \mathrm{conv}\{-e_1, e_1, \dots, -e_d, e_d\}$. Find an analogous statement for B_∞^d.

11. Show that for a set A in $\mathbb{R}^d$ and a hyperplane H such that $A \subset H^-$, we have $\mathrm{conv}(A \cap H) = (\mathrm{conv}\, A) \cap H$.

12. Show that for subsets A, B in $\mathbb{R}^d$, we have $\mathrm{conv}(A + B) = \mathrm{conv}(A) + \mathrm{conv}(B)$.

13. Consider the following vertices of B_∞^4:

$$\begin{aligned} a_1 &= (-1, 1, 1, -1), & a_2 &= (1, -1, 1, -1) & a_3 &= (1, 1, -1, -1) \\ b_1 &= (1, -1, -1, 1), & b_2 &= (-1, 1, -1, 1), & b_3 &= (-1, -1, 1, 1). \end{aligned}$$

    a) Show that $K = \mathrm{conv}\{a_1, a_2, a_3, b_1, b_2, b_3\}$ is a 3-dimensional regular octahedron.

    b) Prove that $B_\infty^4 \cap H = K$, where $H = \{x \in \mathbb{R}^4,\ x_1 + \cdots + x_4 = 0\}$.

    What is the analogous result in $\mathbb{R}^3$, that is what do we obtain by intersecting the cube $[-1, 1]^3$ with the hyperplane $H = \{x \in \mathbb{R}^3,\ x_1 + x_2 + x_3 = 0\}$?

14. Show that for every polynomial P (with complex coefficients), the roots of its derivative P' all lie within the convex hull of the roots of P (the Gauss-Lucas theorem).

3 Separation

3.1 Supporting hyperplanes

Let K be a closed convex subset of $\mathbb{R}^d$. Let H be a hyperplane. We say that H is a **supporting hyperplane** of K if: 1) H *touches* K, that is $K \cap H \neq \varnothing$, and 2) K is contained in one of the half-spaces of H. Then $K \cap H$ is called a **face** of K. For instance, any face of a Euclidean ball is a singleton.

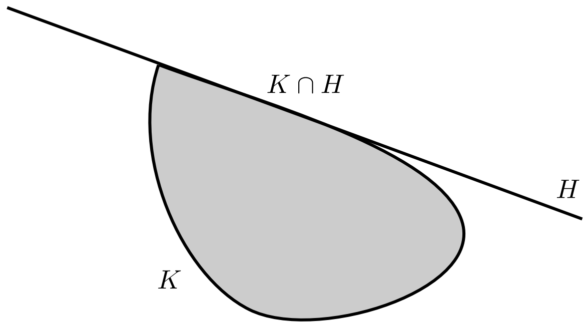

Figure 3.1: H is a supporting plane of K.

3.1 Theorem. *Faces of polytopes are polytopes.*

Proof. It follows immediately from Exercise 2.11 (if, say $K \subset H^-$ for a supporting hyperplane H of a polytope $K = \mathrm{conv}(A)$, then we have $K \cap H = \mathrm{conv}(A) \cap H = \mathrm{conv}(A \cap H)$). □

We give special names to commonly used types of faces of polytopes: 0-dimensional faces are called **vertices**, 1-dimensional ones are called **edges** and 1-codimensional ones are called **facets**. If we denote $f_k(P)$ to be the number of k-dimensional faces of a polytope P in $\mathbb{R}^d$, then

$$\sum_{k=0}^{d-1} (-1)^k f_k(P) = 1 + (-1)^{d-1}. \qquad \text{(Euler's formula)}$$

(We shall not prove this.)

	No. of vertices	No. of edges	No. of facets
B_1^3	8	12	6
B_∞^3	6	12	8

Table 1: The number of faces of the cross-polytope and cube.

Supporting hyperplanes exist along any direction, which can be argued by compactness.

3.2 Theorem. *Let K be a compact convex set in $\mathbb{R}^d$ and let u be a unit vector. Then K has a supporting hyperplane H with equation $\langle x, u\rangle = b$ and $K \subset H^-$ (that is, u is a normal vector pointing outwards).*

Proof. Consider the function $f(x) = \langle x, u\rangle$. Since it is continuous, it attains its maximum on the compact set K, say at $x_0 \in K$. Then $f(x) \leq f(x_0)$ for every $x \in K$, so we define $H = \{x \in \mathbb{R}^d,\ \langle x, u\rangle = \langle x, u\rangle\}$. This inequality means that $K \subset H^-$. Moreover, H touches K because $x_0 \in K \cap H$. $\square$

3.3 Example. Let $K = B_1^3$ and $u = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$. Then $H = \{x \in \mathbb{R}^3, x_1+x_2+x_3 = 1\}$ is a supporting hyperplane of K with an outward pointing unit vector u and the facet $K \cap H$ is the triangle $\mathrm{conv}\{e_1, e_2, e_3\}$.

The above theorem motives the definition of support functions. The **support function** $h_K \colon \mathbb{R}^d \to \mathbb{R}$ of a compact and convex set K in $\mathbb{R}^d$ is defined by

$$h_K(x) = \max_{y\in K}\langle x, y\rangle.$$

Geometrically, if u is a unit vector, then $h_K(u)$ is the (signed) distance from 0 to the supporting hyperplane of K with outward normal u.

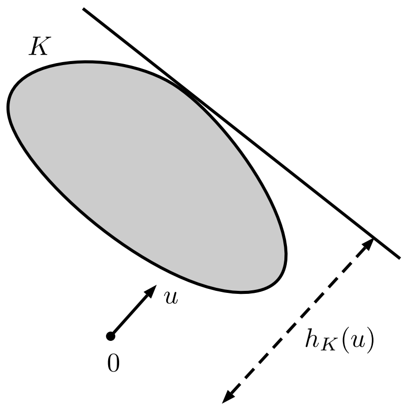

Figure 3.2: The support function $h_K(u)$.

3.4 Example. Let $a \in \mathbb{R}^d$. The support function of the singleton $\{a\}$ is linear, $h_{\{a\}}(x) = \langle x, a\rangle$. The support function of the symmetric interval $[-a, a]$ is also simple, given by $h_{[-a,a]}(x) = |\langle x, a\rangle|$. The support function of the unit ball B_2^d is the Euclidean distance, $h_{B_2^d}(x) = |x|$.

3.5 Example. Let $p \in [1, \infty]$. The support function of the ℓ_p-ball B_p^d is given by the ℓ_q norm, $h_{B_p^d}(x) = \|x\|_q$, where $\frac{1}{p} + \frac{1}{q} = 1$.

We have the following basic properties of the support function h_K of a compact and convex set K in $\mathbb{R}^d$.

1) $h_K(0) = 0$.

2) $h_K(\lambda x) = \lambda h_K(x)$, for every $x \in \mathbb{R}^d$ and $\lambda \geq 0$.

3) $h_K(x+y) \leq h_K(x)+h_K(y)$, for every $x \in \mathbb{R}^d$ (consequently, h_K is a convex function).

4) $h_{-K}(x) = h_K(-x)$, for every $x \in \mathbb{R}^d$.

5) h_K is Lipschitz.

Support functions behave nicely when we take Minkowski sums and convex hulls (we leave proofs as exercises).

3.6 Theorem. *For every compact convex sets K and L in $\mathbb{R}^d$, we have*

$$h_{K+L} = h_K + h_L.$$

3.7 Theorem. *For every family of compact convex sets $\{K_\alpha\}$ in $\mathbb{R}^d$, we have*

$$h_{\mathrm{conv}\{\bigcup K_\alpha\}} = \max_\alpha h_{K_\alpha}.$$

In particular, for a polytope $P = \mathrm{conv}\{x_i\}_{i=1}^k$, we have that the support function of P is piecewise linear, $h_P(x) = \max_{i \leq k}\langle x, x_i\rangle$.

Support functions determine compact convex sets uniquely (we leave proof as an exercise).

3.8 Theorem. *For compact convex sets K and L in $\mathbb{R}^d$, we have $K \subset L$ if and only if $h_K \leq h_L$. In particular, h_K determines K uniquely, $K = L$ if and only if $h_K = h_L$.*

The **width** of a compact convex set K in $\mathbb{R}^d$ in the direction of a unit vector u is defined as

$$w_K(u) = h_K(u) + h_K(-u).$$

The width of K is the minimal width over all directions,

$$w_K = \min_{|u|=1} w_K(u).$$

Geometrically, this is the width of a thinnest plank fully covering K. Note that by the properties of support functions, $w_K = h_{K-K}$, that is the width of K is given by the support function of $K - K$. We say that K is of **constant width** if w_K is a constant function.

3.9 Example. Of course, Euclidean balls are of constant width. The Reuleaux triangle (the intersection of three unit disks centered at vertices of a unit equilateral triangle) is an example of a planar set of constant width.

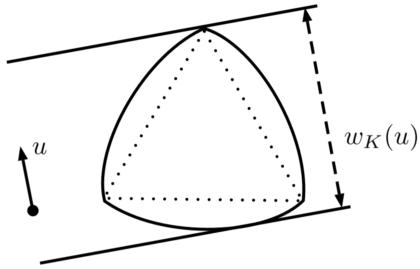

Figure 3.3: The Reuleaux triangle. Its width function $w_K(u)$ is constant.

Let A be a closed subset of $\mathbb{R}^d$. The distance from a point x to A is

$$d(x, A) = \inf_{a \in A} |x - a|.$$

The infimum is attained. Moreover, if A is convex, the minimiser is unique, called the **nearest point** from x to A. These are explained in the following theorem.

3.10 Theorem. *Let K be a closed subset of $\mathbb{R}^d$. For every point x, there is a point y in K realizing the distance from x to K, that is $d(x, K) = |x - y|$. Moreover, if K is convex, this point is unique.*

Proof. Consider a sequence of points y_n from K such that $|x - y_n| \to d(x, K)$ (which exists by the definition of infimum). Since $|y_n| \leq |x| + |x - y_n|$, eventually $|y_n| \leq |x| + d(x, K) + 1$, so the sequence $|y_n|$ is bounded, say by R, that is all the points y_n are in the ball RB_2^d. By its compactness, we can find a convergent subsequence, say $y_{n_k} \to y$. Since K is closed, $y \in K$. By the continuity of the Euclidean distance, $|x - y| = \lim_k |x - y_{n_k}| = d(x, K)$. This proves that y is a closest point in K to x.

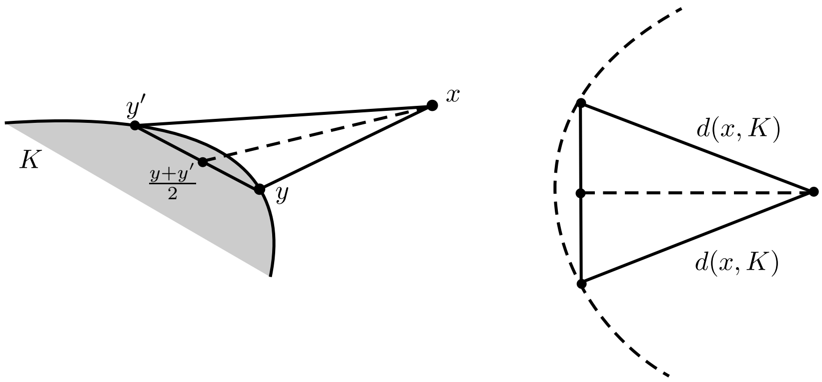

Figure 3.4: An argument explaining why a nearest point is unique.

Suppose now that K is additionally convex. Say there is another point y' in K such that $|x - y'| = d(x, K)$, different that y. By convexity, the whole segment $[y, y']$ is in

K. On the other hand, its midpoint $\frac{y+y'}{2}$ is nearer to x than y (see Figure 3.4): by the parallelogram identity, we have

$$\left|\frac{y+y'}{2} - x\right|^2 = \frac{|(y-x)+(y'-x)|^2}{4} = \frac{2|y-x|^2 + 2|y'-x|^2 - |y-y'|^2}{4}$$
$$= \frac{2d(x,K)^2 + 2d(x,K)^2 - |y-y'|^2}{4} < d(x,K)^2.$$

This gives a contradiction. □

3.2 Separation theorems

We are ready to prove two separation theorems and then present some important consequences. The first theorem is sometimes called the easier supporting theorem which is about separating points outside convex sets by hyperplanes. The second theorem is sometimes called the harder supporting thereom which is about existence of supporting hyperplanes at boundary points.

3.11 Theorem. *Let K be a closed convex set in $\mathbb{R}^d$ and let p be a point outside K, that is $p \in \mathbb{R}^d \setminus K$. Let y the the nearest point in K to p. Then the hyperplane $H = \{x \in \mathbb{R}^d,\ \langle x-y, p-y\rangle = 0\}$ is a supporting hyperplane of K.*

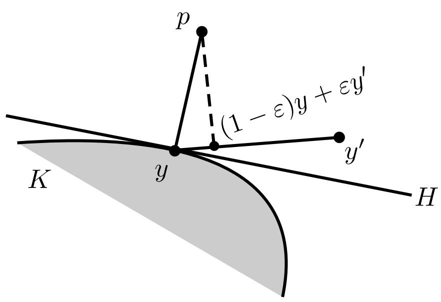

Figure 3.5: The proof of the easier supporting theorem.

Proof. Since $y \in H$, H touches K. If K is not entirely in H^-, say $y' \in K \cap (H^+ \setminus H)$, then points on the segment $[y, y']$ (which is in K by convexity!), which are very close to y, are nearer to p than y (see Figure 3.5). To justify this rigorously, we consider

$$|(1-\varepsilon)y + \varepsilon y' - p|^2 = |(y-p) + \varepsilon(y'-y)|^2$$
$$= |y-p|^2 - 2\varepsilon\langle y'-y, p-y\rangle + \varepsilon^2|y'-y|^2.$$

Since the coefficient at ε is negative (because $y' \in H^+ \setminus H$), for sufficiently small positive ε, the value of this function will be smaller than its value at $\varepsilon = 0$. □

3.12 Theorem. *Let K be a closed convex set in $\mathbb{R}^d$ and let $b \in \partial K$ be a point on its boundary. Then there exists a supporting hyperplane of K containing b.*

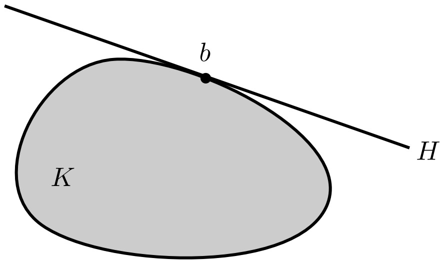

Figure 3.6: The the harder supporting theorem.

Proof. Consider the unit sphere $S = \{x \in \mathbb{R}^d,\ |x - b| = 1\}$ centred at b. Take a point s_0 on S furthest from K, that is $d(s_0, K) = \max_{s \in S} d(s, K)$.

Claim. $d(s_0, K) = 1$.

Since $|s_0 - b| = 1 = d(s_0, K) = \min_{x \in K} |s_0 - x|$, b is the nearest point in K to s_0. It remains to apply Theorem 3.11. We are left with showing the claim.

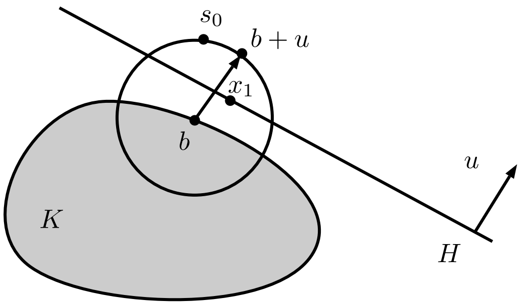

Figure 3.7: The proof of the claim.

Proof of the claim. Let $\varepsilon > 0$, take $x_1 \notin K$ such that $|x_1 - b| < \varepsilon$. Separate x_1 from K by the hyperplane given by Theorem 3.11, and let H be the hyperplane parallel to it passing through x_1, say given by $H = \{x \in \mathbb{R}^d,\ \langle x - x_1, u\rangle = 0\}$, where u is a unit vector oriented such that $H^- \supset K$. Note that for a point x in K, we have

$$\begin{aligned}
|b + u - x| &\geq \langle b + u - x, u\rangle \\
&= \langle b, u\rangle + 1 - \langle x, u\rangle \\
&\geq \langle b, u\rangle + 1 - \langle x_1, u\rangle \\
&= \langle b - x_1, u\rangle + 1 \\
&\geq 1 - |b - x_1| \\
&> 1 - \varepsilon
\end{aligned}$$

(the first and fifth lines follow by the Cauchy-Schwarz inequality and the third one holds because $K \subset H^-$). This means $d(b+u, K) \geq 1 - \varepsilon$. Since u is a unit vector, $b + u \in S$ and we get, $\max_{s \in S} d(s, K) \geq 1 - \varepsilon$. Sending ε to 0 gives $\max_{s \in S} d(s, K) \geq 1$. On the other hand, for every $s \in S$, we clearly have $d(s, K) \leq |s - b| = 1$, so $\max_{s \in S} d(s, K) \leq 1$. This shows that $\max_{s \in S} d(s, K) = 1$ and finishes the proof. $\square$

3.13 Theorem. *Let K and L be compact convex disjoint sets in $\mathbb{R}^d$. Then there is a hyperplane H such that $K \subset H^+$ and $L \subset H^-$.*

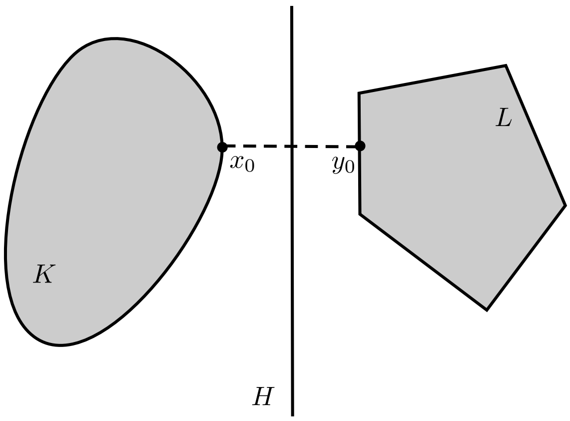

Figure 3.8: The separation theorem for two sets.

Proof. Since $K \times L$ is compact and the function $(x, y) \mapsto |x - y|$ is continuous, the minimum $\min_{x \in K, y \in L} |x - y|$ is attained, say at (x_0, y_0). Choose H to be perpendicular to the segment $[x_0, y_0]$, passing through its midpoint. Since x_0 is the nearest point in K to y_0, we get that K is entirely on one side of H (by Theorem 3.11). Similarly, L is also entirely on one side of H. Since x_0 and y_0 are on different sides, K and L are in fact separated by H. $\square$

3.14 Remark. By additional approximation arguments, the compactness assumption can be removed.

We are ready to show that compact convex sets are intersections of half-spaces.

3.15 Theorem. *Let K be a compact convex set in $\mathbb{R}^d$. Then*

$$K = \bigcap \{H^- : \ H \textit{ is a supporting hyperplane of } K \textit{ oriented such that } K \subset H^-\}.$$

Proof. Let

$$L = \bigcap \{H^- : \ H \text{ is a supporting hyperplane of } K \text{ oriented such that } K \subset H^-\}.$$

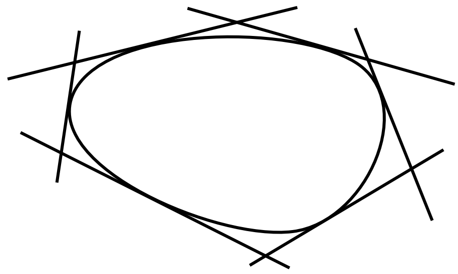

Figure 3.9: Convex sets are intersections of halfspaces.

Since $H^- \supset K$ if H is a supporting hyperplane, we clearly have $L \supset K$. To show the opposite inclusion, $L \subset K$, take $x \notin K$. Separate x from K by the hyperplane H given by Theorem 3.11. Since $x \in H^+$, we get $x \notin L$. $\square$

3.3 Application in linear optimisation – Farkas' lemma

As an application of separation theorems, we show Farkas' lemma concerning solvability of linear systems of inequalities.

3.16 Lemma (Farkas)**.** *For every $d \times n$ matrix A, exactly one of the following occurs*

(i) $Ax = 0$ has a nontrivial nonnegative solution (that is, $Ax = 0$ for some $x \in \mathbb{R}^n$ with nonnegative coordiates, not all equal 0)

(ii) there is $y \in \mathbb{R}^d$ such that $y^\top A$ is a vector with all entries negative

3.17 Remark. Note that if (ii) holds, (i) is ruled out because multiplying jth equation of $Ax = 0$ by y_j and adding we get $\sum [y^\top A]_i x_i = 0$; since all the coefficients $[y^\top A]_i$ are negative, this equation cannot have a nontrivial nonegative solution.

Proof. Let $V = \{v_1, \ldots, v_n\}$ be the column vectors of A. We have two possibilities.

Case 1. $0 \in \mathrm{conv}(V)$, then

$$0 = \sum_{i=1}^{n} \lambda_i v_i = A \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix}$$

for some nonnegative λ_i adding up to 1 (not all zero). We thus can take $x = [\lambda_i]_{i=1}^n$ and (i) holds.

Case 2. $0 \notin \mathrm{conv}(V)$, then we separate 0 from $\mathrm{conv}(V)$ by a hyperplane H, that is there is a vector u such that $[u^\top A]_j = \langle u, v_j \rangle < \langle u, 0 \rangle = 0$ for every j. We thus can take $y = u$ and (ii) holds. $\square$

3.4 Exercises

1. Let P be a polytope. Show that P has finitely many faces.

2. Show that every compact convex set K in $\mathbb{R}^d$ has at least one 0-dimensional face.

   Hint. Consider $\max_{x \in K} |x|$.

3. Show that for the number of k-dimensional faces of the cube, we have $f_k(B_\infty^d) = 2^{d-k}\binom{d}{k}$, $0 \le k \le d-1$. Check that Eurler's formula holds.

4. Show that for the number of k-dimensional faces of the cross-polytope, we have $f_k(B_1^d) = 2^{k+1}\binom{d}{k+1}$, $0 \le k \le d-1$. Check that Eurler's formula holds.

5. Prove properties 1) - 5) of the support function h_K of a compact convex set K in $\mathbb{R}^d$.

6. Prove Theorems 3.6 and 3.7.

7. Find the support function of the ellipse $E = \{(x, y) \in \mathbb{R}^2,\ \frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1\}$. Deduce that all the circumscribed rectangles on E have vertices on a fixed circle (called the *director circle* of E).

8. Prove the claim from Example 3.5 that $h_{B_p^d}(x) = \|x\|_q$.

9. For a compact convex set K in $\mathbb{R}^d$, let $x_1, x_2 \in K$ realise its diameter, that is $|x_1 - x_2| = \mathrm{diam}(K)$. Show that H_i defined as the hyperplane orthogonal to $x_1 - x_2$ and passing through x_i is a supporting hyperplane and $K \cap H_i = \{x_i\}$, $i = 1, 2$. (Consequently, unless K is a singleton, it has at least two 0-dimensional faces.)

   The diameter of a compact set A in $\mathbb{R}^d$ is $\mathrm{diam}(A) = \max_{a,a' \in A} |a - a'|$.

10. Prove that for a compact convex set K in $\mathbb{R}^d$, we have $\mathrm{diam}(K) = \max_{|u|=1} w_K(u)$.

11. Show that for a compact set A in $\mathbb{R}^d$, we have $\mathrm{diam}(\mathrm{conv}(A)) = \mathrm{diam}(A)$.

12. Show that an elephant can be packed inside B_∞^n for n sufficiently large.

    Hint. B_∞^{4n} has 4 vertices:

$$(\underbrace{-1,\dots,-1}_{n},\underbrace{1,\dots,1}_{n},\underbrace{1,\dots,1}_{n},\underbrace{1,\dots,1}_{n})$$
$$(\underbrace{1,\dots,1}_{n},\underbrace{-1,\dots,-1}_{n},\underbrace{1,\dots,1}_{n},\underbrace{1,\dots,1}_{n}), \qquad \text{etc.}$$

13. Prove Theorem 3.8.

14. Prove Remark 3.14.

4 Further aspects of convexity

4.1 Extreme points and Minkowski's theorem

Let K be a compact convex set. A point x in K is an **extreme point** of K if x is *not* "between" two points of K: there is no $x_1, x_2 \in K$, $x_1 \neq x_2$, $\lambda \in (0,1)$ such that $x = \lambda x_1 + (1-\lambda)x_2$.

4.1 Example. Any boundary point of a Euclidean ball is extreme. Only the vertices of the cube B_∞^d are its extreme points. Closed half-spaces have no extremal points.

4.2 Lemma. *Every* 0*-dimensional face of a compact convex set is its extreme point.*

Proof. Consider a 0-dimensional face $\{x\} = K \cap H$ for a supporting hyperplane H. Suppose $x = \lambda x_1 + (1-\lambda)x_2$, for some $x_1, x_2 \in K$, $x_1 \neq x_2$, $\lambda \in (0,1)$. Consider the segment $[x_1, x_2] = \operatorname{conv}\{x_1, x_2\}$ which is entirely in K. Note that $x \in H$ (because $\{x\} = K \cap H$) and clearly $x \in \operatorname{conv}\{x_1, x_2\}$, so $\{x\} \subset \operatorname{conv}\{x_1, x_2\} \cap H$. On the other hand, $\{x\} = K \cap H \supset \operatorname{conv}\{x_1, x_2\} \cap H$, so $\{x\} = \operatorname{conv}\{x_1, x_2\} \cap H$. Thus, by Exercise 2.11,

$$\{x\} = \operatorname{conv}(\{x_1, x_2\} \cap H).$$

As a result, x_1 or x_2 is in H (otherwise the right hand side would be empty). If both x_1 and x_2 were in H, then the right hand side would be the whole segment $[x_1, x_2]$. Therefore, exactly one of these points is in H, say x_1, but then the right hand side is $\{x_1\}$, thus $x = x_1$, a contradiction. $\square$

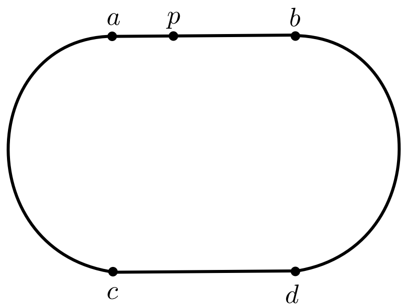

Figure 4.1: A stadium: the "corners" a, b, c, d are extreme points, but they are not 0-dimensional faces. A point p is not extreme.

4.3 Remark. The converse is not true in general: not every extreme point has to be a 0-dimensional face – see Figure 4.1 for the example of a stadium.

4.4 Lemma. *Let F be a face of a compact convex set K in $\mathbb{R}^d$. Let x be a point in F. Then x is an extreme point of F if and only if x is an extreme point of K.*

Proof. Since $F \subset K$, extreme points in K are extreme in F. For the other implication, suppose $F = K \cap H$ for a hyperplane H such that $K \subset H^-$ and x is an extreme point of F. Suppose that x is not extreme in K. Then $x = \lambda x_1 + (1-\lambda)x_2$, for some $x_1, x_2 \in K$, $x_1 \neq x_2$, $\lambda \in (0,1)$. Using Exercise 2.11, we have

$$\operatorname{conv}\{x_1, x_2\} \cap H = \operatorname{conv}(\{x_1, x_2\} \cap H).$$

As in the proof of Lemma 4.2, $x \in \operatorname{conv}\{x_1, x_2\} \cap H$, so the right hand side is nonempty, in which case x_1 or x_2 is in H. Say $x_1 \in H$, but since $x \in H$, the whole line xx_1 is in H, too. In particular, $x_2 \in H$. Since $F = K \cap H$, we get $x_1, x_2 \in F$, which contradicts the fact that x is extremal in F. □

We are ready to prove a fundamental result about extreme points, saying that convex sets are convex hulls of their extreme points. For finite dimensional Euclidean spaces, it is called Minkowski's theorem (in fact, Minkowski proved it in 3 dimensions which was extended by Steinitz to any finite dimension), whereas the general form for topological vector spaces is the famous Krein-Milman theorem.

4.5 Theorem (Minkowski)**.** *For a compact convex set K in $\mathbb{R}^d$, let E be the set of all extreme points of K. Then*

$$K = \operatorname{conv}(E).$$

Proof. Since $E \subset K$, we have $\operatorname{conv}(E) \subset K$. We show the opposite implication, that is $K \subset \operatorname{conv}(E)$ by induction on the dimension d. It is clear for $d = 1$. Fix $d \geq 2$. Suppose the implication holds for all dimensions less than d. Let $x \in K$. If x is a boundary point, then x belongs to a face F of K (by the harder separation theorem, Theorem 3.12). Since F is at most $d-1$ dimensional (as contained in a supporting hyperplane), x is a convex combination of extreme points of F, which are also extreme in K (Lemma 4.4). If x is not a boundary point, it is on an interval whose end points (say y and z) are on the boundary. By the previous argument, each of these end points y and z is a convex combination of the extreme points of K, hence x, too. □

4.2 Extreme points of polytopes

We first prove that a polytope is the convex hull of its vertices (0-dimensional faces). Then we indentify that the set of all vertices (the set of 0-dimensional faces) is the set of its extreme points.

4.6 Theorem. *Let P be a convex polytope in $\mathbb{R}^d$ with vertex set V. Then $P = \operatorname{conv}(V)$.*

Proof. By definition, $P = \operatorname{conv}\{x_1, \ldots, x_k\}$ for some points $x_1, \ldots, x_k$ in $\mathbb{R}^d$. Assume that these points are such that k is minimal (so in particular, no point x_i is a convex combination of the other points). We show $V = \{x_1, \ldots, x_k\}$. Let $P' = \operatorname{conv}\{x_2, \ldots, x_k\}$.

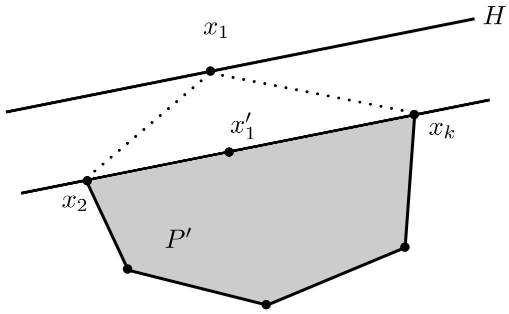

Figure 4.2: The proof of Theorem 4.6.

Then $x_1 \notin P'$. Let x_1' be the nearest point in P' to x_1 and let H be the hyperplane passing through x_1 and parallel to the supporting hyperpane from Theorem 3.11 separating x_1 from P'. Then $P \subset H^-$ and $H \cap P = H \cap \operatorname{conv}\{x_1, \dots, x_k\} = \operatorname{conv}(H \cap \{x_1, \dots, x_k\}) = \operatorname{conv}\{x_1\} = \{x_1\}$. This means that x_1 is a vertex, that is $x_1 \in V$. Similarly for the other points, so $\{x_1, \dots, x_k\} \subset V$. For the opposite inclusion, if $x \in V$, that is $\{x\} = P \cap H$ for a supporting hyperplane H, then

$$\{x\} = \operatorname{conv}\{x_1, \dots, x_k\} \cap H = \operatorname{conv}(\{x_1, \dots, x_k\} \cap H),$$

and as a result, the intersection $\{x_1, \dots, x_k\} \cap H$ is a singleton, say $\{x_i\}$ and then $x = x_i$. □

4.7 Theorem. *Let P be a convex polytope in $\mathbb{R}^d$. Let V be the set of its vertices and let E be the set of its extreme points. Then $V = E$.*

Proof. By Lemma 4.2, $V \subset E$. If $x \in P \setminus V$, then by Theorem 4.6, we can write $x = \sum_{i=1}^k \lambda_i x_i$ for some $x_i \in V$, $\lambda_i \in (0,1)$ and $\sum_{i=1}^k \lambda_i = 1$. Writing,

$$x = (1 - \lambda_1)x_1 + (1 - \lambda_1)\left(\frac{\lambda_2}{1 - \lambda_1}x_2 + \cdots + \frac{\lambda_k}{1 - \lambda_1}x_k\right),$$

we conclude that $x \in P \setminus E$. Thus, $P \setminus V \subset P \setminus E$ and since V and E are subsets of P, we obtain $E \subset V$. □

4.3 Exercises

1. Let $D = \{x \in \mathbb{R}^3,\ \sqrt{x_1^2 + x_2^2} + |x_3| \leq 1\}$. Prove that D is convex, describe the extreme points of D and give an intuitive description as to why Minkowski's theorem holds for D.

2. The set of extreme points of a compact convex set with nonempty interior in $\mathbb{R}^d$, $d \geq 3$, need not be closed.

3. Give an example of two disjoint closed convex sets on the plane which are not strictly separable (K and L are strictly separable if for some hyperplane H, we have $K \subset H^- \setminus H$ and $L \subset H^+ \setminus H$).

4. Let $f\colon \mathbb{R}^d \to \mathbb{R}^n$ be an affine map (that is f is of the form $f(x) = Ax + b$, $x \in \mathbb{R}^d$, for some $n \times d$ matrix A and a vector $b \in \mathbb{R}^n$).

   a) If $K \subset \mathbb{R}^d$ is convex, then $f(K)$ is convex.

   b) Is the preimage of a convex set always convex?

   c) For any subset A of $\mathbb{R}^d$, we have $\operatorname{conv} f(A) = f(\operatorname{conv} A)$.

5 Polytopes I

Combinatorial objects often give rise to interesting polytopes and vice-versa.

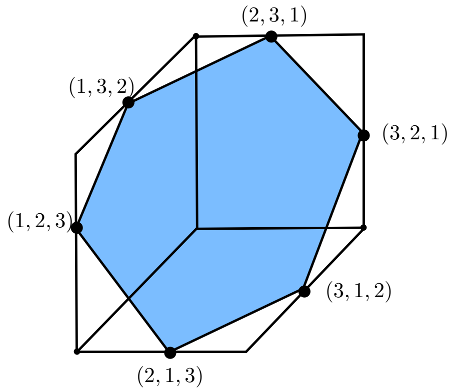

Figure 5.1: A permutohedron of order 3 is a regular hexagon (obtained as a section of the cube $[1,3]^3$).

5.1 Example. Let

$$P = \mathrm{conv}\{(\sigma(1), \sigma(2), \dots, \sigma(n)),\ \sigma \text{ is a permutation on } \{1, 2, \dots, n\}\}.$$

This is a **permutohedron** of order n, an $n-1$-dimensional polytope with $n!$ vertices (all vertices lie on the hyperplane $\{x \in \mathbb{R}^n,\ \sum_{i=1}^n x_i = \frac{n(n+1)}{2}\}$).

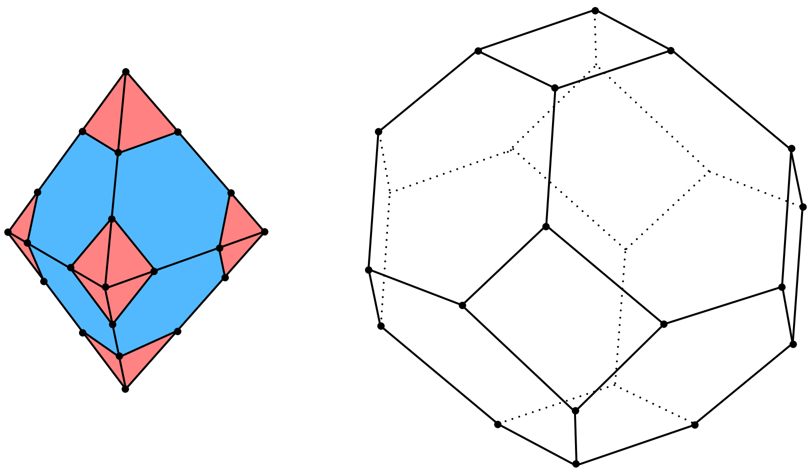

Figure 5.2: A permutohedron of order 4 can be obtained by trimming an octahedron. Each of its 24 vertices has 3 neighbours. Each edge has length $\sqrt{2}$ and connects two vertices which differ by swapping two coordinates with values differing by 1.

5.2 Example. A tetrahedron, cube, octahedron, dodecahedron, icosahedron are platonic solids whose faces are congruent regular polygons (triangles, squares, triangles, pentagons, triangles, respectively). Their graphs are planar.

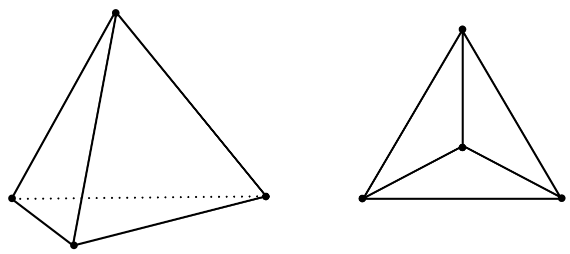

Figure 5.3: A tetrahedron (fire).

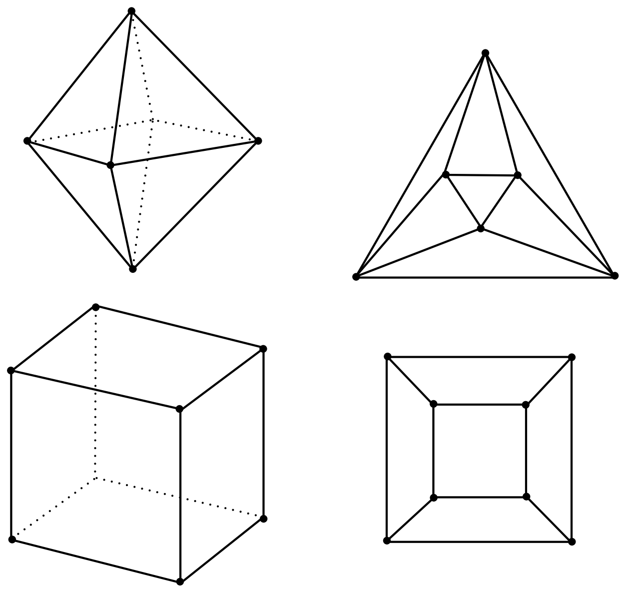

Figure 5.4: An octahedron (air) and cube (earth).

5.1 Duality

Recall that a (convex) polytope in $\mathbb{R}^d$ is the convex hull of a finite set of points in $\mathbb{R}^d$. We showed that it is in fact the convex hull of its 0-dimensional faces, called vertices and these are exactly its extreme points (Theorems 4.6 and 4.7). We shall now develop another view-point, motivated by the fact that convex sets are intersections of half-spaces (Theorem 3.15). We first discuss the powerful concept of duality.

The **duality transform** on $\mathbb{R}^d$ is a mapping which assigns to a point $a \in \mathbb{R}^d \setminus \{0\}$

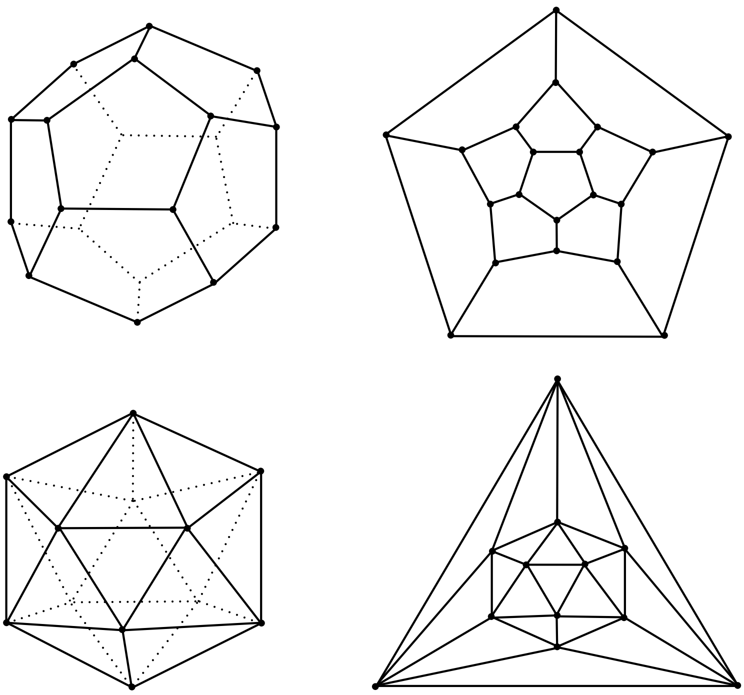

Figure 5.5: A dodacehedron (universe) and icosahedron (water).

the hyperplane

$$\mathcal{D}(a) = \{y \in \mathbb{R}^d,\ \langle a, y\rangle = 1\}$$

and to a hyperplane H not passing through 0, uniquely written as $\{x \in \mathbb{R}^d,\ \langle a, x\rangle = 1\}$, the point

$$\mathcal{D}(H) = a.$$

Duality preserves incidences, as explained in the next lemma.

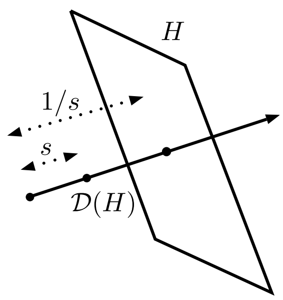

Figure 5.6: If a hyperplane is H is at distance $1/s$ from the origin, then its dual point $\mathcal{D}(H)$ is at distance s from the origin.

5.3 Lemma. *For a point a and a hyperplane H in $\mathbb{R}^d$, we have* *(i)* $a \in H$ *if and only if* $\mathcal{D}(H) \in \mathcal{D}(a)$.

(ii) $a \in H^-$ *if and only if* $\mathcal{D}(H) \in \mathcal{D}(a)^-$.

Proof. Suppose $H = \{x \in \mathbb{R}^d,\ \langle x, p\rangle = 1\}$ for a vector $p \in \mathbb{R}^d$. Then $p = \mathcal{D}(H) \in \mathcal{D}(a)$ if and only if $\langle p, a\rangle = 1$ which holds if and only if $a \in H$. This proves (i). The proof of (ii) is identical modulo changing "=" to "≤". □

5.4 Example. Consider a triangle $\Delta = \mathrm{conv}\{a_1, a_2, a_3\}$ with vertices a_1, a_2, a_3 on the plane. To describe the set of all lines intersecting Δ we can use duality. A line l intersects Δ if and only if there is a point $p \in \Delta$ such that $p \in l$ which is equivalent to $\mathcal{D}(l) \in \bigcup_{p\in\Delta} \mathcal{D}(p)$. The condition $p \in \Delta$ is equivalent to $p \in H_1^- \cap H_2^- \cap H_3^-$, where H_1 is the line a_1a_2, etc. Let $v_i = \mathcal{D}(H_i)$ be the points dual to the lines bounding Δ. Now, by duality again, $p \in H_1^- \cap H_2^- \cap H_3^-$, if and only if $v_1, v_2, v_3 \in \mathcal{D}(p)^-$, which gives that $\bigcup_{p\in\Delta} \mathcal{D}(p)$ is the union of all lines l such that $v_1, v_2, v_3 \in l^-$, which is exactly $\mathrm{conv}\{v_1, v_2, v_3\}^c$. Summarising,

$$l \cap \Delta \neq \varnothing \quad \Leftrightarrow \quad \mathcal{D}(l) \in \bigcup_{p\in\Delta} \mathcal{D}(p) = \mathrm{conv}\{v_1, v_2, v_3\}^c.$$

In words, all lines intersecting Δ are the lines dual to the points $\mathrm{conv}\{v_1, v_2, v_3\}^c$.

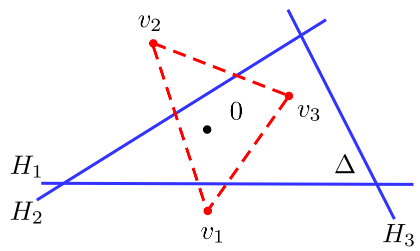

Figure 5.7: The lines intersecting Δ are exactly the ones dual to the points from the complement $\mathrm{conv}\{v_1, v_2, v_3\}^c$ of the dual triangle.

For a set X in $\mathbb{R}^d$ we define its **dual (polar) set** X° (sometimes also denoted $X^\star$) as

$$X^\circ = \{y \in \mathbb{R}^d,\ \forall x \in X\ \langle x, y\rangle \leq 1\}.$$

Note that

$$\begin{aligned} X^\circ &= \bigcap_{x\in X} \{y \in \mathbb{R}^d,\ \langle x, y\rangle \leq 1\} \\ &= \bigcap_{x\in X} \mathcal{D}(x)^-. \end{aligned}$$

In words, the polar set is the intersection of all dual half-spaces $\mathcal{D}(x)^-$ as x ranges over the set X. Straight from the definition we get the following properties:

1) X° contains the origin 0,

2) X° is a closed convex set,

3) if $X \subset Y$, then $Y^\circ \subset X^\circ$.

5.5 Example. Consider a triangle $\Delta = \mathrm{conv}\{a_1, a_2, a_3\}$ with vertices a_1, a_2, a_3 on the plane. We have

$$\Delta^\circ = \bigcap_{p \in \Delta} \mathcal{D}(p)^-.$$

To describe this geometrically, recall that as done in the previous example, $p \in \Delta$ is equivalent to $p \in H_1^- \cap H_2^- \cap H_3^-$, where H_1 is the line a_1a_2, etc. Let $v_i = \mathcal{D}(H_i)$ be the points dual to the lines bounding Δ. Now, by duality again, $p \in H_1^- \cap H_2^- \cap H_3^-$, if and only if $v_1, v_2, v_3 \in \mathcal{D}(p)^-$, which gives that $\bigcap_{p\in\Delta} \mathcal{D}(p)^-$ is the intersection of all half-spaces containing v_1, v_2, v_3, which is exactly $\mathrm{conv}\{v_1, v_2, v_3\}$. Summarising, the dual set $\Delta^\circ = \mathrm{conv}\{v_1, v_2, v_3\}$ is the triangle with vertices being the duals of the edges of the initial set Δ (see again Figure 5.7). This of course extends to polygons.

5.6 Example. Let H be a hyperplane in $\mathbb{R}^d$ such that $0 \in H^-$. Then H° is the ray $(-\infty, \mathcal{D}(H)]$ and $(H^-)^\circ$ is the interval $[0, \mathcal{D}(H)]$.

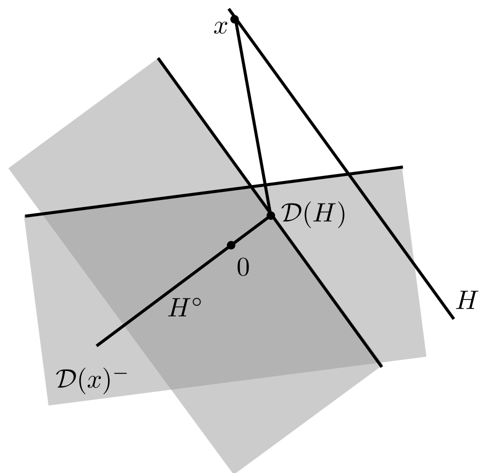

Figure 5.8: The dual set of a hyperplane H is the ray $(-\infty, \mathcal{D}(H)]$.

Duality operations are expected to be involutions (applied twice, they give back the initial input – recall for instance the complex conjugate or matrix transpose). The following theorem explains what happens for our notion of geometric duality for sets ($\mathrm{cl}(A)$ denotes the closure of A).

5.7 Theorem. *For every subset X of $\mathbb{R}^d$, we have $(X^\circ)^\circ = \mathrm{cl}(\mathrm{conv}(X \cup \{0\}))$. In particular, if X is closed, convex and contains 0, then $(X^\circ)^\circ = X$.*

Proof. To show $(X^\circ)^\circ \supset \mathrm{cl}(\mathrm{conv}(X \cup \{0\}))$, it is enough to show $(X^\circ)^\circ \supset X$ (because $(X^\circ)^\circ$ is a closed, convex set which contains 0). To this end, take $x \in X$. Then for every $y \in X^\circ$, we have $\langle x, y\rangle \le 1$, meaning $x \in (X^\circ)^\circ$.

To show the opposite inclusion, we use a separation argument. Take $x \in (X^\circ)^\circ$ and suppose $x \notin \mathrm{cl}(\mathrm{conv}(X \cup \{0\}))$. Then x can be separated from $\mathrm{cl}(\mathrm{conv}(X \cup \{0\}))$ by a hyperplane H and we can assume that 0 does not belong to H (it is enough to translate the supporting hyperplane from Theorem 3.11 a little bit towards x). Say, $H = \{y \in \mathbb{R}^d,\ \langle y, a\rangle = 1\}$ for some vector a and $X \subset H^-$. Then, $\langle y, a\rangle \le 1$ for every $y \in X$, which gives $a \in X^\circ$ and consequently $\langle x, a\rangle \le 1$ because $x \in (X^\circ)^\circ$. On the other hand, $x \in H^+ \setminus H$ gives $\langle x, a\rangle > 1$, a contradiction.

The "in particular" part follows instantly: if X is closed, convex and contains 0, then

$$\mathrm{cl}(\mathrm{conv}(X \cup \{0\})) = \mathrm{cl}(\mathrm{conv}(X)) = \mathrm{cl}(X) = X.$$

□

The next result shows that taking the convex hull will not enlarge the dual set.

5.8 Theorem. *For every subset X of $\mathbb{R}^d$, we have $(\mathrm{conv}(X))^\circ = X^\circ$.*

Proof. Since $\mathrm{conv}(X) \supset X$, we have $(\mathrm{conv}(X))^\circ \subset X^\circ$. For the opposite inclusion, if $x \in X^\circ$, then $\langle x, y\rangle \le 1$ for every $y \in X$, which by linearity gives that $\langle x, y\rangle \le 1$ for every $y \in \mathrm{conv}(X)$, hence $x \in (\mathrm{conv}(X))^\circ$. □

5.9 Remark. By the definition of the dual set, $X^\circ = \bigcap_{x\in X} \mathcal{D}(x)^-$, so

$$(\mathrm{conv}(X))^\circ = \bigcap_{x\in X} \mathcal{D}(x)^-.$$

Combining the last two theorems, we obtain a description of the dual set of a finite intersection of half-spaces. First we need two lemmas.

5.10 Lemma. *Let A be a compact subset of $\mathbb{R}^d$. Then $\mathrm{conv}\, A$ is a closed set.*

Proof. Let $S = \{(\lambda_1, \ldots, \lambda_{d+1}) \in \mathbb{R}^{d+1},\ \lambda_i \ge 0, \sum_{i=1}^{d+1} \lambda_i = 1\}$ be a simplex and define a function $F\colon (\mathbb{R}^d)^{d+1} \times S \to \mathbb{R}^d$, $F(x_1, \ldots, x_{d+1}, \lambda) = \sum_{i=1}^{d+1} \lambda_i x_i$. By Carathéodory's theorem, $\mathrm{conv}\, A = F(\underbrace{A \times \cdots \times A}_{d+1} \times S)$. Since F is continuous and $A \times \cdots \times A \times S$ is compact, the assertion follows. □

5.11 Example. Consider the following subset of the plane $A = \{0\} \times [0,1] \cup [0,\infty) \times \{0\}$ which is closed (but not compact). Its convex hull is not closed (see Figure 5.9).

5.12 Lemma. *For a convex set K in $\mathbb{R}^d$ its polar K° is bounded if and only if 0 is in the interior of K.*

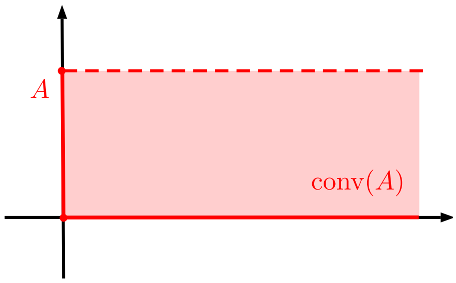

Figure 5.9: An example of a closed set whose convex hull is not closed.

Proof. If 0 is in the interor of K, then $\varepsilon B_2^d \subset K$ for some $\varepsilon > 0$. Then $K^\circ \subset (\varepsilon B_2^d)^\circ = \frac{1}{\varepsilon} B_2^d$, which means that K° is bounded.

If 0 is not in the interior of K, then 0 is either on the boundary of K or is outside K. In any case, there is a hyperplane H passing through 0 such that $H^- \supset K$ (by separation – see Theorems 3.11, 3.12). Then $(H^-)^\circ \subset K^\circ$. But $(H^-)^\circ$ is a ray from ∞ to 0 in the direction normal to H, hence K° is not bounded. $\square$

5.13 Theorem. *Let $\mathcal{H}$ be a finite family of hyperplanes in $\mathbb{R}^d$ not passing through 0 oriented such that $0 \in H^-$ for every $H \in \mathcal{H}$. Suppose $\bigcap_{H \in \mathcal{H}} H^-$ is bounded. Then*

$$\left(\bigcap_{H \in \mathcal{H}} H^- \right)^\circ = \operatorname{conv}\{\mathcal{D}(H),\ H \in \mathcal{H}\}.$$

Proof. Let $X = \{\mathcal{D}(H),\ H \in \mathcal{H}\}$ and $K = \operatorname{conv}(X)$ (the right hand side of the statement). Then, by Theorem 5.8

$$K^\circ = (\operatorname{conv}(X))^\circ = X^\circ = \bigcap_{x \in X} \mathcal{D}(x)^- = \bigcap_{H \in \mathcal{H}} \mathcal{D}(\mathcal{D}(H))^- = \bigcap_{H \in \mathcal{H}} H^-.$$

In particular, K° is bounded, so by Lemma 5.12, K contains 0. Moreover, K is compact (as the convex hull of a finite set – see also Lemma 5.10). By Theorem 5.7, $(K^\circ)^\circ = K$. Consequently, $\left(\bigcap_{H \in \mathcal{H}} H^-\right)^\circ = (K^\circ)^\circ = K$, as desired. $\square$

5.2 Bounded polyhedra are polytopes

The identities $(\operatorname{conv}(X))^\circ = \bigcap_{x \in X} \mathcal{D}(x)^-$ and $\left(\bigcap_{H \in \mathcal{H}} H^-\right)^\circ = \operatorname{conv}\{\mathcal{D}(H),\ H \in \mathcal{H}\}$ say that the duals of polytopes can be described by finite intersections of closed half-spaces and their duals are polytopes (if they are bounded). This motivates the following definition (of the dual of a polytope).

A **polyhedron** in $\mathbb{R}^d$ is the intersection of finitely many closed half-spaces.

5.14 Example. The cube $[-1, 1]^d$ is the intersection of the half-spaces $\{x \in \mathbb{R}^d,\ x_i \leq 1\}$ and $\{x \in \mathbb{R}^d,\ x_i \geq -1\}$, $i = 1, \ldots, d$.

As suggested by Example 5.5, it turns out that the duals of polytopes are polytopes, that is polyhedra (if bounded) are polytopes, and vice versa.

5.15 Theorem. *Every bounded polyhedron in $\mathbb{R}^d$ is a polytope and vice versa.*

Proof. We split the proof into two parts.

Part 1. A bounded polyhedron is a polytope. We do this by induction on dimension. The base case $d = 1$ is clear. Fix $d \geq 2$ and consider a bounded polyhedron P given as the intersection of a finite family of half-spaces, say $P = \bigcap_{H \in \mathcal{H}} H^-$. For each hyperplane $H \in \mathcal{H}$, let $F_H = P \cap H$. Each F_H is a bounded polyhedron in dimension $d - 1$ (for $H' \in \mathcal{H}$, $H' \neq H$, the intersection $H' \cap H$ is a half-space in H). By induction, F_H is a polytope, say $F_H = \mathrm{conv}(V_H)$ for some finite subset V_H of F_H. We claim that $P = \mathrm{conv}(\cup_{H \in \mathcal{H}} V_H)$. Since $V_H \subset P$, the inclusion $\mathrm{conv}(\cup_{H \in \mathcal{H}} V_H) \subset P$ is clear. To prove the opposite inclusion, take $x \in P$. Let l be a line passing through x. Then $l \cap P$ is a segment with end points y, z which are on the boundary of P (otherwise the segment can be extended within P). Thus $y \in F_H$, $z \in F_G$ for some $H, G \in \mathcal{H}$. Consequently, $x \in \mathrm{conv}\{y, z\} \subset \mathrm{conv}(V_H \cup V_G)$, which finishes the argument.

Part 2. A polytope is a bounded polyhedron. Let P be a polytope in $\mathbb{R}^d$, say $P = \mathrm{conv}(V)$ for a finite set V in $\mathbb{R}^d$. Without loss of generality, let $0 \in \mathrm{int} P$. By Theorem 5.8,

$$P^\circ = \bigcap_{v \in V} \mathcal{D}(v)^-.$$

By Lemma 5.12, P° is bounded because 0 is in the interior of P, hence the intersection on the right hand side is a bounded polyhedron. By Part 1, it is a polytope, so we can write it as $\mathrm{conv}(\tilde{V})$ for a finite set $\tilde{V}$ in $\mathbb{R}^d$. Now we have,

$$P = (P^\circ)^\circ = (\mathrm{conv}(\tilde{V}))^\circ = \bigcap_{v \in \tilde{V}} \mathcal{D}(v)^-$$

(the first equality holds because P is compact and contains 0 and the third equality follows from Theorem 5.9). This shows that P is a polyhedron, which finishes the proof. $\square$

5.3 Exercises

1. Find the dual of rB_2^d, a centred Euclidean ball of radius r.

2. Show that $(B_p^n)^\circ = B_q^n$, for $p, q \in [1, \infty]$, $\frac{1}{p} + \frac{1}{q} = 1$.

3. Find the dual set of a hyperplane passing through the origin and a corresponding half-space.

4. Let K be a subset of $\mathbb{R}^d$. Show that $K = K^\circ$ if and only if $K = B_2^d$.

5. Show that every d-dimensional simplex in $\mathbb{R}^d$ is an intersection of $d+1$ half-spaces.

6. Prove that every polytope in $\mathbb{R}^d$ is an orthogonal projection of a simplex of a sufficiently large dimension N onto $\mathbb{R}^d \times \{0\}^{N-d}$.

7. Prove that every symmetric polytope is a linear image of a cross-polytope of a sufficiently large dimension.

6 Polytopes II

The main goal of this chapter is to present a construction of polytopes with many (in fact, maximal) number of faces in a given dimension with a given number of vertices.

6.1 Faces

Recall the definition of a face of a polytope P: it is P itself or $P \cap H$ for some supporting hyperplane H. The dimension of the face $P \cap H$ is the dimension of its affine hull. Zero-dimensional faces are called vertices.

We begin with a description of vertices and faces of faces. Recall that we showed that a polytope is the convex hull of its vertices (Theorem 4.6) and the vertices are exactly the extreme points (Theorem 4.7).

6.1 Theorem. *Let P be a polytope in $\mathbb{R}^d$ and let F be its face.*

(i) F is a polytope.

(ii) The vertices of F are exactly those vertices of P that lie in F.

(iii) The faces of F are exactly those faces of P that are contained in F ("faces of faces are faces").

Proof. (i) This was exactly Theorem 3.1. Equipped with the equivalent description of polytopes as bounded polyhedra (Theorem 5.15), we can give a different proof. If $P = \bigcap_{H \in \mathcal{H}} H^-$ and $F = P \cap H_0$ for a supporting hyperplane H_0, then $F = P \cap H_0 = \left(\bigcap_{H \in \mathcal{H}} H^-\right) \cap H_0^- \cap H_0^+$, so F is a bounded polyhedron, so it is a polytope.

(ii) By Theorem 4.7, the vertices of F are its extremal points. Moreover, by Lemma 4.4, an extreme point of a face of P is extreme in P and consequently, by Theorem 4.7, it is its vertex.

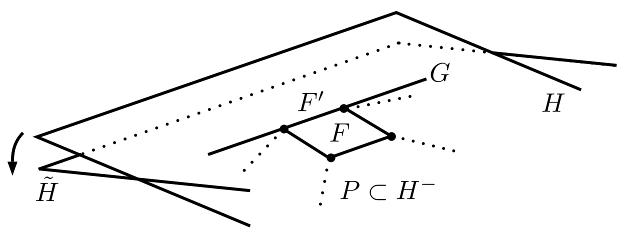

Figure 6.1: Proof of Theorem 6.1 (iii).

(iii) Suppose $F = P \cap H$ for a supporting hyperplane H, $P \subset H^-$. Let $F' \subset F$ be a face of F, that is $F' = F \cap G$ for a supporting hyperplane of G in H (so it is $d-2$-dimensional). We rotate H around G in a direction such that the rotated half-space $\tilde{H}^-$ still contains P. If we rotate only a bit, then the vertices of P not in F are still in

$\tilde{H}^-$. Moreover, $\tilde{H}^-$ also contains all the vertices of F which are not in F'. Therefore $\tilde{H}$ defines the face of P, namely $P \cap \tilde{H}$ which has exactly those vertices of P which are in G. By (ii), these are all vertices of F' and this shows that F' is a face of P. $\square$

6.2 Example. Let P be a d-dimensional simplex in $\mathbb{R}^d$ with vertex set V. By Theorem 6.1 (ii), if F is a face of P, then $F = \mathrm{conv}(S)$ for $S \subset V$. Thus, P has at most $\binom{d+1}{k+1}$ faces of dimension k, $k \in \{0, 1, \ldots, d\}$. In fact, it has exactly $\binom{d+1}{k+1}$ faces of dimension k (exercise).

6.3 Example. For the cross-polytope $B_1^d = \mathrm{conv}\{\pm e_i\}_{i=1}^d$, a subset $S \subsetneq \{\pm e_i\}_{i=1}^d$ determines a face $F = \mathrm{conv}(S)$ if and only if there is no i such that both e_i and $-e_i$ are in F (exercise). Consequently, B_1^d has 3^d faces (S arises by making a choice for each i: take e_i or $-e_i$ or neither, so excluding $S = \varnothing$, there are $3^d - 1$ choices for S).

6.4 Example. For the cube $B_\infty^d = \mathrm{conv}(\{-1, 1\}^d)$, each face F corresponds to a vector $v \in \{-1, 0, 1\}^d$, $F = \mathrm{conv}\{u \in \{-1, 1\}^d,\ u_i = v_i \text{ for each i such that } v_i \neq 0\}$ and v is the barycentre of F_v (exercise). Consequently, B_∞^d has 3^d faces.

Recall that for a polytope in $\mathbb{R}^d$, its $d-1$-dimensional faces are called facets. A polytope P is called **simplicial** if each of its facets is a simplex and is called **simple** if each of its vertices is in d facets.

6.5 Example. Tetrahedron, octahedron, icosahedron are simplicial. Tetrahedron, cube, dodecahedron are simple. The cross-polytope B_1^d is simplicial. The cube B_∞^d is simple. A pyramid $B_1^3 \cap \{x_1 \geq 0\}$ is neither simplicial nor simple. Simplices (in any dimension) and two-dimensional polygons are both simplicial and simple.

A simplicial polytope in $\mathbb{R}^3$ contains only triangular faces. Its graph is a maximal planar graph (a simple graph is maximal planar if it is planar but adding any edge destroys this property). Maximal planar graphs have necessarily triangular faces. The deep theorem of Steinitz asserts that every 3-dimensional polytope forms a 3-connected planar graph and every 3-connected planar graph can be represented as the graph of a 3-dimensional polytope. In view of this corresponedence, 3-dimensional simplicial polytopes correspond to maximal planar graphs.

As we showed, the dual of a polytope P is a polytope. The vertices of the dual correspond to the facets of P and vice-versa. Moreover, the dual of a simplicial polytope is simple and vice-versa. For example, the dual of an octahedron is a cube, the dual of a dodecahedron is an icosahedron, the dual of a simplex is a simplex.

6.2 Cyclic polytopes have many faces

The **f-vector** of a polytope P in $\mathbb{R}^d$ is

$$f(P) = (f_0(P), \ldots, f_d(P)), \qquad f_k(P) = \text{the number of } k\text{-dimensional faces of } P.$$

In particular, $f_0(P)$ is the number of vertices of P, $f_{d-1}(P)$ is the number of its facets and $f_d(P) = 1$.

6.6 Example. If P is a d-dimensional simplex in $\mathbb{R}^d$, then $f(P) = \left(\binom{d+1}{k+1}\right)_{k=0}^{d}$.

In a given dimension d, for a given number of vertices n, that is if $f_0(P) = n$, how large can the number of facets $f_{d-1}(P)$ get? How about the total number of faces $\sum_{k=0}^{d} f_k(P)$?

6.7 Example. In dimension 2, if $f_0(P) = n$, then $f(P) = (n, n, 1)$. In dimension 3, by looking at the graph of P, which is planar, Euler's formula gives $f_1(P) \leq 3n - 6$ and $f_2(P) \leq 2n - 4$ (see (A.3)).

We shall now construct examples for which, given d and $f_0(P) = n$, $f_{d-1}(P)$ is of the order $n^{\lfloor d/2 \rfloor}$. On the other hand, the behaviour for random polytopes is quite different: if P is the convex hull of n independently chosen points uniformly in B_2^d, then $f_{d-1}(P) = o(n)$ with high probability (as $n \to \infty$), as shown in [4].

The **moment curve** in $\mathbb{R}^d$ is

$$\gamma = \{(t, t^2, \ldots, t^d),\ t \in \mathbb{R}\}.$$

6.8 Lemma. *Every hyperplane H intersect the moment curve γ in $\mathbb{R}^d$ in at most d points. If there are d points of intersection, then H cannot be tangent to γ, so at each intersection γ passes from one side of H to the other side.*

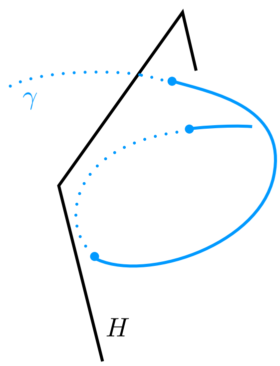

Figure 6.2: The moment curve γ in $\mathbb{R}^d$ intersect a hyperplane in at most d points.

Proof. Let $H = \{x \in \mathbb{R}^d,\ \langle x, a \rangle = b\}$. Then

$$H \cap \gamma = \left\{ (t, \ldots, t^d),\ \sum_{k=1}^{d} a_k t^k = b \right\}.$$

The polynomial $p(t) = \sum_{k=1}^{d} a_k t^k - b$ is of degree d, so it has at most d roots. Consequently, $H \cap \gamma$ has at most d points. If there are d distinct points, p has d distinct roots, they are all simple and $p(t)$ changes sign at each root. This explains the second part of the assertion. $\square$

6.9 Corollary. *Every d points on γ are affinely independent.*

Proof. If some d points on γ are affinely dependent, then there is a hyperlane passing through them plus one more point on γ, which contradict Lemma 6.8. $\square$

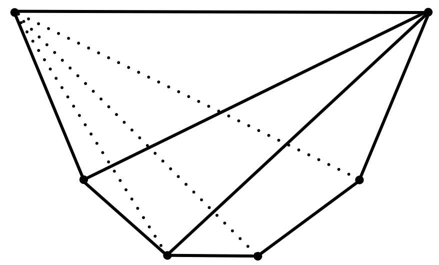

Figure 6.3: A cyclic polytope in $\mathbb{R}^3$ with 6 vertices.

The convex hull of finitely many points on γ is called a **cyclic polytope**. Let us count the number of facets of a cyclic polytope. Each facet is determined by a set of d vertices and by Corollary 6.9, distinct d-sets cannot determine the same face. Which d-sets give rise to facets? There is a convenient criterion due to Gale. For two distinct points u, v on γ, we write $u \prec v$ if v corresponds to a larger parameter t of γ than u.

6.10 Lemma (Gale)**.** *Let V be the vertex set of a cyclic polytope P in $\mathbb{R}^d$. Let $F = \{v_1, \ldots, v_d\} \subset V$ be a d-set of vertices of P labelled such that $v_1 \prec \cdots \prec v_d$. Then F determines a facet of P if and only if for every two vertices u, v not in F, $u, v \in V \setminus F$, the number of vertices v_i from F such that $u \prec v_i \prec v$ is even.*

Proof. Let $H = \operatorname{aff}(F)$. Note that this is a hyperplane by Corollary 6.9. This hyperplane determines a facet if and only if all points $V \setminus F$ lie one the same side of H. We have $H \cap \gamma = F$ and γ is partitioned into $d+1$ consecutive pieces $\gamma_0, \ldots, \gamma_d$. By Lemma 6.8,

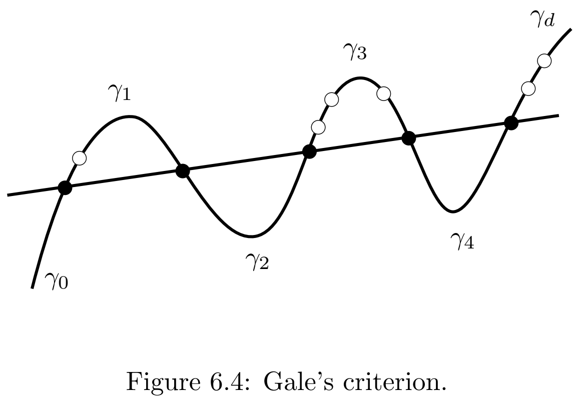

Figure 6.4: Gale's criterion.

each piece is contained completely in either H^- or H^+. Therefore, $V \setminus F$ must be all on $\gamma_1 \cup \gamma_3 \cup \ldots$ or on $\gamma_0 \cup \gamma_2 \cup \ldots$. It remains to observe that these are equivalent to Gale's criterion. □

6.11 Theorem. *The number of facets of a d-dimensional cyclic polytope with n vertices equals*

$$\begin{cases} \binom{n-\lfloor d/2 \rfloor}{\lfloor d/2 \rfloor} + \binom{n-\lfloor d/2 \rfloor - 1}{\lfloor d/2 \rfloor - 1}, & \text{if } d \text{ is even,} \\ 2\binom{n-\lfloor d/2 \rfloor - 1}{\lfloor d/2 \rfloor}, & \text{if } d \text{ is odd.} \end{cases} \tag{6.1}$$

6.12 Remark. For a fixed dimension d, (6.1) is asymptotic to $n^{\lfloor d/2 \rfloor}$ as $n \to \infty$.

Proof of Theorem 6.11. By Gale's criterion (Lemma 6.10), the number of facets is the number of ways of placing d black beads and $n-d$ white beads in a row in such a way that we have an even number of black beads between each two white beads.

Case 1. $d = 2k+1$ is odd. We examine the run of black beads before the first white bead and the run of black beads after the last white bead. Since there is an even number of black beads in between and the total number is odd, exactly one of these two runs is of odd size. Say the first run is of odd size. Then remove the first bead (black); what remains is a sequence of $2k$ black beads and $n-2k-1$ white beads such that every run of black beads is even. There are $\binom{(n-2k-1)+k}{k} = \binom{n-k-1}{k}$ such sequences (to see that take out every other black bead, arrange the remaining k black beads among $(n-2k-1)+k$ possible positions and put back the removed beads). The same happens when the last run is odd and we get the second case of (6.1).

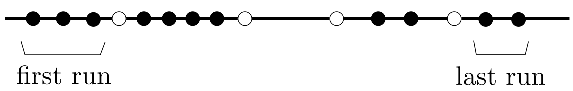

Figure 6.5: Proof of Theorem 6.11, Case 1, d is odd.

Case 2. $d = 2k$ is even. We again examine the run of black beads before the first white bead and the run of black beads after the last white bead. If the first run is of even length, then the last run is also even and we have a sequence of $2k$ black beads and $n-2k$ white beads where each black run is even. There are $\binom{(n-2k)+k}{k} = \binom{n-k}{k}$ such sequences. If the first run is odd, then the last run is also odd. By removing the first and the last bead (which are both black), we get a sequence of $2k-2$ black and $n-2k$ white beads where each black run is even. There are $\binom{(n-2k)+(k-1)}{k-1} = \binom{n-k-1}{k-1}$ such sequences. In total, we get $\binom{n-k}{k} + \binom{n-k-1}{k-1}$ possible sequences, which gives the first case of (6.1). □

6.13 Remark. The so-called upper bound theorem of McMullen asserts that for every polytope P in $\mathbb{R}^d$, we have $f_k(P) \leq f_k(C)$, $k = 0, \ldots, d$, where C is a cyclic polytope in $\mathbb{R}^d$ with the same number of vertices as P, that is $f_0(P) = f_0(C)$. In other words, cyclic polytopes maximise the numbers of facets for a given number of vertices. We refer to [19] or [33] for McMullen's proof and to [16] or [26] for a slick and short proof of a weaker bound of the optimal order.

6.3 Exercises

1. Show that a d-dimensional simplex has $\binom{d+1}{k+1}$ k-dimensional faces, $k = 0, 1, \dots, d$.

2. Show that for the cross-polytope $B_1^d = \operatorname{conv}\{\pm e_i\}_{i=1}^d$, a subset $S \subsetneq \{\pm e_i\}_{i=1}^d$ determines its face $F = \operatorname{conv}(S)$ if and only if there is no i such that both e_i and $-e_i$ are in F.

3. Show that for the cube $B_\infty^d = \operatorname{conv}(\{-1, 1\}^d)$ every face F corresponds to a vector $v \in \{-1, 0, 1\}^d$, $F = \operatorname{conv}\{u \in \{-1, 1\}^d,\ u_i = v_i$ for each i such that $v_i \neq 0\}$ and v is the barycentre of F_v.

4. Show that a permutohedron of order n has $n!$ vertices.

5. If F and G are faces of a polytope P in $\mathbb{R}^d$, then so is $F \cap G$.

6. Let V be the vertex set of a polytope P in $\mathbb{R}^d$. Let U be a subset of V. Show that U is the vertex set of a face of P if and only if $\operatorname{aff}(U) \cap \operatorname{conv}(V \setminus U) = \varnothing$.

7. Show that the graph of a 3-dimensional polytope is (vertex) 3-connected (removing any 2 vertices leaves the graph connected).

8. Consider the curve $\psi = \left\{ \left(\frac{1}{t+1}, \dots, \frac{1}{t+d} \right),\ t > 0 \right\}$ in $\mathbb{R}^d$. Show that ψ intersects every hyperplane in at most d points and if there d points of intersections, the hyperplane is not tangent to ψ.

7 Combinatorial convexity

7.1 Radon's theorem

Radon's theorem says that there are some good partitions of sets having *enough* points.

7.1 Theorem (Radon)**.** *Let A be a subset in $\mathbb{R}^d$ with $|A| \geq d+2$. Then there is a partition, $A = X \cup Y$ ($X \cap Y = \varnothing$), such that* $\operatorname{conv} X \cap \operatorname{conv} Y \neq \varnothing$.

7.2 Remark. The constant $d+2$ is the best possible because for a d-dimensional simplex in $\mathbb{R}^d$, there is no such partition (exercise).

7.3 Remark. When $d = 1$ the theorem is clear as considering three points on a line, there is always one, say x between some two others, say y, z, so it suffices to take $X = \{x\}$ and $Y = A \setminus X \supset \{y, z\}$.

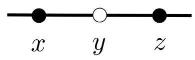

Figure 7.1: Radon's theorem in dimension 1.

7.4 Remark. When $d = 2$, considering 4 points in the plane, there are two possibilities. Either certain three of them are the vertices of a triangle containing the fourth point, or the points are the vertices of a convex quadrilateral. In any case, it is clear what to take for the partition (see Figure 7.2).

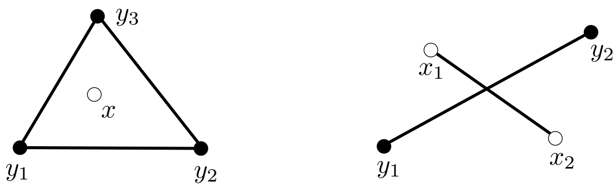

Figure 7.2: Radon's theorem in dimension 2.

Proof of Radon's theorem. Since $|A| \geq d+2$, the set $\{\left[\begin{smallmatrix} a \\ 1 \end{smallmatrix}\right],\ a \in A\}$ of vectors in $\mathbb{R}^{d+1}$ is *not* linearly independent. Therefore there are $a_i \in A$ and nonzero coefficients α_i such that

$$\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right] = \sum \alpha_i \left[\begin{smallmatrix} a_i \\ 1 \end{smallmatrix}\right].$$

Because the sum of the α_i is 0, some of them are positive, some are negative. Let I be the set of all the indices i for which $\alpha_i > 0$ and J for which $\alpha_i < 0$ (neither I nor J is empty). Breaking the sum into two pieces yields

$$\sum_{i \in I} \alpha_i \left[\begin{smallmatrix} a_i \\ 1 \end{smallmatrix}\right] = \sum_{j \in J} (-\alpha_j) \left[\begin{smallmatrix} a_j \\ 1 \end{smallmatrix}\right].$$

Dividing this through $t = \sum_{i\in I} \alpha_i = \sum_{j\in J}(-\alpha_j)$, which is positive, shows that we can take $X = \{a_i,\ i \in I\}$ and $Y = A \setminus X$, for then

$$\operatorname{conv} X \ni \sum_{i\in I} \frac{1}{t}\alpha_i \left[\begin{smallmatrix} a_i \\ 1 \end{smallmatrix}\right] = \sum_{j\in J} \frac{1}{t}(-\alpha_j) \left[\begin{smallmatrix} a_j \\ 1 \end{smallmatrix}\right] \in \operatorname{conv} Y.$$

□

We should remark that there is a generalisation of Radon's theorem, Tverberg's theorem.

7.5 Theorem (Tverberg)**.** *Let $r \geq 2$ be an integer. Let X be a subset of $\mathbb{R}^d$ with $|X| = (r-1)(d+1)+1$. Then there is a partition $X = X_1 \cup \ldots \cup X_r$ (X_i's are pairwise disjoint) such that $\bigcap_{i=1}^r \operatorname{conv} X_i \neq \varnothing$.*

7.6 Remark. Taking here $r = 2$ recovers Radon's theorem.

7.7 Remark. The constant $(r-1)(d+1)+1$ is the best possible. To see this, consider $(r-1)(d+1)$ points in $\mathbb{R}^d$ in general position, meaning that no $d+1$ of them lie in the same hyperplane.

7.2 Helly's theorem

We shall now discuss Helly's theorem. We say that a family of sets in $\mathbb{R}^d$ has **Helly's property** if every $d+1$ of them have a nonempty intersection.

7.8 Theorem (Helly)**.** *Let $K_1, K_2, \ldots, K_n$, $n \geq d+1$, be convex sets in $\mathbb{R}^d$ with Helly's property. Then*

$$\bigcap_{i=1}^{n} K_i \neq \varnothing.$$

In words, having empty intersection has a finite reason.

Proof. By induction on n. Case $n = d+1$ is trivial. Now suppose that $n \geq d+2$ and the theorem holds for smaller n.

For each $j = 1, \ldots, n$ set

$$G_j = \bigcap_{\substack{i=1 \\ i\neq j}}^{n} K_i.$$

By induction, $G_j \neq \varnothing$. Take arbitrary $z_j \in G_j$. By Radon's Theorem, there is a partition of $z_1, \ldots, z_n$ into two sets X, Y such that $\operatorname{conv} X \cap \operatorname{conv} Y \neq \varnothing$. Take z in this intersection. We claim that $z \in G_i$ for all $i = 1, \ldots, n$. Without loss of generality, focus on K_1 and suppose that $z_1 \in X$. Then all $z_j \in Y$ belong to K_1, hence $\operatorname{conv} Y \subset K_1$ (because K_1 is convex), so $z \in K_1$ (because $z \in \operatorname{conv} Y$). □

7.9 Remark. In a special case $d = 1$ (intervals on a line), we can give a different argument. Sketch: Take the rightmost left-endpoint x of these intervals. Then every interval starts to the left of x and ends to the right of x.

Note that Helly's Theorem in general fails for infinite families. For example, take the family of intervals of the form $I_n = [n, \infty)$, or the family of intervals of the form $J_n = (0, 1/n]$. However, if all the sets are compact, the theorem holds.

7.10 Theorem (Helly's Theorem, infinite version)**.** *Let $K_1, K_2, \ldots$ be compact convex sets in $\mathbb{R}^d$ with Helly's property. Then $\bigcap_{i=1}^{\infty} K_i \neq \varnothing$.*

Proof. Fix $n \geq 1$. Then the family of the sets $K_1, \ldots, K_n$ has Helly's property, hence, by usual Helly's Theorem, there exists $z_n \in \bigcap_{i=1}^{n} K_i$. Of course, $z_n \in K_1$ for every n. By compactness, take a convergent subsequence (z_{n_k}) of (z_n) and assume it tends to some z_0. Then for a fixed l, since eventually $z_{n_k} \in K_l$ (for k such that $n_k \geq l$), we have $z_0 \in K_l$ as K_l being compact is closed. Thus, $\bigcap_{i=1}^{\infty} K_i \neq \varnothing$. $\square$

Helly's Theorem has many applications. We present several now and defer many others to exercises.

7.11 Example. Let $\mathcal{F}$ be a finite family of convex sets in $\mathbb{R}^d$, $|\mathcal{F}| = n \geq d+1$ and let $C \subset \mathbb{R}^d$ be convex. Then there exists a translate of C intersecting every set from $\mathcal{F}$ if and only if there exists a translate of C intersecting every $(d+1)$-tuple from $\mathcal{F}$.

Proof. For $K \in \mathcal{F}$, let $\tilde{K} = \{x \in \mathbb{R}^d,\ (x + C) \cap K \neq \varnothing\}$ be the set of translates of C intersecting K. The key observation is that every $\tilde{K}$ is convex and then we are done by Helly's Theorem. We leave the details as an exercise. $\square$

7.12 Example. Let $\mathcal{H}$ be a finite family of closed half-spaces in $\mathbb{R}^d$ and let C be a convex set in $\mathbb{R}^d$ such that $C \subset \bigcup_{H \in \mathcal{H}} H$. Then there exists a subfamily $\mathcal{H}' \subset \mathcal{H}$, $|\mathcal{H}'| = d+1$ such that $C \subset \bigcup_{H \in \mathcal{H}'} H$.

Proof. For every $H \in \mathcal{H}$, set $\tilde{H} = C \setminus H$. Then $\tilde{H}$ is convex and $\bigcap_{H \in \mathcal{H}} \tilde{H} = \varnothing$. By Helly's Theorem, there exists a $(d+1)$-tuple $\tilde{H}_1, \ldots, \tilde{H}_{d+1}$ with empty intersection. Then $C \subset H_1 \cup \cdots \cup H_{d+1}$. $\square$

7.13 Example. (Kirchberger's Theorem) Let R and B be finite sets of points in $\mathbb{R}^d$ (of say red and blue points). Then R and B can be strictly separated by a hyperplane if and only if for every $Y \subset R \cup B$ with $|Y| \leq d+2$, we can separate the sets $Y \cap R$ and $Y \cap B$. (A hyperplane H *strictly separates* sets A and B if A lies in one open half-space determined by H and B lies in the opposite open half-space.)

Proof. With every $r \in R$ we associate a half-space

$$C_r = \left\{ \left[\begin{smallmatrix} a \\ \alpha \end{smallmatrix}\right] \in \mathbb{R}^{d+1},\ \langle \left[\begin{smallmatrix} a \\ \alpha \end{smallmatrix}\right], \left[\begin{smallmatrix} r \\ -1 \end{smallmatrix}\right] \rangle > 0 \right\}.$$

Likewise, with every $b \in B$ we associate

$$D_b = \left\{ \left[\begin{smallmatrix} a \\ \alpha \end{smallmatrix}\right] \in \mathbb{R}^{d+1},\ \left\langle \left[\begin{smallmatrix} a \\ \alpha \end{smallmatrix}\right], \left[\begin{smallmatrix} b \\ -1 \end{smallmatrix}\right] \right\rangle < 0 \right\}.$$

By the assumption, every $d+2$ half-spaces have a point in common, hence by Helly's Theorem all the half-spaces have a point in common. This point determines a strictly separating hyperplane. $\square$

7.3 Centrepoint

Helly's theorem provides an elegant argument justifying existence of a centrepoint, an important and useful notion in computational geometry (especially in divide-and-conquer type algorithms).

Let X be an n-element set in $\mathbb{R}^d$. A point x in $\mathbb{R}^d$ is called a **centrepoint** of X if every closed half-space containing x contains at least $\frac{1}{d+1}n$ points of X. When $d=1$, it is just a median of X. A centrepoint need not belong to X.

7.14 Theorem (Rado)**.** *Every finite set in $\mathbb{R}^d$ has a centrepoint.*

7.15 Remark. Replacing the factor $\frac{1}{d+1}$ in the definition of a centrepoint with $\theta > \frac{1}{d+1}$ would be too restrictive for the existence, for consider the example of a simplex. For $\theta = \frac{1}{d+1}$, there are no efficient algorithms for finding exact cetnrepoints (needless to say that our proof, based on Helly's theorem, cannot be constructive). Efficient algorithms often use θ-centrepoints with an appropriate $\theta < \frac{1}{d+1}$, which are easier to find.

Proof of Theorem 7.14. Let $X \subset \mathbb{R}^d$ have size n. It suffices to find a point x such that x is in every open half-space H such that $|X \cap H| > \frac{d}{d+1}n$. Indeed, take F, a closed half-space containing x. If F contains fewer than $\frac{n}{d+1}$ points from X, then $H = F^c$ is an open half-space containing strictly more than $\frac{d}{d+1}n$ points, but not containing x, contradicting the definition of x.

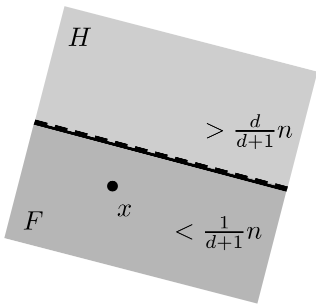

Figure 7.3: An explanation of an equivalent definition of a centrepoint.

To show that such x exists we use Helly's theorem: consider the family

$$\mathcal{C} = \left\{ \mathrm{conv}(X \cap H),\ H \text{ is an open half-space such that } |X \cap H| > \frac{d}{d+1}n \right\}.$$

Clearly this is a finite family (of size at most $2^{|X|}$). The intersection of any $d+1$ sets from $\mathcal{C}$ misses less than $(d+1)\frac{n}{d+1} = n$ points from X, so it is nonempty. By Helly's theorem, there is $x \in \bigcap_{C \in \mathcal{C}} C$. This is a desired point. $\square$

7.16 Remark. Using the infinite version of Helly's theorem, This theorem can be generalised to Borel probability measures (exercise).

We mention in passing the ham-sandwich theorem, a result of similar flavour but without convexity. It is proved using topological arguments.

7.17 Theorem (Banach's ham-sandwich theorem)**.** *Every d finite sets in $\mathbb{R}^d$ can be simultaneously bisected by a hyperplane.*

(A hyperplane H bisects a finite set A if each of the two open half-spaces defined by Hcontains at most $\lfloor \frac{|A|}{2} \rfloor$ points of A.) This theorem also generalises to Borel probability measures.

7.4 Exercises

1. Fill out the details of Remark 7.2.

2. Fill out the details of the proof of Example 7.11.

3. Find an example of 4 convex sets on the plane such that the intersection of every 3 of them contains an interval of length 1, but the intersection of all of them does not contain such an interval.

4. Let $I_1, \dots, I_n$, $n \geq 3$, be vertical intervals in $\mathbb{R}^2$ such that for every 3 of them there is a line intersecting them. Show that then there is a line intersecting all of the intervals.

5. Show that every Borel probability measure has a centrepoint.

   Hint: first show that for a Borel probability measure μ on $\mathbb{R}^d$ and an open half-space H such that $\mu(H) > \alpha$ for some $\alpha \in [0, 1)$, there is a compact convex set C such that $\mu(C) > \alpha$ and then use the infinite version of Helly's theorem.

6. Let $A_1, \dots, A_{n+1}$ be nonempty subsets of the n-element set $\{1, \dots, n\}$. Show that there are nonempty disjoint subsets I, J of $\{1, \dots, n+1\}$ such that

   $$\bigcup_{i \in I} A_i = \bigcup_{j \in J} A_j.$$

   (This can be viewed as a discrete analogue of Radon's theorem, without any convexity.)

   Hint: consider the $n \times n + 1$ matrix whose columns are indicator vectors of the sets A_i (which are nonzero) and use the fact that the columns are linearly dependent. There is a generalisation for partitions to more than two sets due to Lindström (see [15]).

7. (a) Let X be subset of $\mathbb{R}^d$. If every $d+1$ points from X can be covered by a (closed) ball of radius r, then X can be covered by such a ball.

   (b) Every set of $d+1$ points in $\mathbb{R}^d$ of diameter at most 2 can be covered by a closed ball of radius $r \leq \sqrt{\frac{2d}{d+1}}$ (which is sharp for a regular simplex).

   (c) If X is a subset of $\mathbb{R}^d$ with diameter at most 2, then X can be covered by a closed ball of radius at most $\sqrt{\frac{2d}{d+1}}$ (Jung's theorem).

8. (a) A compact convex set in $\mathbb{R}^d$ of width 1 contains a segment of length 1 of every direction.

   (b) Let $\mathcal{C}$ be a finite family of compact convex sets in $\mathbb{R}^d$ such that the intersection of every $d+1$ of them is of width at least 1. Then $\bigcap_{C \in \mathcal{C}} C$ has width at least 1.

8 Arrangements and incidences

8.1 Arrangements

For a finite set $\mathcal{H}$ of hyperplanes in $\mathbb{R}^d$, its **arrangement** is the partition of $\mathbb{R}^d$ into relatively open convex sets called the **faces** of the arrangement of $\mathcal{H}$. Their dimensions are 0 (vertices), through d (cells).

8.1 Remark. The cells are the connected components of $\mathbb{R}^d \setminus \bigcup_{H\in\mathcal{H}} H$. To obtain the facets, we take an element H from $\mathcal{H}$, the induced arrangement in H by $H \cap H'$, $H' \in \mathcal{H} \setminus \{H\}$ and the cells of this arrangement, etc.

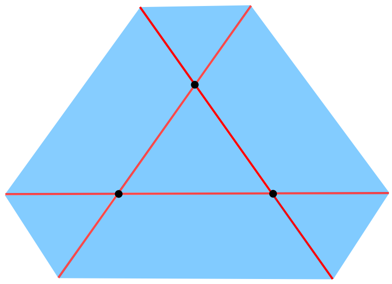

Figure 8.1: An arrangement of 3 lines in $\mathbb{R}^2$ with: 7 cells (blue), 9 rays (red), 3 vertices (black).

8.2 Example. Let $\mathcal{H} = \{\{x \in \mathbb{R}^3,\ x_i = 0\},\ i = 1, 2, 3\}$ be the family of the coordinate hyperplanes in $\mathbb{R}^3$. The arrangement of $\mathcal{H}$ consists of 8 cells: a) $(0,\infty)\times(0,\infty)\times(0,\infty)$, $(-\infty,0)\times(0,\infty)\times(0,\infty)$, etc., b) 12 facets $(0,\infty)\times(0,\infty)\times\{0\}$, etc, c) 6 rays (the oriented axes) $(0,\infty)\times\{0\}\times\{0\}$, etc, and d) 1 vertex, the origin.

The arrangement of $\mathcal{H}$ is called **simple** if the hyperplanes from $\mathcal{H}$ satisfy: the intersection of every k of them is $(d-k)$-dimensional, $k = 2, 3, \ldots, d, d+1$ (with the convention that only the empty set is of dimension -1). If $|\mathcal{H}| \geq d+1$, for the arrangement of $\mathcal{H}$ to be simple, it suffices that every d hyperplanes intersect at a single point and no $d+1$ hyperplanes have a common point. In particular, such an arrangement has $\binom{|\mathcal{H}|}{d}$ vertices.

How complex are simple arrangements? For instance, we can find the number of cells in simple arrangements.

8.3 Theorem. *The number of cells in a simple arrangement of n hyperplanes in $\mathbb{R}^d$ equals*

$$\Phi_d(n) = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{d}. \tag{8.1}$$

Proof. For $d = 1$, we have n points on the line and they divide it into $n+1$ components, so $\Phi_1(n) = n + 1$. Suppose we have a family $\mathcal{H}$ of $n-1$ hyperplanes in $\mathbb{R}^d$ forming a simple arrangement (with $\Phi_d(n-1)$ cells). We insert a new hyperplane H. Then H gets divided by $\mathcal{H}$ into $\Phi_{d-1}(n-1)$ cells. Each cell of this arrangement in H divides one d-cell of $\mathcal{H}$ into two. Therefore,

$$\begin{aligned}\Phi_d(n) &= (\text{number of cells initially in } \mathcal{H}) + (\text{cells gained by inserting } H) \\ &= \Phi_d(n-1) + \Phi_{d-1}(n-1).\end{aligned}$$

This recurrence and the initial condition $\Phi_1(n) = n + 1$ determine the sequences $\Phi_d(n)$ uniquely. It remains to check that the values $\Phi_d(n) = \sum_{k=0}^{d} \binom{n}{k}$ satisfy the recurrence. □

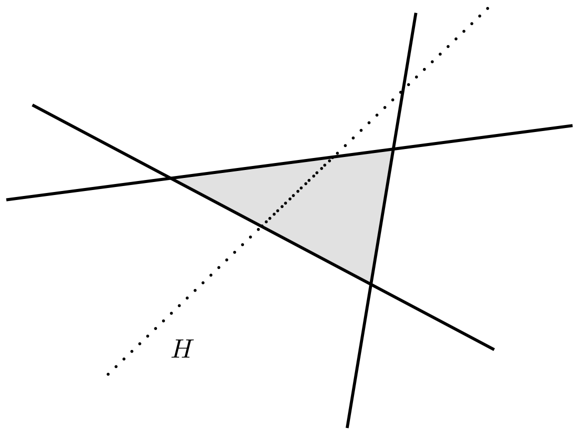

Figure 8.2: A new hyperplane divides existing cells into two.

8.4 Remark. For a fixed d, we have $\Phi_d(n) = O(n^d)$ (as $n \to \infty$).

8.5 Remark. By Remark, 8.1, knowing $\Phi_d(n)$, we obtain the number of k-faces of a simple arrangement of n hyperplanes in $\mathbb{R}^d$. We have,

$$\begin{aligned}\text{Number of } d\text{-faces } &= \Phi_d(n), \\ \text{Number of } d-1\text{-faces } &= n\Phi_{d-1}(n-1), \\ \text{Number of } d-2\text{-faces } &= \binom{n}{2}\Phi_{d-2}(n-2), \\ &\cdots\cdots\cdots \\ \text{Number of } d-k\text{-faces } &= \binom{n}{k}\Phi_{d-k}(n-k), \qquad k = 0, 1, \ldots, d,\end{aligned}$$

with the convention that $\Phi_0(n) = 1$.

Moreover, it follows by a standard perturbation-type argument that the number of k-faces of a non-simple arrangement is upper-bounded by the number of k-faces of a simple arrangement.

8.6 Remark. The number of vertices in a d-cell of an arrangement of n hyperplanes in $\mathbb{R}^d$ is $O(n^{\lfloor d/2 \rfloor})$. Explanation: a cell is the intersection of at most n-hyperplanes, so its dual is a polytope with at most n vertices which has at most $O(n^{\lfloor d/2 \rfloor})$ facets (by the upper bound theorem – see Remark 6.13), so by duality again, the cell has at most this many vertices.

Of course, it is interesting to ask about combinatorial complexity of arrangements of other geometric objects, say segments, or spheres. This has many applications, is important for analysis of geometric algorithms and has extensively been studied. We refer to the survey [1].

8.2 Incidences

Suppose we have a finite set L of lines on the plane and a finite set P of points on the plane. Let $I(P, L)$ be the **number of incidences**, that is pairs (p, l) such that p is a point in P, l is a line in L and p is on l. What is the maximal number of incidences given that there are say m points and n lines? In other words, we define

$$I(m, n) = \max_{P,L:|P|=m,|L|=n} I(P, L)$$

and ask about bounds on $I(m, n)$.

8.7 Example. Considering 3 lines bounding a triangle, we get $I(3, 3) \geq 6$.

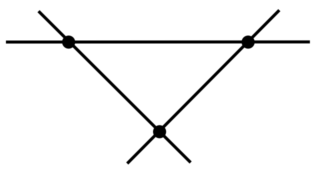

Figure 8.3: This configuration shows that $I(3, 3) \geq 6$.

There is of course a trivial upper bound $I(m, n) \leq mn$, but it is never attained unless $m = 1$ or $n = 1$. An optimal bound is provided by the Szemerédi-Trotter theorem.

8.8 Theorem (Szemerédi-Trotter, [30])**.** *For the maximal number $I(m, n)$ of incidences for sets of m points and n lines on the plane, we have*

$$I(m, n) = O(m^{2/3}n^{2/3} + m + n).$$

8.9 Remark. Our proof will give $I(m,n) \le 4(m^{2/3}n^{2/3} + m + n)$. Currently the best bound is $2.5m^{2/3}n^{2/3} + m + n$ and it is known that the factor 2.5 cannot be replaced with 0.42 (see [23], [25]).

There are several other related problems of similar flavour:

1) What is the maximal number of unit distances in the plane,

$$U(n) = \max_{S \subset \mathbb{R}^2, |S|=n} |\{(x,y) \in S \times S,\ |x-y| = 1\}| \ ?$$

2) What is the minimal number of distinct distances in the plane,

$$g(n) = \min_{S \subset \mathbb{R}^2, |S|=n} |\{(|x-y|,\ x,y \in S\}| \ ?$$

3) What is the maximal total number $K(m,n)$ of vertices of m distinct cells in an arrangement of n lines in the plane?

8.10 Example. There is an example of sets of n points and n lines with at least $\Omega(n^{4/3})$ incidences, hence $I(n,n) = \Omega(n^{4/3})$. Consequently, the Szemerédi-Trotter theorem is asymptotically tight. Let $n = 4k^3 = k \cdot 4k^2 = 2k \cdot 2k^2$ and consider the sets of grid points

$$P = \{(i,j),\ i = 0,1,\ldots,k-1,\ j = 0,1,\ldots,4k^2-1\}$$

and lines

$$L = \{y = ax + b,\ a = 0,1,\ldots,2k-1,\ b = 0,1,\ldots,2k^2-1\}.$$

For $x \in [0,k)$ and every line in L, we have $ax + b < ak + b < 2k^2 + 2k^2 = 4k^2$. As a result, for each $i = 0,1,\ldots,k-1$, each line of L contains a point $(i,j) \in P$ and thus $I(n,n) = I(P,L) \ge k \cdot |L| = kn = \left(\frac{n}{4}\right)^{1/3} \cdot n = \Omega(n^{4/3})$.

We shall present Szekely's proof (see [29]) of the Szemerédi-Trotter theorem, who ingeniously employed the notion of the crossing number of a graph (introduced by Ajtai et al and independently by Leighton – see [2] and [14]).

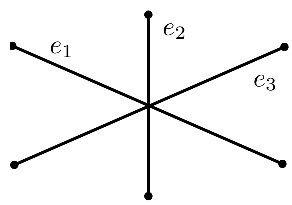

Figure 8.4: There are 3 crossings for this drawing: $\{e_1, e_2\}, \{e_1, e_3\}, \{e_2, e_3\}$.

8.3 The crossing number of a graph

A **drawing** of a graph $G = (V, E)$ is a set of points on the plane corresponding to the vertices of G and arcs between these points (continuous curves without self-intersections) corresponding to the edges of G. The **crossing number** of a drawing of the graph is the number of unordered pairs of different arcs which intersect (nontrivially – outside the vertices). The **crossing number** $\mathrm{cr}(G)$ of the graph G is the minimal crossing number over all of its drawings. Of course, we always have $\mathrm{cr}(G) \leq \binom{|E|}{2}$.

8.11 Example. The crossing number $\mathrm{cr}(G)$ is 0 if and only if G is planar. The clique K_5 is not planar, so $\mathrm{cr}(K_5) \geq 1$. In fact, $\mathrm{cr}(K_5) = 1$. Similarly for the complete bipartite graph $K_{3,3}$, $\mathrm{cr}(K_{3,3}) = 1$. (see Figures 8.5 and 8.6).

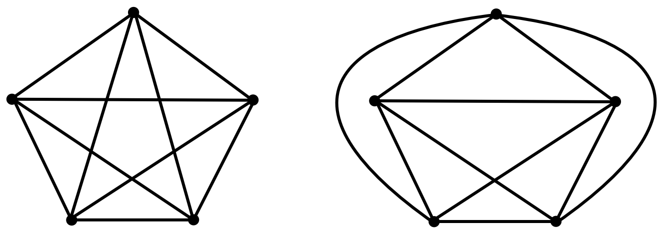

Figure 8.5: Two drawings of K_5: the left one has many crossings, the right one is optimal and it has only 1 crossing.

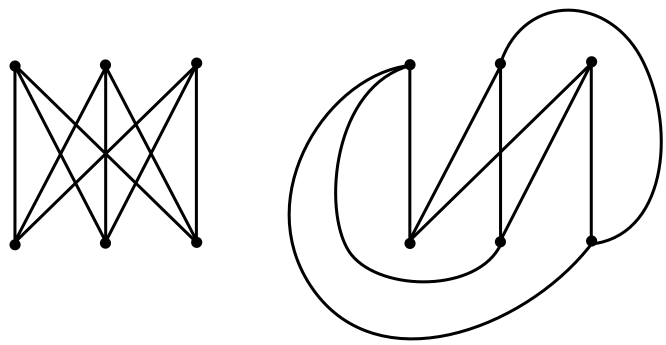

Figure 8.6: Two drawings of $K_{3,3}$: the left one has many crossings, the right one is optimal and it has only 1 crossing.

Recall a known fact about planar graphs: if a graph $G = (V, E)$ has n vertices and is planar, then G has at most $3n-6$ edges. In other words, if $|E| \geq 3n-5$, then $\mathrm{cr}(G) > 0$. In general, if G has relatively many edges with respect to the number of vertices, its crossing number should be large. This intuition can be quantified which is done in the following theorem (discovered independently by Ajtai et al and Leighton).

8.12 Theorem (The crossing number inequality)**.** *For a simple graph $G = (V, E)$, we have*

$$\mathrm{cr}(G) \geq \frac{1}{64}\frac{|E|^3}{|V|^2} - |V|. \tag{8.2}$$

8.13 Remark. This bound is tight. For instance, let G be the complete graph. Then $|E| = \binom{|V|}{2}$, so $\mathrm{cr}(G) \leq |E|^2 = \frac{|E|^3}{|E|} = O\left(\frac{|E|^3}{|V|^2}\right)$.

We shall present a proof based on the so-called *amplification trick*. First we amplify the statement "if G is planar, then $|E| \leq 3|V|$" to arbitrary graphs by exploiting the freedom to delete edges. This gives a (suboptimal) lower bound on the crossing number stated in the following lemma.

8.14 Lemma. *For a simple graph $G = (V, E)$, we have*

$$\mathrm{cr}(G) \geq |E| - 3|V|.$$

Proof. Take a drawing of G which has $\mathrm{cr}(G)$ crossings. By deleting at most $\mathrm{cr}(G)$ arcs, we remove all the crossings, so we obtain a planar graph with at least $|E| - \mathrm{cr}(G)$ edges, thus $|E| - \mathrm{cr}(G) \leq 3|V|$. □

Now we amplify this lower bound by exploiting the freedom to remove vertices. For the edges, it was clear which edges should be removed to make G planar. For vertices the situation is more complex and we make a random choice.

Proof of Theorem 8.12. Consider a drawing of G with n vertices, m edges and x crossings. Without loss of generality, we can assume that $m > 4n$ (otherwise the right hand side of (8.2) is nonpositive. Let $p \in (0, 1)$ and let $V' \subset V$ be selected at random including each element $v \in V$ into V' independently with probability p. Let n', m', x' be the inherited parameters for the induced (random) graph and its induced drawing. We compute the expectations

$$\begin{aligned}
\mathbb{E}n' &= np, \\
\mathbb{E}m' &= \mathbb{E}\left(\sum_{e \in E} \mathbf{1}_e\right) = \sum_{e \in E} \mathbb{E}\mathbf{1}_e = mp^2, \\
\mathbb{E}x' &= \mathbb{E}\left(\sum_{\text{crossings } c=\{e,e'\}} \mathbf{1}_c\right) = \sum_{\text{crossings } c=\{e,e'\}} \mathbb{E}\mathbf{1}_c = xp^4,
\end{aligned}$$

where $\mathbf{1}_e$ and $\mathbf{1}_c$ are the indicator random variables of an edge e and a crossing c respectively. By Lemma 8.14, we have $x' \geq m' - 3n'$, thus $\mathbb{E}x' \geq \mathbb{E}m' - 3\mathbb{E}n'$ which becomes $xp^4 \geq mp^2 - 3np$, or $x \geq \frac{m}{p^2} - \frac{3n}{p^3}$. We choose positive $p < 1$ to maximise the right hand side (which is increasing for $p \in (0, \frac{9n}{2m})$ decreasing for $p > \frac{9n}{2m}$). For simplicity, we take $p = \frac{4n}{m}$ (which we know is less than 1) and obtain $x \geq \frac{m^3}{4^2 n^2} - \frac{3m^3}{4^3 n^2} = \frac{1}{64}\frac{m^3}{n^2}$. This finishes the proof. □

8.15 Remark. Carrying out the optimisation over p carefully, we can slightly improve the constant $\frac{1}{64}$ and get $\mathrm{cr}(G) \geq \frac{4}{243}\frac{|E|^3}{|V|^2} - \frac{3}{2}|V|$.

8.4 Proof of the Szemerédi-Trotter theorem

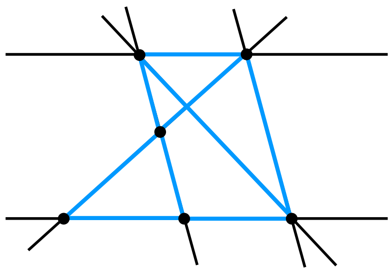

Figure 8.7: A construction of a graph (blue edges) based on sets of points P and lines L (in black).

Proof of Theorem 8.8. Let P and L, $|P| = m$, $|L| = n$ be the sets of points and lines realising the maximal number of incidences, $I(P, L) = I(m, n)$. Define a graph $G = (V, E)$ as follows: $V = P$ and

$$\begin{aligned} E = \{\{p, q\},\ p, q, \in P,\ \ p, q \in \ell \text{ for some } \ell \in L \\ \text{and the segment } [p, q] \text{ does not contain any other point } r \in P\}. \end{aligned}$$

Note that

$$|E| = \sum_{\ell \in L}(\text{number of points on } \ell - 1) = I(P, L) - |L|.$$

By looking at the drawing of G given by P and L, we get a trivial upper bound

$$\mathrm{cr}(G) \leq \binom{|L|}{2}$$

(a possible crossing arises only if two lines from L cross). On the other hand, by the crossing number inequality (8.2),

$$\mathrm{cr}(G) \geq \frac{1}{64}\frac{|E|^3}{|P|^2} - |P|.$$

Combining yields

$$\frac{1}{64}\frac{(I(m, n) - n)^3}{m^2} - m \leq \binom{n}{2},$$

hence

$$\begin{aligned} I(m, n) \leq n + 4m^{2/3}\left[\binom{n}{2} + m\right]^{1/3} &\leq n + 4m^{2/3}\left[\binom{n}{2}^{1/3} + m^{1/3}\right] \\ &\leq n + 4m^{2/3}(n^{2/3} + m^{1/3}) \\ &\leq 4(m^{2/3}n^{2/3} + m + n). \end{aligned}$$

□

8.5 Application in additive combinatorics

We show an application of the Szemerédi-Trotter theorem to sum-product estimates from additive combinatorics. Suppose A is a finite subset of $\mathbb{R} \setminus \{0\}$. We form the **sum set**

$$A + A = \{a + b,\ a, b, \in A\}$$

and the **product set**

$$A \cdot A = \{ab,\ a, b \in A\}.$$

Typically, if A is not appropriately *structured* to cause cancellations in sums or products, the size of $A + A$ as well as $A \cdot A$ will be roughly $|A|^2$. However, they can be significantly smaller: if the elements of A form an arithmetic progression, then $|A + A|$ will be comparable to just $|A|$, or similarly, if the elements of A form a geometric progression, then $|A \cdot A|$ will be small. However, A cannot form an arithmetic and geometric progression simultaneously (unless it is very short). Therefore, it is reasonable to believe that $|A + A|$ or $|A \cdot A|$ is always much larger than $|A|$. Erdös and Szemerédi quantified such a conjecture and asked in [11] whether for every positive δ there is n_0 such that

$$\max\{|A + A|, |A \cdot A|\} \geq |A|^{2-\delta},$$

provided that $|A| \geq n_0$. This conjecture remains open. In that paper, Erdös and Szemerédi showed that there are positive constants ε, c and C such that for every $A \subset \mathbb{R} \setminus \{0\}$, we have

$$c|A|^{1+\varepsilon} \leq \max\{|A + A|, |A \cdot A|\} \leq C|A|^2 \exp\left\{-\frac{c \log |A|}{\log \log |A|}\right\}.$$

The (inexplicit) constant ε in the exponent in the lower bound has been improved, first to $\varepsilon = \frac{1}{31}$ by Nathanson in [22], then to $\varepsilon = \frac{1}{4}$ by Elekes in [10], who used the Szemerédi-Trotter theorem in a simple and powerful way. Currently the best lower bound is of the order $\frac{|A|^{1+\frac{3}{11}}}{\log^{\frac{3}{11}} |A|}$, due to Solymosi from [27], who also used the Szemerédi-Trotter theorem (in a sophisticated way). We present Elekes' slick argument.

8.16 Theorem ([10]). *For every subset A of $\mathbb{R} \setminus \{0\}$ with $|A| \geq 8$, we have*

$$\max\{|A + A|, |A \cdot A|\} \geq \frac{1}{8}|A|^{5/4}.$$

Proof. Define the set of points in the plane

$$P = (A \cdot A) \times (A + A)$$

and the set of lines

$$L = \{y = \frac{1}{a_1}x + a_2,\ a_1, a_2 \in A\}.$$

We have $|P| = |A \cdot A||A + A|$ and $|L| = |A|^2$. Note that every line from L contains at least $|A|$ points from P, namely for a line $\ell : y = \frac{1}{a_1}x + a_2$, we have that a point

$(a_1 a, a + a_2)$ is on ℓ for every $a \in A$. Thus the number of incidences $I(P, L)$ is at least $|L||A| = |A|^3$. On the other hand, by the Szemerédi-Trotter theorem (Theorem 8.8), we have $I(P, L) \leq 4(|P|^{2/3}|L|^{2/3} + |P| + |L|)$. Let $m = \max\{|A + A|, |A \cdot A|\}$. Since $|P| = |A \cdot A||A + A| \leq m^2$ and crudely $m \leq |A|^2$, we also have $|P| \leq m^{4/3}|A|^{4/3}$. Thus,

$$|A|^3 \leq I(P, L) \leq 4(m^{4/3}|A|^{4/3} + m^{4/3}|A|^{4/3} + |A|^2) = 8m^{4/3}|A|^{4/3} + 4|A|^2,$$

so

$$m^{4/3} \geq \frac{|A|^3 - 4|A|^2}{8|A|^{4/3}} = |A|^{2/3}\frac{|A| - 4}{8} \geq |A|^{2/3}\frac{|A|}{16} = \frac{|A|^{5/3}}{16},$$

which gives $m \geq \frac{|A|^{5/4}}{16^{3/4}} = \frac{|A|^{5/4}}{8}$. □

8.6 Exercises

1. Justify the claim made in Remark 8.5 that the number of k-faces in a simple arrangement of n hyperplanes in $\mathbb{R}^d$ is $\binom{n}{k}\Phi_{d-k}(n-k)$.

2. Show that the maximal number of incidences for m points and n lines such that every point is on at least 2 lines is at most a half the maximal number of vertices of m distinct cells in an arrangement of n lines in the plane.

3. Show that the number of incidences of n lines in $\mathbb{R}^3$ and m points in $\mathbb{R}^3$ is at most $O(m^{2/3}n^{2/3}+m+n)$.

4. Show that the number of lines in $\mathbb{R}^2$ such that each contains at least k points from a given set of m points in $\mathbb{R}^2$ is $O(m^2/k^3+m/k)$.

5. Show that n points on the plane determine at most

   (a) $O(n^{7/3})$ triangles with a given angle α,

   (b) $O(n^{7/3})$ triangles with area 1,

   (c) $O(n^{7/3})$ isosceles triangles.

   (These results are from [24].)

9 Volume

We shall denote Lebesgue measure (volume) on $\mathbb{R}^d$ by $\mathrm{vol}_d(\cdot)$ or, for brevity, $|\cdot|$ (whenever it is not ambiguous). If a set A in $\mathbb{R}^d$ is lower dimensional, say it is contained in a k-dimensional affine subspace, by $\mathrm{vol}_k(A)$, or $|A|$, we mean its Lebesgue measure on that subspace. We write $\mathrm{vol}_{d-1}(\partial A)$ or simply $|\partial A|$ to denote the surface measure of the boundary of A. Recall two crucial properties of volume: it is translation invariant, $|A + x| = |A|$ for any $x \in \mathbb{R}^d$, it is d-homogeneous, that is $|\lambda A| = \lambda|A|$ for any $\lambda \geq 0$, and more generally, if $f\colon \mathbb{R}^d \to \mathbb{R}^d$ is a linear map, then $|f(A)| = |\det(f)| \cdot |A|$.

9.1 The Brunn-Minkowski inequality

We begin with a simple example. Let K be a convex compact set in $\mathbb{R}^2$ and let u be a unit vector in $\mathbb{R}^2$. Consider the function

$$f_u(t) = \mathrm{vol}_1\left(K \cap (tu + u^\perp)\right), \qquad t \in \mathbb{R},$$

which gives the length of the section of K by a line perpendicular to u passing through tu. Since K is convex, the function f_u is concave on its support (the support of a

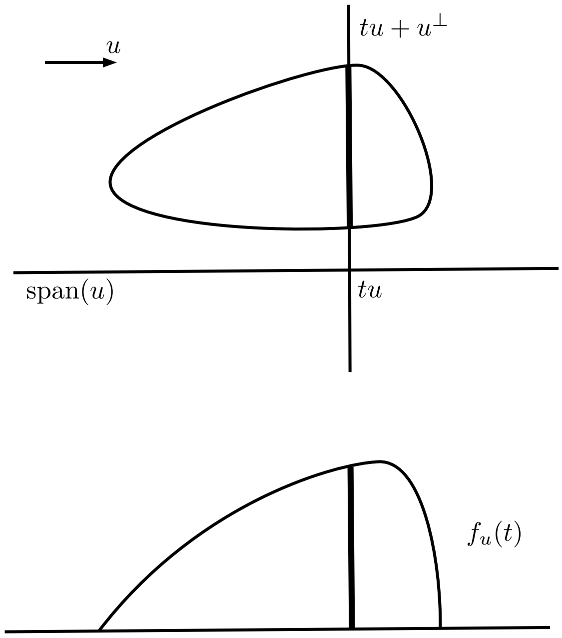

Figure 9.1: Brunn's principle.

function f is the closure of the set $\{x, f(x) \neq 0\}$ where it is nonzero). Note that the support of f_u is the projection of K onto the line spanned by u. Suppose $u = e_1$. The concavity of f_u can be seen for instance by viewing K as the region between two functions: $K = \{(x, y),\ x \in [a, b], f_1(x) \leq y \leq f_2(x)\}$. Since K is convex, f_1 is convex and f_2 is concave. Then $f_u(x) = f_2(x) - f_1(x)$, $x \in [a,b]$ is concave as the sum of two concave functions. This can be generalised to higher dimensions and is usually referred to as Brunn's principle. Before stating it, let us consider one more example.

9.1 Example. Consider the cone $K = \mathrm{conv}(\{(0,0,0)\} \cup \{0\} \times B_2^d)$ in $\mathbb{R}^3$. Let $u = e_1$ and consider the function of the volume of sections of K by hyperplanes perpendicular to u defined as

$$f_u(t) = \mathrm{vol}_2\Big(K \cap (tu + u^\perp)\Big), \qquad t \in \mathbb{R}.$$

Plainly, $f_u(t) = \pi t^2 \mathbf{1}_{[0,1]}(t)$. This is not a concave function on its support, but $f_u^{1/2}$ is concave, even linear, on $[0,1]$.

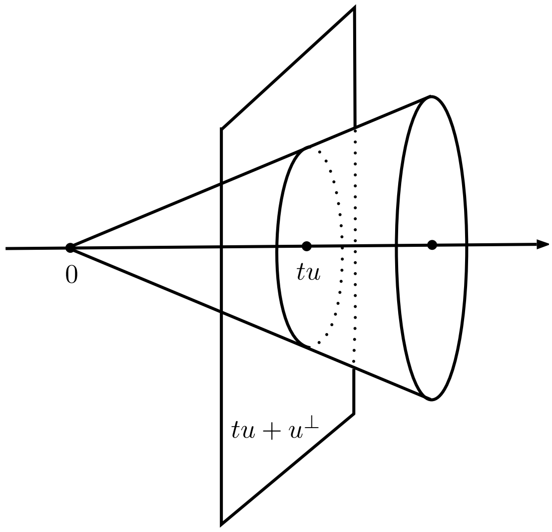

Figure 9.2: Brunn's principle for a cone.

9.2 Theorem (Brunn's principle)**.** *Let K be a compact convex set in $\mathbb{R}^d$. Let u be a unit vector in $\mathbb{R}^d$. Then the function*

$$f_u(t) = \mathrm{vol}_{d-1}\Big(K \cap (tu + u^\perp)\Big), \qquad t \in \mathbb{R},$$

satisfies: $f_u^{\frac{1}{d-1}}$ is concave on its support.

Historically, this was shown by Brunn for $d = 2$ and $d = 3$ and later by Minkowski for any d. He established a more general result, traditionally called the Brunn-Minkowski inequality.

9.3 Theorem (Brunn-Minkowski inequality)**.** *Let A and B be compact nonempty sets in $\mathbb{R}^d$. Then*

$$|A+B|^{\frac{1}{d}} \geq |A|^{\frac{1}{d}} + |B|^{\frac{1}{d}}. \tag{9.1}$$

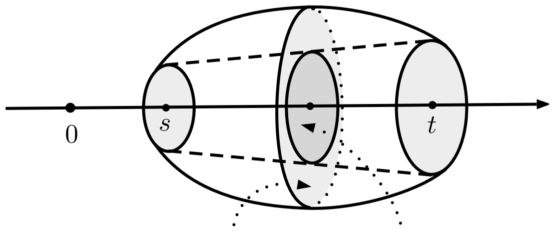

Figure 9.3: Proof of Brunn's principle: by convexity $K_{\lambda s+(1-\lambda)t} \supset \lambda K_s + (1-\lambda)K_t$.

Note that we do not assume the convexity of A or B. First we show how to deduce Brunn's principle from the Brunn-Minkowski inequality and then prove the latter.

Proof of Theorem 9.2 using Theorem 9.3. Define $K_t = \{x \in u^\perp,\ x+t\theta \in K\}$, $t \in \mathbb{R}$ and let $f(t)$ be the $d-1$-dimensional volume (on $u^\perp$) of K_t. Note that K_t is the translation of $K \cap (tu + u^\perp)$ so that it is contained in the subspace $u^\perp$, hence $f(t) = f_u(t)$. Take $\lambda \in [0,1]$, s, t in the support of f and set $A = \lambda K_s$ and $B = (1-\lambda)K_t$. By convexity, $K_{\lambda s+(1-\lambda)t}$ contains $\lambda K_s + (1-\lambda)K_t = A + B$, thus, using (9.1),

$$\begin{aligned} f(\lambda s + (1-\lambda)t)^{\frac{1}{d-1}} &\geq |A+B|^{\frac{1}{d-1}} \geq |A|^{\frac{1}{d-1}} + |B|^{\frac{1}{d-1}} = \lambda|K_s|^{\frac{1}{d-1}} + (1-\lambda)|K_t|^{\frac{1}{d-1}} \\ &= \lambda f(s)^{\frac{1}{d-1}} + (1-\lambda)f(t)^{\frac{1}{d-1}}, \end{aligned}$$

which shows that f is $\frac{1}{d-1}$-concave on its support. □

Proof of Theorem 9.3. By a box in $\mathbb{R}^d$, we mean a set of the form $[a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_d, b_d]$ (a closed axis-parallel parallelopiped). Note that the volume of A is $|A| = \prod_{i=1}^d (b_i - a_i)$, where $\alpha_i = b_i - a_i$ are the side lengths of A. First we claim that (9.1) holds for boxes.

Claim 1. If A and B are boxes, then (9.1) holds.

Proof. Suppose A has side lengths $\alpha_1, \ldots, \alpha_d > 0$ and B has side lengths $\beta_1, \ldots, \beta_d > 0$. Then $A + B$ is a box with side lengths $\alpha_1 + \beta_1, \ldots, \alpha_d + \beta_d$ and (9.1) becomes

$$\left(\prod_{i=1}^d (\alpha_i + \beta_i)\right)^{1/d} \geq \left(\prod_{i=1}^d \alpha_i\right)^{1/d} + \left(\prod_{i=1}^d \beta_i\right)^{1/d}.$$

Showing this is left as an exercise. □

We call a set in $\mathbb{R}^d$ a brick set if it is a union of finitely many boxes with disjoint interiors. Now we claim that it is enough to establish (9.1) for brick sets.

Claim 2. If (9.1) holds for all sets A, B which are nonempty brick sets in $\mathbb{R}^d$, then it holds for all nonempty compact sets A, B in $\mathbb{R}^d$.

Proof. We use the following fact from measure theory: if measurable sets $X_1, X_2, \dots$ of $\mathbb{R}^d$ satisfy $X_1 \supset X_2 \supset \dots$, then $\lim_{n\to\infty} |X_n| = |\bigcap_{i=1}^{\infty} X_i|$. Suppose A, B are nonempty compact sets in $\mathbb{R}^d$. Fix an integer $n \ge 1$ and consider the tiling of $\mathbb{R}^d$ by the translates of the cube $[0, 2^{-n}]^d$ by the lattice $2^{-n}\mathbb{Z}^d$. Let A_n be the union of all such cubes which intersect A. Then $A_1 \supset A_2 \supset \dots$ and $\bigcap_{i=1}^{\infty} A_i = A$ (exercise). Thus $\lim_{n\to\infty} |A_n| = |\bigcap_{i=1}^{\infty} A_i| = |A|$. Similarly for B_n and B. By compactness, $A + B \supset \bigcap_{n=1}^{\infty} A_n + B_n$ (if $x = a_n + b_n$ for every n and some $a_n \in A_n$, $b_n \in B_n$, then by passing to convergent subsequences, we have $a_n \to a$ and $b_n \to b$ for some $a \in A$, $b \in B$, thus $x = a+b \in A+B$). By this inclusion, for every n, we get $|A+B| \ge |A_n+B_n|$. Since A_n and B_n are nonempty brick sets, (9.1) yields $|A_n + B_n|^{1/d} \ge |A_n|^{1/d} + |B_n|^{1/d}$, thus

$$|A + B|^{1/d} \ge |A_n|^{1/d} + |B_n|^{1/d}.$$

Letting $n \to \infty$, we have $|A_n| \to |A|$, $|B_n| \to |B|$ and we get (9.1) for A and B, as desired. $\square$

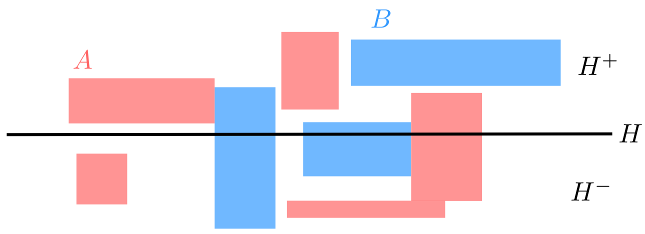

Figure 9.4: Proof of the Brunn-Minkowski inequality for brick sets.

It remains to show (9.1) if A and B are brick sets. Suppose A and B contain in total k boxes. We argue by induction on k than (9.1) holds. The base case $k = 2$ has been done in Claim 1. Suppose $k > 2$ and (9.1) holds for all brick sets A, B having in total fewer than k boxes. Suppose A, B are brick sets having in total k boxes, with say A having at least 2 boxes. There is a coordinate hyperplane H (that is a hyperplane of the form $\{x \in \mathbb{R}^d,\ x_i = t\}$ for some $i \le d$ and $t \in \mathbb{R}$) such that one box of A is fully in H^- and one in H^+ (why?). Consider $A^- = A \cap H^-$ and $A^+ = A \cap H^+$. These are also brick sets. By the choice of H, each of them has at least one brick less than A. Let $\lambda = \frac{|A^-|}{|A|}$. Since the interiors of A^- and A^+ are disjoint, $|A| = |A^-| + |A^+|$ and, consequently, $1 - \lambda = \frac{|A^+|}{|A|}$. We also define $B^- = B \cap H^-$ and $B^+ = B \cap H^+$. Translating B if necessary, we can assume that $\lambda = \frac{|B^-|}{|B|}$ and then $1 - \lambda = \frac{|B^+|}{|B|}$. Note that

$$A + B \supset (A^- + B^-) \cup (A^+ + B^+)$$

(simply because $A \supset A^-$, $B \supset B^-$, so $A + B \supset A^- + B^-$ and similarly for $A^+ + B^+$). Moreover, $A^- + B^- \subset H^-$ and $A^+ + B^+ \subset H^+$, so $A^- + B^-$ and $A^+ + B^+$ have disjoint interiors. Thus,

$$|A+B| \geq |A^- + B^-| + |A^+ + B^+|.$$

By induction, (9.1) holds for A^-, B^- (they have altogether fewer than k boxes), as well as for A^+, B^+. Thus,

$$\begin{aligned}
|A+B| &\geq |A^- + B^-| + |A^+ + B^+| \\
&\geq \Big(|A^-|^{1/d} + |B^-|^{1/d}\Big)^d + \Big(|A^+|^{1/d} + |B^+|^{1/d}\Big)^d \\
&= \Big(\lambda|A|^{1/d} + \lambda|B|^{1/d}\Big)^d + \Big((1-\lambda)|A|^{1/d} + (1-\lambda)|B|^{1/d}\Big)^d \\
&= \lambda\Big(|A|^{1/d} + |B|^{1/d}\Big)^d + (1-\lambda)\Big(|A|^{1/d} + |B|^{1/d}\Big)^d = \Big(|A|^{1/d} + |B|^{1/d}\Big)^d.
\end{aligned}$$

□

9.2 Isoperimetric and isodiametric inequality

The isoperimetric problem asks to determine which sets of fixed perimeter have largest volume (*isos* from Greek means equal and *perimetros* means perimeter). The isodiametric problem asks to determine which sets of fixed diameter have largest volume. The Brunn-Minkowski inequality gives a way to solve both of these questions.

For a compact set A in $\mathbb{R}^d$, we define its *perimeter* (the surface area of its boundary), denoted $|\partial A|$, as follows

$$|\partial A| = \liminf_{\varepsilon \to 0+} \frac{|A + \varepsilon B_2^d| - |A|}{\varepsilon}.$$

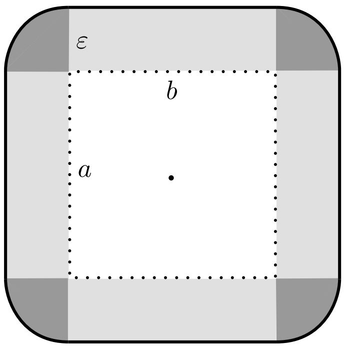

Figure 9.5: The perimeter of a rectangle of side lengths a and b of course equals $\liminf_{\varepsilon\to 0+} \frac{2a\varepsilon + 2b\varepsilon + 4\frac{1}{4}\pi\varepsilon^2}{\varepsilon} = 2a + 2b$.

In particular, for the unit Euclidean ball,

$$\begin{aligned}
|\partial B_2^d| = \liminf_{\varepsilon \to 0+} \frac{|B_2^d + \varepsilon B_2^d| - |B_2^d|}{\varepsilon} &= \liminf_{\varepsilon \to 0+} \frac{|(1+\varepsilon)B_2^d| - |B_2^d|}{\varepsilon} \\
&= \liminf_{\varepsilon \to 0+} \frac{(1+\varepsilon)^d - 1}{\varepsilon}|B_2^d| = d|B_2^d|.
\end{aligned}$$

It is left as an exercise to compute the volume of B_2^d.

9.4 Theorem (The isoperimetric inequality)**.** *Let A be a compact set in $\mathbb{R}^d$. Let B be a Euclidean ball in $\mathbb{R}^d$ with the same volume as A, $|B| = |A|$. Then for every nonnegative t, we have*

$$|A + tB_2^d| \geq |B + tB_2^d|. \tag{9.2}$$

In particular,

$$|\partial A| \geq |\partial B|. \tag{9.3}$$

Proof. By translation invariance, we can assume that B is centred, say $B = rB_2^d$. By (9.1),

$$\begin{aligned}|A + tB_2^d|^{1/d} \geq |A|^{1/d} + t|B_2^d|^{1/d} = |B|^{1/d} + +t|B_2^d|^{1/d}& \\ = (r+t)|B_2^d|^{1/d} = |(r+t)B_2^d|^{1/d} = |B + tB_2^d|^{1/d}&.\end{aligned}$$

□

Inequality (9.3) says that among all compact sets with the same volume as B, the ball B has the smallest surface area. In other words, the solution to the isoperimetric problem is a Euclidean ball (exercise).

9.5 Theorem (The isodiametric inequality)**.** *Let A be a compact set in $\mathbb{R}^d$. Let B be a Euclidean ball such that* $\operatorname{diam}(A) = \operatorname{diam}(B)$. *Then*

$$|\operatorname{conv}(A)| \leq |B|. \tag{9.4}$$

Proof. Since $\operatorname{diam}(\operatorname{conv}(A)) = \operatorname{diam}(A)$, we can assume that A is convex. By the Brunn-Minkowski inequality,

$$\left|\frac{A-A}{2}\right| \geq \left(\left|\frac{A}{2}\right|^{1/d} + \left|\frac{-A}{2}\right|^{1/d}\right)^d = |A|.$$

Note also that

$$\operatorname{diam}\left(\frac{A-A}{2}\right) = \operatorname{diam}(A).$$

Indeed, one inequality follows by the triangle inequality, that is if $a, a' \in \frac{A-A}{2}$, then $a = \frac{a_1 - a_2}{2}$ and $a' = \frac{a_1' - a_2'}{2}$ for some $a_1, a_2, a_1', a_2' \in A$, so

$$\begin{aligned}|a - a'| = \left|\frac{a_1 - a_2}{2} - \frac{a_1' - a_2'}{2}\right| = \left|\frac{a_1 - a_1' + a_2' - a_2}{2}\right| &\leq \frac{|a_1 - a_1'| + |a_2' - a_2|}{2} \\ &\leq \operatorname{diam}(A),\end{aligned}$$

which gives $\operatorname{diam}\left(\frac{A-A}{2}\right) \leq \operatorname{diam}(A)$. The opposite inequality holds because if we have that $\operatorname{diam}(A) = |a_1 - a_2|$ for some $a_1, a_2 \in A$, then $a = \frac{a_1 - a_2}{2}$ and $a' = -a$ are in $\frac{A-A}{2}$ and $|a - a'| = |a_1 - a_2|$. Since $\frac{A-A}{2}$ is symmetric, we have

$$\frac{A-A}{2} \subset \frac{\operatorname{diam}(A)}{2} B_2^d.$$

Thus,

$$|A| \le \left|\frac{A-A}{2}\right| \le \left|\frac{\operatorname{diam}(A)}{2} B_2^d\right|$$

and $\frac{\operatorname{diam}(A)}{2} B_2^d$ is a Euclidean ball with the same diameter as A. $\square$

Theorem 9.5 says that the solution to the isodiametric problem is a Euclidean ball.

We will need later a generalisation of Theorem 9.5 to arbitrary norms. Note that in the proof, we only used the triangle inequality, thus repeating the whole argument verbatim, we obtain the following result.

9.6 Theorem (The isodiametric inequality for arbitrary norms)**.** *Let* $\|\cdot\|$ *be a norm on* $\mathbb{R}^d$*. Let* A *be a compact set in* $\mathbb{R}^d$*. Let* B *be a ball with respect to* $\|\cdot\|$ *such that* $\operatorname{diam}_{\|\cdot\|}(A) = \operatorname{diam}_{\|\cdot\|}(B)$*, say* $B = \{x \in \mathbb{R}^d,\ \|x\| \le r\}$ *with* $r = \frac{1}{2}\operatorname{diam}_{\|\cdot\|}(A)$*. Then*

$$|\operatorname{conv}(A)| \le |B|.$$

Here, for a compact set X *in* $\mathbb{R}^d$*,* $\operatorname{diam}_{\|\cdot\|}(X) = \max\{\|x - x'\|,\ x, x' \in X\}$*.*

9.3 Epsilon nets

We finish this chapter by explaining the existence of small nets, which is a very useful fact. Recall that a δ-net of a metric space (M, d) is a subset X of M such that that for every point y from M, there is a point x in X such that $d(x, y) < \delta$. In other words, M is covered with the (open) balls with radius r centred at the points in X, $M \subset \bigcup_{x \in X} B(x, \delta)$.

9.7 Theorem. *Let* $\|\cdot\|$ *be a norm on* $\mathbb{R}^d$*. For every* $\delta > 0$*, the unit sphere* $\{x \in \mathbb{R}^d,\ \|x\| = 1\}$ *admits a* δ*-net, with respect to the distance measured by* $\|\cdot\|$*, of size at most* $(1 + 2/\delta)^d$*.*

Proof. Let $B = \{x \in \mathbb{R}^d,\ \|x\| < 1\}$ be the unit ball and let $S = \{x \in \mathbb{R}^d,\ \|x\| = 1\}$ be the unit sphere with respect to $\|\cdot\|$. Let X be a subset of S of maximal cardinality with the property that every two points of X are at least δ-apart in distance measured by $\|\cdot\|$, equivalently, the balls $\{x + \frac{\delta}{2}B\}_{x \in X}$ are disjoint. Note that by its maximality, X is also a δ-net of S (otherwise, we could add a point to X). By a volume argument, X cannot be too large,

$$\begin{aligned} |X| \cdot (\delta/2)^d \operatorname{vol}_d(B) = \operatorname{vol}_d\left(\bigcup_{x \in X} (x + \frac{\delta}{2}B)\right) &\le \operatorname{vol}_d\left(\left(1 + \frac{\delta}{2}\right)B\right) \\ &= \left(1 + \frac{\delta}{2}\right)^d \operatorname{vol}_d(B), \end{aligned}$$

hence $|X| \le (1 + 2/\delta)^d$. $\square$

9.4 Concentration of measure on the sphere

We equip the unit Euclidean sphere $S^{d-1} = \partial B_2^d$ in $\mathbb{R}^d$ with a probability measure σ defined as follows

$$\sigma(A) = \frac{|\tilde{A}|}{|B_2^d|}, \qquad \tilde{A} = \{ta \in \mathbb{R}^d,\ a \in A, t \in [0,1]\}$$

for all Borel subsets A of S^{d-1}.

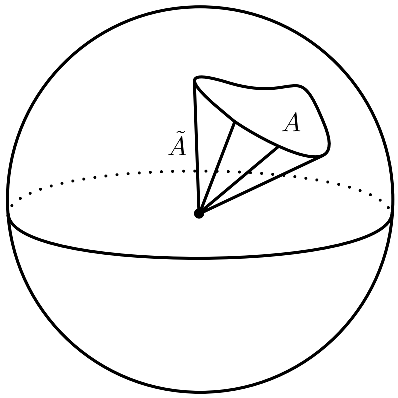

Figure 9.6: For a set A in S^{d-1}, we define a cone $\tilde{A}$. Its normalised volume is $\sigma(A)$, the normalised surface measure of A.

Note that σ is rotationally invariant. In fact, it is the usual surface measure on S^{d-1} normalised to be a probability measure. For a subset A of S^{n-1} and $t \geq 0$, we define its **t-enlargement** A_t by

$$A_t = \{x \in S^{n-1},\ \mathrm{dist}(x, A) \leq t\} = (A + tB_2^d) \cap S^{d-1}.$$

Note that for $t \geq 2$, A_t becomes the whole sphere.

The concentration of measure phenomenon asserts that enlargements of sets result in sets of measure *almost* 1. Formally, we have the following result.

9.8 Theorem. *For a Borel subset A of the unit Euclidean sphere S^{d-1} with measure at least one-half, $\sigma(A) \geq 1/2$, we have for positive t,*

$$\sigma(A_t) \geq 1 - 2e^{-dt^2/4}.$$

Proof. We can assume that $t < 2$. Let B be the complement of the t-enlargement A_t of A, $B = S^{d-1} \setminus A_t$. For $x \in A$ and $y \in B$, we have $|x - y| \geq t$, so

$$\left|\frac{x+y}{2}\right| = \sqrt{1 - \left(\frac{|x-y|}{2}\right)^2} \leq \sqrt{1 - \frac{t^2}{4}} \leq 1 - \frac{t^2}{8}.$$

Let $\tilde{A}$ be the part in B_2^d of the cone built on A, $\tilde{A} = \{\alpha x,\ \alpha \in [0,1], x \in A\}$, so that $\sigma(A) = |\tilde{A}|/|B_2^d|$; similarly for B and $\tilde{B}$. Consider $\tilde{x} \in \tilde{A}$ and $\tilde{y} \in \tilde{B}$, say $\tilde{x} = \alpha x$ and

$\tilde{y} = \beta y$, for some $\alpha, \beta \in [0,1]$ and $x \in A$, $y \in B$. If, say $\alpha \leq \beta$, we have

$$\begin{aligned}\left|\frac{\tilde{x}+\tilde{y}}{2}\right| = \left|\frac{\alpha x + \beta y}{2}\right| = \beta\left|\frac{\frac{\alpha}{\beta}x + y}{2}\right| &= \beta\left|\frac{\alpha}{\beta}\frac{x+y}{2} + \left(1-\frac{\alpha}{\beta}\right)\frac{y}{2}\right| \\ &\leq \left|\frac{\alpha}{\beta}\frac{x+y}{2} + \left(1-\frac{\alpha}{\beta}\right)\frac{y}{2}\right| \\ &\leq \frac{\alpha}{\beta}\left|\frac{x+y}{2}\right| + \left(1-\frac{\alpha}{\beta}\right)\left|\frac{y}{2}\right|.\end{aligned}$$

Since $\left|\frac{x+y}{2}\right| \leq 1 - \frac{t^2}{8}$ and $\left|\frac{y}{2}\right| \leq \frac{1}{2} \leq 1 - \frac{t^2}{8}$, we get

$$\left|\frac{\tilde{x}+\tilde{y}}{2}\right| \leq 1 - \frac{t^2}{8},$$

thus

$$\frac{\tilde{A}+\tilde{B}}{2} \subset \left(1 - \frac{t^2}{8}\right) B_2^d.$$

By the Brunn-Minkowski inequality,

$$\left(1-\frac{t^2}{8}\right)^d |B_2^d| \geq \left|\frac{\tilde{A}+\tilde{B}}{2}\right| \geq \sqrt{|\tilde{A}| \cdot |\tilde{B}|} = |B_2^d|\sqrt{\sigma(A)\sigma(B)}.$$

Using, $\sigma(A) \geq \frac{1}{2}$, $\sigma(B) = 1 - \sigma(A_t)$, $1 - \frac{t^2}{8} \leq e^{-t^2/8}$ and rearranging finishes the proof. $\square$

9.5 Exercises

1. Prove that for every nonnegative numbers $\alpha_1, \ldots, \alpha_d$ and $\beta_1, \ldots, \beta_d$, we have
$$\left(\prod_{i=1}^{d}(\alpha_i+\beta_i)\right)^{1/d} \geq \left(\prod_{i=1}^{d}\alpha_i\right)^{1/d} + \left(\prod_{i=1}^{d}\beta_i\right)^{1/d}.$$

2. Let A be a nonempty compact set in $\mathbb{R}^d$. Let $n \geq 1$ be an integer and consider the tiling of $\mathbb{R}^d$ by the translates of the cube $[0, 2^{-n}]^d$ by the lattice $2^{-n}\mathbb{Z}^d$. Let A_n be the union of all such cubes which intersect A. Show that $A_1 \supset A_2 \supset \ldots$ and $\bigcap_{n=1}^{\infty} A_n = A$.

3. Justify the existence of the hyperplane H from the inductive argument for brick sets.

4. Using Theorem 9.4, show that if A is a compact set in $\mathbb{R}^d$ and B is a Euclidean volume of the same perimeter, $|\partial A| = |\partial B|$, then $|A| \leq |B|$. In other words,
$$|\partial A| \geq d|B_2^d|^{\frac{1}{d}}|A|^{\frac{d-1}{d}}.$$

5. Recall that $\Gamma(t) = \int_0^\infty x^{t-1}e^{-x}\mathrm{d}x$, $t > 0$ is the Gamma function. In particular, $\Gamma(t+1) = t\Gamma(t)$ and for a positive integer n, $\Gamma(n) = (n-1)!$.

   (a) Show that $\int_{\mathbb{R}^d} e^{-(x_1^2+\cdots+x_d^2)}\mathrm{d}x = d|B_2^d|\int_0^\infty r^{d-1}e^{-r^2}\mathrm{d}r = |B_2^d|\Gamma(\frac{d}{2}+1)$.

   (b) Using Fubini's theorem, note that $\int_{\mathbb{R}^d} e^{-(x_1^2+\cdots+x_d^2)}\mathrm{d}x = \left(\int_{\mathbb{R}} e^{-x^2}\mathrm{d}x\right)^d$ and using the case $d = 2$ evaluate $\int_{\mathbb{R}} e^{-x^2}\mathrm{d}x$.

   (c) Conclude that $|B_2^d| = \frac{\sqrt{\pi}^d}{\Gamma(\frac{d}{2}+1)}$.

6. We say that a function $f\colon \mathbb{R}^d \to \mathbb{R}$ is locally Lipschitz if the quantity $\|\nabla f(x)\| = \limsup_{y\to x}\frac{|f(y)-f(x)|}{|y-x|}$ is bounded on $\mathbb{R}^d$. Note that if f is a C^1 function, then $\|\nabla f(x)\|$ is simply the Euclidean norm of $\nabla f(x)$. Prove the so-called co-area inequality
$$\int_{\mathbb{R}^d}\|\nabla f(x)\|\mathrm{d}x \geq \int_{\mathbb{R}}|\partial A_t|\mathrm{d}t,$$
where $A_t = \{x \in \mathbb{R}^d,\ f(x) > t\}$.

7. Using the isoperimetric inequality and the co-area inequality prove the L_1–Sobolev inequality with sharp constant: for every locally Lipschitz function $f\colon \mathbb{R}^d \to \mathbb{R}$ with a compact support, we have
$$\left(\int_{\mathbb{R}^d}|f|^{\frac{d}{d-1}}\right)^{\frac{d-1}{d}} \leq \frac{1}{d|B_2^d|^{1/d}}\int_{\mathbb{R}^d}\|\nabla f(x)\|\mathrm{d}x.$$
Show that this inequality when applied to f approximating the indicator function of a measurable set A in $\mathbb{R}^d$, recovers the isoperimetric inequality: $|\partial A| \geq d|B_2^d|^{\frac{1}{d}}|A|^{\frac{d-1}{d}}$.

10 Equilateral and equiangular sets

Let $\|\cdot\|$ be a norm on $\mathbb{R}^d$. A set X in $\mathbb{R}^d$ is called $\|\cdot\|$**-equilateral** if there is a positive number α such that $\|x-y\| = \alpha$ for every two distinct elements x and y in X. In particular, ℓ_p-equilateral means equilateral with respect to the ℓ_p-norm, that is $\|x\| = \|x\|_p = (\sum_{i=1}^d |x_i|^p)^{1/p}$. The basic question we are interested in is: given a norm $\|\cdot\|$, what is the size of the largest (in terms of the cardinality) $\|\cdot\|$-equilateral set?

10.1 Example. Let X be the set of the vertices of a regular d-dimensional simplex in $\mathbb{R}^d$. Then X is ℓ_2-equilateral and $|X| = d+1$.

10.2 Example. Let $X = \{-1,1\}^d$ (the vertices of the cube $[-1,1]^d$, that is the unit ball in ℓ_∞-norm). Then X is ℓ_∞-equilateral and $|X| = 2^d$.

10.3 Example. Let $X = \{-e_1, e_1, \ldots, -e_d, e_d\}$ (the vertices of the cross-polytope, that is the unit ball in ℓ_1-norm). Then X is ℓ_1-equilateral and $|X| = 2d$.

To address the above question, we shall take a geometric view-point on norms and then equilateral sets. Norms are intimately connected to convex sets. A **convex body** in $\mathbb{R}^d$ is a convex set which is compact and has nonempty interior.

10.4 Theorem. *Given a norm $\|\cdot\|$ on $\mathbb{R}^d$, its (closed) unit ball, $\{x \in \mathbb{R}^d,\ \|x\| \leq 1\}$ is a symmetric convex body. Conversely, given a symmetric convex body K in $\mathbb{R}^d$, there is a norm $\|\cdot\|$ on $\mathbb{R}^d$ whose unit ball is K.*

The proof is the content of two exercises. We have the following geometric characterisation of equilateral sets. We say that two convex bodies K and L **touch** if they intersect, but only along their boundaries, that is $K \cap L \neq \varnothing$ and $K \cap L \subset \partial K \cap \partial L$.

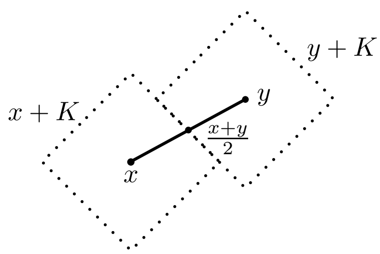

Figure 10.1: Two translates $x+K$, $y+K$ of the unit ball K touch if and only if x and y are exactly 2 apart.

10.5 Theorem. *Let $\|\cdot\|$ be a norm on $\mathbb{R}^d$. Let $K = \{x \in \mathbb{R}^d,\ \|x\| \leq 1\}$ be its unit ball. If X is a $\|\cdot\|$-equilateral set such that $\|x-y\| = 2$ for every two distinct x, y in X, then the family $\{x+K\}_{x \in X}$ is a family of pairwise touching translates of K. Conversely, every such family defines an equilateral set.*

Proof. If $\|x-y\|=2$, then $\frac{x+y}{2}$ is a common point of $x+K$ and $y+K$. Moreover, if there was a common point in the interior of one of them, say $p \in \operatorname{int}(x+K)$, meaning $\|p-x\|<1$, then $\|x-y\| \leq \|x-p\|+\|p-y\|<2$. Conversely, if $x+K$ and $y+K$ intersect, say p is their common point, then $\|x-y\| \leq \|x-p\|+\|p-y\| \leq 2$. Moreover, if $\|x-y\|<2$, then $\frac{x+y}{2}$ satisfies $\|x-\frac{x+y}{2}\|<1$ and $\|y-\frac{x+y}{2}\|<1$, that is $\frac{x+y}{2}$ is in the interior of both $x+K$ and $y+K$. $\square$

10.1 Upper bound for equilateral sets for arbitrary norms

10.6 Theorem. *If $\|\cdot\|$ is a norm on $\mathbb{R}^d$ and X is a $\|\cdot\|$-equilateral set in $\mathbb{R}^d$, then X is finite with cardinality satisfying $|X| \leq 2^d$.*

10.7 Remark. The example of the vertices of the cube (see Example 10.2) shows that this upper bound is sharp for the ℓ_∞-norm. In other words, the largest ℓ_∞-equilater set is of size 2^d.

Proof of Theorem 10.6. Let K be the unit ball of $\|\cdot\|$ and suppose that X is such that $\|x-y\|=2$ for every two distinct points x, y in X (rescale X if needed). We shall use a volume argument. Consider the set

$$L=\bigcup_{x \in X}\{x+K\}.$$

By Theorem 10.5, the sets $x+K$ pairwise touch, so in particular, they have disjoint interiors, hence the volume of L satisfies

$$|L|=|X| \cdot|K|,$$

where $|X|$ is the cardinality of X and $|K|$ is the volume of K ($|X|=\infty$, hence $|L|=\infty$ if X is not finite). On the other hand, note that

$$\operatorname{diam}(L) \leq 4.$$

Explanation: take two elements from L, say $x+a$ and $y+b$ for $x, y \in X$ and $a, b \in K$, and then $\|x+a-(y+b)\| \leq\|x-y\|+\|a\|+\|b\| \leq 2+1+1=4$. By the isodiametric inequality for the norm $\|\cdot\|$ (see Theorem 9.6), we get the upper bound on the volume of L,

$$|L| \leq|2 K|=2^d|K|,$$

which combined with $|L|=|X| \cdot|K|$ gives the desired $|X| \leq 2^d$. $\square$

10.2 Upper bound for equilateral sets for Euclidean norms

10.8 Theorem. *If X is an ℓ_2-equilateral set in $\mathbb{R}^d$, then $|X| \leq d+1$.*

10.9 Remark. The example of the vertices of a regular simplex (see Example 10.1) shows that this upper bound is sharp. In other words, the largest ℓ_2-equilateral set is of size $d+1$.

10.10 Lemma. *For every $d \times n$ matrix B, we have* $\ker(B^\top B) = \ker(B)$ *and, consequently,* $\operatorname{rank}(B^\top B) = \operatorname{rank}(B)$. *In particular,* $\operatorname{rank}(B^\top B) \leq \min\{d, n\}$.

Proof. The statement about the ranks follows from the statement about the kernels thanks to the rank-nullity theorem. Clearly, $\ker(B) \subset \ker(B^\top B)$. To prove the reverse inclusion, if $x \in \ker(B^\top B)$, then $B^\top Bx = 0$, so $0 = x^\top B^\top Bx = (Bx)^\top(Bx) = \langle Bx, Bx\rangle = |Bx|^2$, so $Bx = 0$, that is $x \in \ker(B)$. □

Proof of Theorem 10.8. Suppose $X = \{x_0, \dots, x_n\}$ is such that $|x_i - x_j| = 1$ for every $i \neq j$ (rescale X if needed). We can assume that $x_0 = 0$ (translate X if needed). Then $1 = |x_i - x_0| = |x_i|$ for every $1 \leq i \leq n$. We consider the Gram matrix A of $x_1, \dots, x_n$: $a_{ij} = \langle x_i, x_j\rangle$, $1 \leq i, j \leq n$. For $i = j$, we have $a_{ii} = |x_i|^2 = 1$. For $i \neq j$, we have $a_{ij} = \frac{1}{2}$ because $1 = |x_i - x_j|^2 = |x_i|^2 + |x_j|^2 - 2\langle x_i, x_j\rangle = 2 - 2\langle x_i, x_j\rangle$. Thus A has 1 on the diagonal and $\frac{1}{2}$ at every other entry and it can be checked that A is of full rank, that is $\operatorname{rank}(A) = n$. On the other hand, $\operatorname{rank}(A) \leq d$ because A is the Gram matrix of vectors in $\mathbb{R}^d$ ($A = B^\top B$, where the columns of B are the vectors x_i – see Lemma 10.10). Therefore, $|X| = n + 1 \leq d + 1$, as desired. □

There is a robust version of the above result providing an upper bound on the size of *approximately* equilateral sets (we shall need it later to discuss the ℓ_1 case). Its proof also uses algebraic methods.

10.11 Theorem. *If $x_1, \dots, x_n$ are vectors in $\mathbb{R}^d$ such that*

$$1 - \frac{1}{\sqrt{n}} \leq |x_i - x_j|^2 \leq 1 + \frac{1}{\sqrt{n}}, \qquad i \neq j,$$

then $n \leq 2(d+2)$.

10.12 Lemma. *If A is an $n \times n$ symmetric nonzero matrix, then* $\operatorname{rank}(A) \geq \frac{(\operatorname{tr}(A))^2}{\operatorname{tr}(A^\top A)}$.

Proof. Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of A (listed with multiplicities). If $r = \operatorname{rank}(A)$, then exactly r eigenvalues are nonzero, say these are $\lambda_1, \dots, \lambda_r$. Then by the Cauchy-Schwarz inequality,

$$\operatorname{tr}(A) = \sum_{j=1}^{n} \lambda_j = \sum_{j=1}^{r} \lambda_j \leq \sqrt{r}\sqrt{\sum_{j=1}^{r} \lambda_j^2} = \sqrt{r}\sqrt{\operatorname{tr}(A^\top A)}.$$

□

Proof of Theorem 10.11. Consider a matrix A with entries $a_{ij} = 1 - |x_i - x_j|^2$. We have $a_{ii} = 1$ and $|a_{ij}| \leq \frac{1}{\sqrt{n}}$, $i \neq j$. As a result,

$$\operatorname{tr}(A) = \sum_{i=1}^{n} a_{ii} = n$$

and

$$\operatorname{tr}(A^\top A) = \sum_{i,j=1}^{n} a_{ij}^2 = n + \sum_{i\neq j} a_{ij}^2 \le n + \frac{1}{n}(n^2 - n) = 2n - 1,$$

so, by Lemma 10.12,

$$\operatorname{rank}(A) \ge \frac{n^2}{2n-1} \ge \frac{n}{2}.$$

We now upper bound the rank of A in terms of d. For $j = 1, \ldots, n$, consider the functions

$$f_j(y) = 1 - |y - x_j|^2, \qquad y \in \mathbb{R}^d.$$

We have, $A = [f_j(x_i)]_{i,j=1}^n$. Observe that each f_j is a linear combination of $d+2$ functions: $g_k(y) = y_k$, $k = 1, \ldots, d$ and $g_{d+1}(y) = |y|^2$, $g_{d+2}(y) = 1$, say

$$f_j = \sum_{k=1}^{d+2} \lambda_{j,k} g_k.$$

Then the jth column of A can be written as

$$\begin{bmatrix} f_j(x_1) \\ \vdots \\ f_j(x_n) \end{bmatrix} = \sum_{k=1}^{d+2} \lambda_{j,k} \begin{bmatrix} g_k(x_1) \\ \vdots \\ g_k(x_n) \end{bmatrix} = \sum_{k=1}^{d+2} \lambda_{j,k} v_k,$$

where $v_k = \begin{bmatrix} g_k(x_1) \\ \vdots \\ g_k(x_n) \end{bmatrix}$, $k = 1, \ldots, d+2$. In other words, each column of A is in the span of $v_1, \ldots, v_{d+2}$, hence $\operatorname{rank}(A) \le d+2$. Since, as we saw, $\operatorname{rank}(A) \ge \frac{n}{2}$, we conclude $n \le 2(d+2)$. ☐

10.3 Upper bound for equilateral sets for the ℓ_1 norm

10.13 Theorem. *If X is an ℓ_1-equilateral set in $\mathbb{R}^d$, then $|X| < 100d^4$.*

The idea here is to first embed such a set into a higher dimensional space to make it approximately ℓ_2-equilateral and then use the robust results for the Euclidean norm (Theorem 10.11).

10.14 Lemma. *For every positive integers d, N, there is a function $f_{d,N}\colon [0,1]^d \to \mathbb{R}^{Nd}$ such that for every $x, y \in [0,1]^d$, we have*

$$\|x - y\|_1 - \frac{2d}{N} \le \frac{1}{N}|f_{d,N}(x) - f_{d,N}(y)|^2 \le \|x - y\|_1 + \frac{2d}{N}. \tag{10.1}$$

Proof. First we consider the case $d = 1$. For $x \in [0,1]$, we define

$$f_{1,N}(x) = (\underbrace{1, 1, \ldots, 1}_{\lfloor xN \rfloor}, \underbrace{0, 0, \ldots, 0}_{N - \lfloor xN \rfloor}).$$

Then, for $x, y \in [0,1]$,

$$|f_{1,N}(x) - f_{1,N}(y)|^2 = \Big| \lfloor xN \rfloor - \lfloor yN \rfloor \Big|.$$

Since $\|x-y\|_1 = |x-y|$, it remains to check that

$$|x-y| - \frac{2}{N} \le \frac{1}{N}\Big|\lfloor xN\rfloor - \lfloor yN\rfloor\Big| \le |x-y| + \frac{2}{N}.$$

This follows since $\lfloor xN\rfloor$ differs from xN by at most 1 and $\lfloor yN\rfloor$ differs from yN by at most 1, so $\lfloor xN\rfloor - \lfloor yN\rfloor$ differs from $xN - yN$ by at most 2.

When $d > 1$, we define $f_{d,N}(x)$ by concatenating $f_{1,N}(x_1), \ldots, f_{1,N}(x_d)$ and since $\|x-y\|_1 = \sum_{i=1}^d |x_i - y_i|$, the error bounds add up. □

Proof of Theorem 10.13. Suppose, to argue by contradiction, that X is an ℓ_1-equilateral set in $\mathbb{R}^d$ with $n = |X| = 100d^4$ and $\|x-y\|_1 = \frac{1}{2}$ for every two distinct points $x, y \in X$. By translating if needed, we can assume that one of the points from X is $(\frac{1}{2}, \ldots, \frac{1}{2})$. Then $X \subset [0,1]^d$. Let $N = 40d^3$. We apply Lemma 10.14 to get a function $f_{d,N}\colon [0,1]^d \to \mathbb{R}^{Nd}$ satisfying (10.1). Let $Y = f_{d,N}(X)$. This is an n-element set in $\mathbb{R}^{Nd}$. Since X is ℓ_1-equilateral, for every $u, v \in Y$, we have

$$\frac{1}{2} - \frac{1}{20d^2} \le \frac{1}{N}|u-v|^2 \le \frac{1}{2} + \frac{1}{20d^2}.$$

Equivalently, since $10d^2 = \sqrt{n}$,

$$1 - \frac{1}{\sqrt{n}} \le \left|\sqrt{\frac{2}{N}}u - \sqrt{\frac{2}{N}}v\right|^2 \le 1 + \frac{1}{\sqrt{n}}.$$

This means that the set $\sqrt{\frac{2}{N}}Y$ satisfies the hypothesis of Theorem10.11. Thus, $100d^4 = n \le 2(Nd+2) = 2(40d^4+2) = 80d^4 + 2$, a contradiction. □

10.4 Upper bound for equiangular sets

Here, by S^{d-1} we mean the unit centred Euclidean sphere in $\mathbb{R}^d$, that is $S^{d-1} = \partial B_2^d$.

A set X of unit vectors in $\mathbb{R}^d$ ($X \subset S^{d-1}$) is called **equiangular** if there is $0 \le \alpha < 1$ such that for every two distinct points x, y in X, we have $|\langle x, y\rangle| = \alpha$. This means that the lines $\{l = \mathrm{span}(x),\ x \in X\}$ are equiangular, that is the angle between every two of them is the same. We are interested in the maximal number of such lines,

$$N(d) = \sup\{|X|,\ X \subset S^{d-1}, X \text{ is equiangular}\}.$$

For instance, in $\mathbb{R}^2$ there are 3 equiangular lines meeting at 120°. In $\mathbb{R}^3$, there are 6 equiangular lines: the longest diagonals of a regular icosahedron. These are in fact optimal because we have the following general upper bound.

10.15 Theorem. *If X is an equiangluar set of unit vectors in $\mathbb{R}^d$, then its cardinality satisfies $|X| \le \binom{d+1}{2}$.*

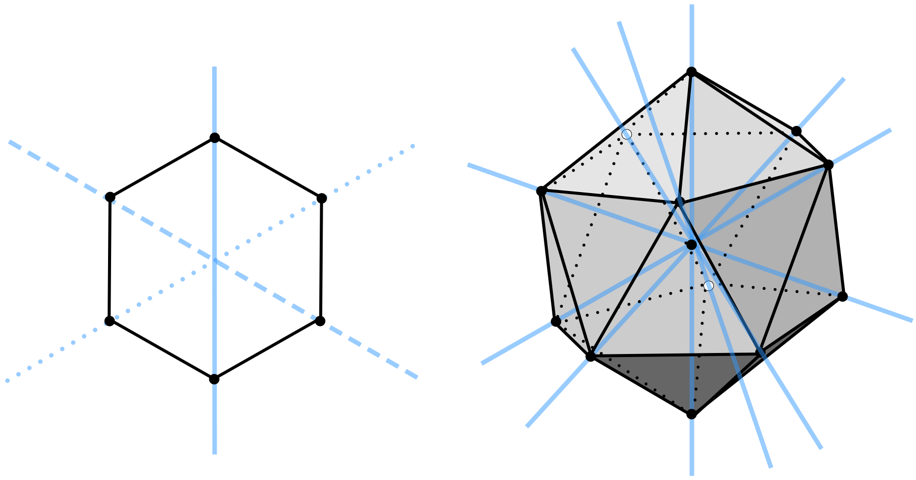

Figure 10.2: There are 3 equiangular lines in $\mathbb{R}^2$ (the main diagonals of a regular hexagon) and 6 equiangular lines in $\mathbb{R}^3$ (the main diagonals of a regular icosahedron).

Table 2: The values of $N(d)$ in small dimensions in comparison to $\binom{d+1}{2}$.

d	**1**	**2**	**3**	4	5	6	**7**	8	9	10	11	12	13
$N(d)$	**1**	**3**	**6**	10	16	28	**28**	28	28	28	28	28	28
$\binom{d+1}{2}$	**1**	**3**	**6**	10	15	21	**28**	36	45	55	66	78	91

Proof. Consider the set of $d \times d$ symmetric matrices $\{xx^\top, \ x \in X\}$. We show it is linearly independent. This suffices to conclude the assertion because the space of all $d \times d$ real symmetric matrices is $\binom{d+1}{2}$ dimensional. Suppose

$$\sum_{i=1}^{n} \lambda_i x_i x_i^\top = 0$$

for some reals λ_i. Fix $j \in \{1, \dots, n\}$. Multiplying both sides on the left by $x_j^\top$ and on the right by x_j, since $|\langle x_i, x_j \rangle| = \alpha$ for every $i \neq j$, we get

$$0 = \sum_{i=1}^{n} \lambda_j \langle x_i, x_j \rangle^2 = \lambda_j |x_j|^2 + \alpha^2 \sum_{i \neq j} \lambda_j |\langle x_i, x_j \rangle|^2 = \lambda_j + \alpha^2 \sum_{i \neq j} \lambda_i = [A\lambda]_j,$$

where $\lambda = (\lambda_1, \dots, \lambda_n$ is the vector of the coefficients λ_i, A is the $n \times n$ matrix with 1 on the diagonal and α^2 on the off-diagonal and $[A\lambda]_j$ is the jth component of the vector $A\lambda$. It can be checked that for every $\alpha \in [0, 1)$ matrix A is nonsingular (in fact, it is positive semi-definite – exercise), hence $\lambda = 0$, which shows that the considered set of matrices is linearly independent. □

10.16 Remark. The exact value of $N(d)$ has been found only for finitely many d. The upper bound from Theorem 10.15 gives $N(d) = O(d^2)$ which turns out to be of the correct order. There are constructions giving $N(d) \geq \frac{2}{9}(d+1)^2$ for every $d = 6 \cdot 4^k - 1$, $k = 0, 1, 2 \dots$ and this in fact implies that $N(d) = \Theta(d^2)$ for all d. For the known examples of sets giving $N(d) = \Omega(d^2)$, the common value α of the inner product tends to 0 as $d \to \infty$. It is of interest to consider a different question: what is the maximal number of equiangular lines in high dimensions, that is as $d \to \infty$, but with a *fixed* common angle? Formally, for $\alpha \in [0, 1)$, define

$$N_\alpha(d) = \sup\{|X|,\ X \subset S^{d-1}, \forall x, y \in X, x \neq y\ |\langle x, y\rangle| = \alpha\}.$$

Note that then

$$N(d) = \max_{\alpha \in [0,1)} N_\alpha(d). \tag{10.2}$$

Bounds on $N_\alpha(d)$ have a long history and the behaviour is quite different. Here we only mention a recent breakthrough result of Bukh from [8], whose theorem in particular gives $N_\alpha(d) \leq C_\alpha d$ with a constant C_α which depends only on α (and, of course, $C_\alpha \to \infty$ as $\alpha \to 0$). See also [12] for latest results and a nice brief historical account.

Using similar algebraic arguments as in the proof of Theorem 10.15, but for carefully chosen polynomials instead of rank 1 matrices, we can prove a generalisation for sets which allow several angles.

10.17 Theorem. *Let $A \subset [0, 1)$ be a finite set and let m be the cardinality of $A \cup (-A)$. Suppose X is a set of unit vectors in $\mathbb{R}^d$ such that $|\langle x, y\rangle| \in A$ for every two distinct points x, y in X. Then the cardinality of X satisfies $|X| \leq \binom{d+m-1}{m}$.*

10.18 Remark. Suppose A has only one element. If it is 0, then $m = 1$ and the theorem gives $|X| \leq \binom{d+1-1}{1} = d$, as expected (when $A = \{0\}$, X is a set of unit vectors which are orthogonal to each other). If $A = \{\alpha\}$ for some $\alpha \in (0, 1)$, then $m = 2$ and the theorem gives $|X| \leq \binom{d+2-1}{2} = \binom{d+1}{2}$, that is $N_\alpha(d) \leq \binom{d+1}{2}$. Since this bound does not depend on α, taking the maximum over all $a \in (0, 1)$ (see (10.2)) recovers Theorem 10.15.

Proof of Theorem 10.17. Let $\varepsilon = 1$, if $0 \in A$ and $\varepsilon = 0$, if $0 \notin A$. Then the cardinality of $A \cup (-A)$ is $m = 2k - \varepsilon$. For every $x \in X$, define a polynomial F_x of d variables $z = (z_1, \dots, z_d)$ by

$$F_x(z) = \langle x, z\rangle^\varepsilon \prod_{\alpha \in A \setminus \{0\}} \frac{\langle x, z\rangle^2 - \alpha^2|z|^2}{1 - \alpha^2}.$$

Note that this is a homogeneous polynomial of degree m. Every such polynomial is a linear combination of monomials $z_1^{j_1} \cdot \ldots \cdot z_d^{j_d}$ for nonnegative integers $j_1, \dots, j_d$ with $j_1 + \cdots + j_d = m$ and the linear space of homogeneous polynomial of degree m has dimension $\binom{d-1+m}{d-1}$. Thus it suffices to show that the set $\{F_x\}_{x \in X}$ is linearly independent. If

$$\sum_{i=1}^n \lambda_i F_{x_i} = 0$$

for some $x_1, \ldots, x_n \in X$, then evaluating this at $z = x_j$ (for a fixed j) gives $\lambda_j = 0$ because we check that $F_{x_i}(x_j) = \delta_{i,j}$ (here $\delta_{i,j}$ is the Kronecker delta symbol, $\delta_{i,j} = 1$ if $i = j$ and $\delta_{i,j} = 0$ if $i \neq j$). $\square$

10.5 Exercises

1. Show that all norms on $\mathbb{R}^d$ are equivalent, that is if $\|\cdot\|$ and $\|\cdot\|'$ are two norms on $\mathbb{R}^d$, then there are positive finite constants α, β such that for every x in $\mathbb{R}^d$, we have

$$\alpha\|x\| \leq \|x\|' \leq \beta\|x\|.$$

2. Let $\|\cdot\|$ be a norm on $\mathbb{R}^d$ and let $K = \{x \in \mathbb{R}^d,\ \|x\| \leq 1\}$ be its unit ball. Show that K is symmetric, convex, compact, with nonempty interior (K is a symmetric convex body).

3. Let K be symmetric convex body in $\mathbb{R}^d$. Define for $x \in \mathbb{R}^d$,

$$p_K(x) = \inf\{t > 0,\ x \in tK\}$$

(the so-called **Minkowski's functional** of K). Show that p_K is a norm on $\mathbb{R}^d$ and its unit ball is K.

4. Let $p \in (1, \infty)$. Find an ℓ_p-equilateral set in $\mathbb{R}^d$ of size $d+1$.

5. Let $\alpha \in [0, 1)$ and let $A = [a_{ij}]_{i,j=1}^n$ be an $n \times n$ matrix with $a_{ii} = 1$ for every $i \leq n$ and $a_{ij} = \alpha$ for every $i \neq j$. Then A is positive semi-definite, that is $\langle Ax, x\rangle > 0$ for every vector $x \neq 0$. In particular, A is nonsingular.

11 Diameter reduction – Borsuk's question

Borsuk asked in 1933 (see [7]): *can every set X in $\mathbb{R}^d$ of finite diameter D be partitioned into $d+1$ subsets, each of diameter strictly less than D?* It will be convenient to say that a partition is **diameter reducing**, if every piece of the partition has a strictly smaller diameter than the initial set. For example, if X is the set of the vertices of a regular d-dimensional simplex in $\mathbb{R}^d$, then every partition of X into at most d sets will not reduce the diameter (because there will be a piece with at least 2 points from X and every two points in X achieve its diameter). Borsuk showed that for the ball, $X = B_2^d$, there are no partitions into at most d parts which reduce the diameter, but there is one into $d+1$ parts.

Borsuk's question has an affirmative answer when

⋆ $X \subset \mathbb{R}^3$ (Borsuk, 1933)

⋆ $X \subset \mathbb{R}^d$ and X is convex with smooth boundary (Hadwiger, 1946)

⋆ $X \subset \mathbb{R}^d$ and X is symmetric convex (Riesling, 1971)

⋆ $X \subset \mathbb{R}^d$ and X is a body of revolution (Dekster 1995) However, for sufficiently large dimensions, the answer is negative!

11.1 Theorem (Kahn, Kalai, [13])**.** *For every prime p, there is a set X in $\mathbb{R}^{d^2}$, $d = 4p$ with no diameter deducing partition into fewer than 1.1^d parts.*

In particular, if $1.1^d > d^2 + 1$, then such X provides a negative answer in dimension d^2. Since the smallest such d is $d = 96$, we need $p \geq 96/4 = 24$, so the choice $p = 29$, $d = 126$ gives an example of a set X in $\mathbb{R}^{13456}$, thus answering Borsuk's question in the negative.

11.1 A result from extremal set theory

Kahn and Kalai's ingenious proof of Theorem 11.1 uses a result in extremal set theory.

11.2 Lemma. *Let p be a prime and let $\mathcal{F}$ be a family of subsets of size $2p-1$ of an n-element set. If for every distinct sets A and B in $\mathcal{F}$, we have $|A \cap B| \neq p-1$, then*

$$|\mathcal{F}| \leq \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{p-1}. \tag{11.1}$$

Note that there are $\binom{n}{2p-1}$ $2p-1$-element subsets of an n-element set. The size of the intersection of every two distinct such subsets is of size $0, 1, \ldots, 2p-2$. This lemma says that forbidding one size intersection, the middle one $p-1$, implies that there are *much fewer* such sets.

11.3 Corollary. *Under the assumptions of Lemma 11.2, if additionally, $n = 4p$, then*

$$|\mathcal{F}| < \frac{\binom{n}{2p-1}}{1.1^n}.$$

Proof. Since $\binom{n}{k-1} = \frac{k}{n-k}\binom{n}{k} \le \frac{1}{3}\binom{n}{k}$ if $n \ge 4k$, we get from (11.1) (recall that $n = 4p$),

$$|\mathcal{F}| \le \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{p-1} \le \left(\frac{1}{3^p} + \frac{1}{3^{p-1}} + \cdots + \frac{1}{3}\right)\binom{n}{p} < \frac{1}{2}\binom{n}{p}.$$

Therefore,

$$\begin{aligned}\frac{\binom{n}{2p-1}}{|\mathcal{F}|} \ge 2\frac{\binom{n}{2p-1}}{\binom{n}{p}} &= 2\frac{(4p)!}{(2p-1)!(2p+1)!}\frac{p!(3p)!}{(4p)!}\\ &= 2\frac{3p}{2p-1}\cdot\frac{3p-1}{2p-2}\cdot\ldots\cdot\frac{2p+2}{p+1}\\ &\ge 2\left(\frac{3}{2}\right)^{p-1} = \frac{4}{3}\left(\frac{3}{2}\right)^{n/4} > 1.1^n.\end{aligned}$$

□

Proof of Lemma 11.2. Given $A \in \mathcal{F}$, we define its characteristic vector $\mathbf{1}_A \in \{0,1\}^n$ which has 1 at i-th coordinate if and only if $i \in A$. We also define the polynomial $f_A : \{0,1\}^n \to F_p$ over the finite field F_p,

$$f_A(x) = \prod_{k=0}^{p-2}\left(\left(\sum_{i\in A} x_i\right) - k\right).$$

For instance, when $p = 3$, $n = 7$ and $A = \{1,2,3,6,7\}$, then $\mathbf{1}_A = (1,1,1,0,0,1,1)$ and $f_A(x) = (x_1 + x_2 + x_3 + x_6 + x_7)(x_1 + x_2 + x_3 + x_6 + x_7 - 1)$.

Consider the vector space V of all functions $f : \{0,1\}^n \to F_p$ (over F_p) and its subspace $V_{\mathcal{F}} = \operatorname{span}\{f_A, A \in \mathcal{F}\}$. We make the following two observations.

Claim 1. The set $\{f_A, A \in \mathcal{F}\}$ is linearly independent. Hence, $\dim(V_{\mathcal{F}}) = |\mathcal{F}|$.

Claim 2. $\dim(V_{\mathcal{F}}) \le \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{p-1}$.

It remains to prove the claims. For Claim 1, first note that for $A \in \mathcal{F}$ (recall that we do computations in F_p)

$$f_A(\mathbf{1}_A) = \prod_{k=0}^{p-2}(|A| - k) = \prod_{k=0}^{p-2}(2p-1-k) = \prod_{k=0}^{p-2}(p-1-k) = (p-1)! \ne 0$$

and for distinct $A, B \in \mathcal{B}$,

$$f_A(\mathbf{1}_B) = \prod_{k=0}^{p-2}(|A \cap B| - k) = 0,$$

since $|A \cap B| \ne p-1$, so $|A \cap B| \in \{0, 1, \ldots, p-2\}$ (in F_p). Consequently, if

$$\sum_{B\in\mathcal{F}} \lambda_B f_B = 0$$

for some scalars λ_B, then evaluating this at $\mathbf{1}_A$ gives $\lambda_A f_A(\mathbf{1}_A) = 0$, so $\lambda_A = 0$ and this holds for each $A \in \mathcal{F}$.

For Claim 2, note that each f_A is a multinomial of degree at most $p-1$, so f_A is a linear combination of monomials $x_1^{j_1}\cdot\ldots\cdot x_n^{j_n}$ with $j_1+\cdots+j_n \le p-1$. For $x_k \in \{0,1\}$, we have $x_k^{j_k} = x_k$, so each f_A is in fact a linear combination of monomials $x_1^{j_1}\cdot\ldots\cdot x_n^{j_n}$ with $j_1+\cdots+j_n \le p-1$ and $j_1,\ldots,j_n \in \{0,1\}$, or, in other words, $\prod_{j\in J} x_j$ for a subset J of $\{1,\ldots,n\}$ of size at most $p-1$. Thus, there are exactly $\binom{n}{0}+\binom{n}{1}+\cdots+\binom{n}{p-1}$ of them. $\square$

11.2 Construction via tensor product

Recall that the tensor product $x_1 \otimes x_2$ of two vectors $x_1 \in \mathbb{R}^{d_1}$ and $x_2 \in \mathbb{R}^{d_2}$ is a vector in $\mathbb{R}^{d_1 d_2}$ whose components are all the products $x_{1,i}x_{2,j}$, $i \le d_1, j \le d_2$,

$$x_1 \otimes x_2 = [x_{1,i}x_{2,j}]_{i\le d_1, j\le d_2}.$$

The standard scalar product behaves nicely: for $x_1, y_1 \in \mathbb{R}^{d_1}$ and $x_2, y_2 \in \mathbb{R}^{d_2}$, we have

$$\begin{aligned}\langle x_1 \otimes x_2, y_1 \otimes y_2\rangle &= \sum_{i,j} x_{1,i}x_{2,j}y_{1,i}y_{2,j} = \sum_i x_{1,i}y_{1,i}\sum_j x_{2,j}y_{2,j}\\ &= \langle x_1, y_1\rangle\langle x_2, y_2\rangle.\end{aligned}$$

Proof of Theorem 11.1. Let p be a prime, $d = 4p$ and let $\mathcal{A}$ be the set of all $2p-1$ element subsets of the set $\{1,\ldots,d\}$. For $A \in \mathcal{A}$, define the vectors in $\mathbb{R}^d$ ($\mathbf{1}$ denotes the vector $(1,\ldots,1)$ of all 1s)

$$u_A = 2\mathbf{1}_A - \mathbf{1}, \qquad [u_A]_i = \begin{cases} 1, & \text{if } i \in A,\\ -1, & \text{if } i \notin A.\end{cases}$$

Then define the vectors in $\mathbb{R}^{d^2}$,

$$q_A = u_A \otimes u_A.$$

The set X in $\mathbb{R}^{d^2}$ with no diameter reducing partitions into fewer than 1.1^d parts is set to be

$$X = \{q_A, A \in \mathcal{A}\}.$$

To check that, first we note that for every $A, B \in \mathcal{A}$, we have

$$\langle u_A, u_B\rangle = 4[|A\cap B| - p + 1].$$

Explanation: since $\langle \mathbf{1}_A, \mathbf{1}_B\rangle = |A\cap B|$, we obtain $\langle u_A, u_B\rangle = \langle 2\mathbf{1}_A - \mathbf{1}, 2\mathbf{1}_B - \mathbf{1}\rangle = 4\langle \mathbf{1}_A, \mathbf{1}_B\rangle - 2\langle \mathbf{1}_A, \mathbf{1}\rangle - 2\langle \mathbf{1}_B, \mathbf{1}\rangle + \langle \mathbf{1}, \mathbf{1}\rangle = 4|A\cap B| - 2|A| - 2|B| + d = 4|A\cap B| - 2(2p-1) - 2(2p-1) + 4p$.
In particular,

$$\langle u_A, u_A\rangle = d$$

and

$$\langle u_A, u_B\rangle = 0 \qquad \text{if and only if} \qquad |A\cap B| = p-1.$$

Finally note that

$$\begin{aligned}|q_A - q_B|^2 &= \langle q_A, q_A\rangle + \langle q_B, q_B\rangle - 2\langle q_A, q_B\rangle \\ &= \langle u_A, u_A\rangle^2 + \langle u_B, u_B\rangle^2 - 2\langle u_A, u_A\rangle^2 \\ &= 2d^2 - 2\langle u_A, u_B\rangle^2.\end{aligned}$$

Since $\langle u_A, u_B\rangle^2 \geq 0$ with equality if and only if $|A \cap B| = 0$, the diameter of X is $2d^2$ and any subset of X has diameter $2d^2$ as long as it contains two points q_A, q_B corresponding to sets $A, B \in \mathcal{A}$ with $|A \cap B| \neq p - 1$.

Consequently, if we partition X, equivalently $\mathcal{A}$ into fewer than 1.1^d parts, then (the pigeon-hole principle) at least one of parts is of size greater than $\frac{|\mathcal{A}|}{1.1^d} = \frac{\binom{d}{2p-1}}{1.1^d}$. By Corollary 11.3, this part has two elements A, B with $|A \cap B| = p - 1$, so such a partition of X is not diameter reducing. □

11.3 A positive answer in dimension 2

A convex set K in $\mathbb{R}^2$ is called a **universal cover** if every set X in $\mathbb{R}^2$ of diameter 1 can be covered by a congruent copy of K (i.e. a possibly translated and rotated copy of K). For instance, plainly a disk of radius 1 is a universal cover. Jung's theorem (Exercise 7.7) says that a disk of radius $\frac{1}{\sqrt{3}}$ is a universal cover. Pál's theorem asserts that a regular hexagon of side length $\frac{1}{\sqrt{3}}$ is a universal cover. This can be used to explain why Borsuk's question has an affirmative answer in $\mathbb{R}^2$ (exercise).

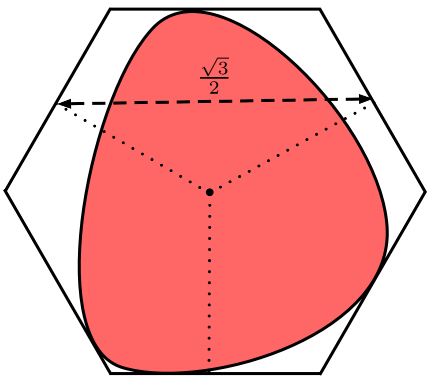

Figure 11.1: A set of diameter 1 can be covered by a regular hexagon of side length $\frac{1}{\sqrt{3}}$ (Pál's theorem). Consequently, it can be partitioned into 3 sets, each of diameter at most $\frac{\sqrt{3}}{2}$

To prove Pál's theorem, it is convenient to have the following important result whose proof we defer to exercises.

11.4 Theorem. *If X is a bounded set in $\mathbb{R}^d$, then X is a subset of a compact convex set of constant width having the same diameter as X.*

Recall that by Exercise 3.10, the diameter of a convex set of constant width equals its width.

11.5 Theorem (Pál). *If X is a set in $\mathbb{R}^2$ with diameter 1, then X is a subset of a regular hexagon of side length $\frac{1}{\sqrt{3}}$.*

Proof. Let K be a compact convex set of constant width 1 which contains X, as provided by Theorem 11.4. It suffices to argue that K can be covered by a regular hexagon of side length $\frac{1}{\sqrt{3}}$. To this end, consider three unit vectors v_1, v_2, v_3 equally spaced on the unit circle, that is the angles between v_1, v_2 and between v_2, v_3 and between v_3, v_1 are all $\frac{2\pi}{3}$. Let ℓ_1, ℓ_1' be parallel supporting lines of K in direction v_1 such that K is between them. The lines are distance 1 apart because K is of constant width. Define similarly ℓ_2, ℓ_2' and ℓ_3, ℓ_3' for directions v_2 and v_3. These lines give two equilateral triangles T and T' both of which contain K (see Figure 11.2). If T and T' are of the same size, then their intersection gives the desired hexagon (because the lines are distance 1 apart – check!). If not, we simultaneously rotate the vectors v_1, v_2, v_3 and follow what happens to the triangles T and T' which also rotate. After rotating by π, the triangles T and T' interchange, thus by continuity, at some point they are of the same size. □

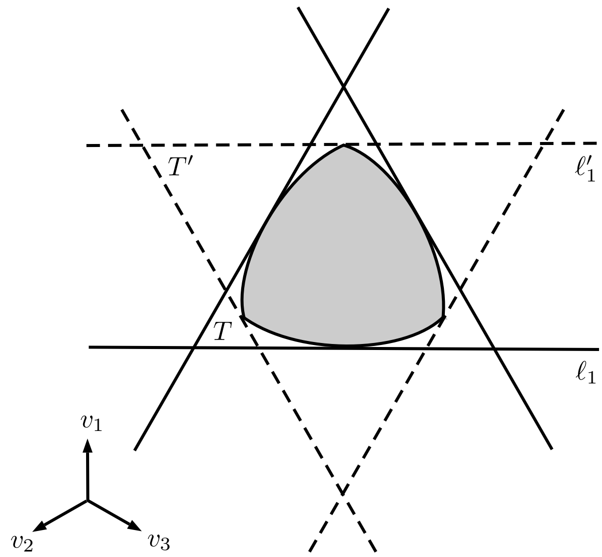

Figure 11.2: Proof of Pál's theorem.

11.4 Exercises

1. Show that there is a partition of the unit ball B_2^d in $\mathbb{R}^d$ into $d+1$ parts such that each part has diameter smaller than 2.

2. Using Theorem 11.5, show that every set X in $\mathbb{R}^2$ of diameter 1 can be partitioned into 3 parts with each part having diameter at most $\frac{\sqrt{3}}{2}$, hence showing that Borsuk's question has an affirmative answer in $\mathbb{R}^2$.

The goal of the next exercises is to prove Theorem 11.4.

3. Show that a compact set K in $\mathbb{R}^d$ is of constant width if and only if for every set L in $\mathbb{R}^d$ such that $L \supset K$, we have $L = K$ or $\operatorname{diam}(L) > \operatorname{diam}(K)$.

4. Let X be a compact convex set in $\mathbb{R}^d$ of diameter 1. Define

$$\mathcal{U}(X) = \{x \in \mathbb{R}^d,\ \operatorname{diam}(X \cup \{x\}) = \operatorname{diam}(X)\},$$
$$\rho(X) = \sup_{x \in \mathcal{U}(X)} \operatorname{dist}(x, X),$$
$$\mathcal{B}(X) = \{x \in \mathcal{U}(x),\ \operatorname{dist}(x, X) = \rho(X)\}.$$

Argue that if X is not of constant width, then $\mathcal{U}(x)$ is nonempty and take $x_1 \in \mathcal{B}(X)$ and set $X_1 = \operatorname{conv}\{x, X\}$. Iterate this procedure and show that $\bigcup_{i=1}^{\infty} X_i$ is of constant width (by using the equivalence from the previous exercise). Deduce Theorem 11.4.

12 Zonotopes and projections of the cube

A **zonotope** in $\mathbb{R}^d$ is the Minkowski sum of finitely many compact segments. Consequently (see Exercise 2.6), zonotopes are convex. They are also compact.

12.1 Example. The Minkowski sum of 3 segments on the plane is a hexagon (see Figure 1.3).

12.2 Example. Recall Exercise 2.7: $B_\infty^n = [-e_1, e_1] + \cdots + [-e_d, e_d]$, so the cube is a zonotope.

Recall that we denote a segment between two points a, b in $\mathbb{R}^d$ by $[a, b]$ which is $\mathrm{conv}\{a, b\} = \{\lambda a + (1-\lambda)b,\ \lambda \in [0,1]\}$. Thus a zonotope K in $\mathbb{R}^n$ is a set of the form

$$K = \sum_{i=1}^n [a_i, b_i]$$

for some $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{R}^d$. Note that $[a_i, b_i] = a_i + [0, v_i]$ with $v_i = b_i - a_i$, so putting $a = a_1 + \ldots + a_n$, K can be written as

$$K = a + \sum_{i=1}^n [0, v_i],$$

If we let $A : \mathbb{R}^n \to \mathbb{R}^d$ to be the linear map sending e_i to v_i (which is surjective onto $\mathrm{span}\{v_1, \ldots, v_n\}$), then $A[0, e_i] = [0, v_i]$, thus

$$K = a + A([0, e_1] + \cdots + [0, e_n]) = a + A([0,1]^n).$$

This establishes the following fact.

12.3 Theorem. *Every zonotope is an affine image of the cube* $[0,1]^n$ *for some* n*. In particular, every zonotope is a centrally symmetric polytope.*

Since zonotopes are polytopes, it is reasonable to ask about their faces. They are zonotopes as well.

12.4 Theorem. *The faces of a zonotope are zonotopes.*

Proof. Let K be a zonotope in $\mathbb{R}^d$ and let F be its face given by a supporting hyperplane $H = \{x \in \mathbb{R}^d,\ \langle x, u\rangle = t\}$, that is $F = K \cap H$ and $K \subset H^-$. By Theorem 12.3, $K = a + A([0,1]^n)$ for some n, a linear map $A\colon \mathbb{R}^n \to \mathbb{R}^d$ and some $a \in \mathbb{R}^d$. Consider $\tilde{H} = \{x \in \mathbb{R}^n,\ \langle Ax, u\rangle = t - \langle a, u\rangle\}$. This is a hyperplane in $\mathbb{R}^n$ (because $\langle Ax, u\rangle = \left\langle x, A^\top u\right\rangle$), which defines a face $\tilde{F}$ of the cube $[0,1]^n$. Indeed, for every $x \in [0,1]^n$, we have $\langle Ax, u\rangle + \langle a, u\rangle = \langle Ax + a, u\rangle \le t$ because $Ax + a \in K$ and we have equality if and only if $Ax + a \in F$, that is $F = A(\tilde{F}) + a$. Finally, $\tilde{F}$ as a face of $[0,1]^n$ is a cube (recall Exercise 6.3), thus F, as its projection by A, is a zonotope. $\square$

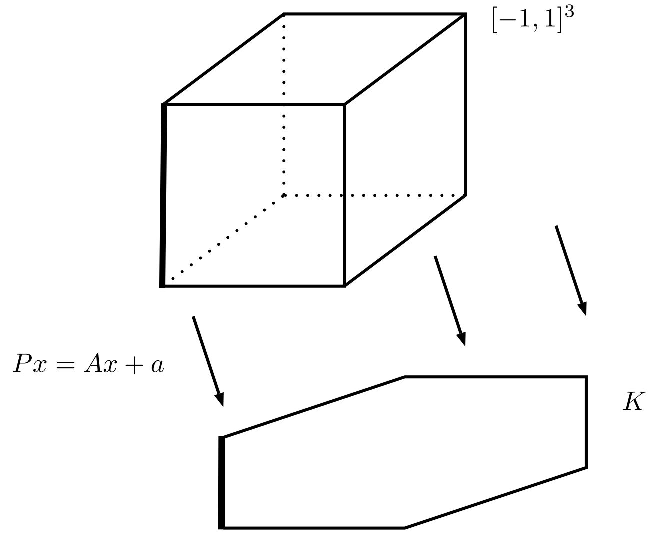

Figure 12.1: Zonotopes are projections of cubes. Preimages of their faces are faces of cubes (Theorem 12.4

12.1 Dissections and volume of zonotopes

For linearly independent vectors $v_1, \dots, v_k$ in $\mathbb{R}^d$ and numbers $\sigma_1, \dots, \sigma_k \in \{0, 1\}$ define the **parallelopiped**

$$P^{\sigma_1,\dots,\sigma_k}_{v_1,\dots,v_k} = \left\{ \sum_{j=1}^{k} \lambda_j v_j,\ \lambda_j \in [0,1) \text{ if } \sigma_j = 0 \text{ and } \lambda_j \in (0,1] \text{ if } \sigma_j = 1 \right\}.$$

and the trivial one $P_\varnothing = \{0\}$, corresponding to the empty set is just the origin. Every zonotope can be dissected (partitioned) into such parallelopipeds.

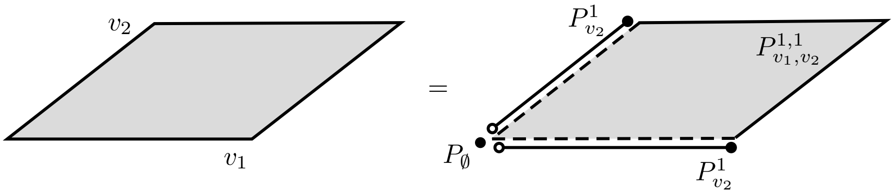

Figure 12.2: A dissection of a parallelopiped.

12.5 Theorem (Shephard, [28]). *Let $v_1, \dots, v_d$ be vectors in $\mathbb{R}^d$. Then the zonotope $Z = \sum_{i=1}^{n} [0, v_i]$ is a disjoint union of translates of parallelopipeds $P^{\sigma_1,\dots,\sigma_k}_{v_{i_1},\dots,v_{i_k}}$, where $\{v_{i_1}, \dots, v_{i_k}\}$ runs over all subsets of $\{v_1, \dots, v_n\}$ which are linearly independent and $\sigma_1, \dots, \sigma_k$ are appropriately chosen signs.*

Proof. We proceed by induction on n and $\dim(Z)$. When $n = 1$ or $\dim(Z) = 1$, the statement is clear. Suppose n and $\dim(Z)$ are both at least 2. Let $Z' = \sum_{i=1}^{n-1} [0, v_i]$. By induction, $Z' = P_1 \cup \dots \cup P_m$ for some parallelopipeds $P_1, \dots, P_m$ given by subsets of $\{v_1, \dots, v_{n-1}\}$ which are linearly independent. Let B be the union of those facets of Z whose outer normal u is such that $\langle u, v_n \rangle > 0$. Let π be the orthogonal projection onto $v_n^\perp$. Note that $\pi(Z)$ is a zonotope of dimension smaller than Z. By induction, $\pi(B) = \pi(Z) = Q_1 \cup \dots Q_l$ for some parallelopipeds $Q_1, \dots, Q_m$. Define $\tilde{P}_j = \pi^{-1}(Q_j) \cap B - [0, v_n]$. We obtain $\tilde{P}_1, \dots, \tilde{P}_l$ which form a partition of $Z \setminus Z'$. These are paralellopipeds corresponding to subsets containing v_n. Together with $P_1, \dots, P_m$ which form partition of Z', we thus obtain a desired partition of Z. □

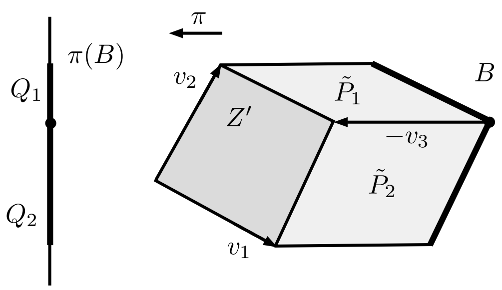

Figure 12.3: Proof of Theorem 12.5.

12.6 Corollary (Shephard's formula). *Let $v_1, \dots, v_d$ be vectors in $\mathbb{R}^d$. The volume of the zonotope $Z = \sum_{i=1}^n [0, v_i]$ equals*

$$|Z| = \sum_{1 \le i_1 < \dots < i_d \le n} |\det[v_{i_1} \ \dots \ v_{i_d}]|, \tag{12.1}$$

where $\det[v_{i_1} \ \dots \ v_{i_d}]$ *is the determinant of the $d \times d$ matrix with columns $v_{i_1}, \dots, v_{i_d}$.*

Proof. We partition Z according to Theorem 12.5. Only the d-dimensional parallelopipeds contribute to the volume of Z (the lower dimensional ones have d-dimensional volume 0). The volume of a parallelopiped $P_{w_1, \dots, w_d}$ is given by the determinant of the matrix with columns $w_1, \dots, w_d$, hence the formula. □

12.2 Volume of projections of the cube on orthogonal subspaces

Using the dissection theorem for zonotope and a fact about orthogonal matrices, we shall prove the following result about projections of the cube.

12.7 Theorem (McMullen, [20]). *Let V be a subspace of $\mathbb{R}^d$. Let P_V and $P_{V^\perp}$ be the orthogonal projections onto V and $V^\perp$. Then we have*

$$|P_V([0,1]^d)| = |P_{V^\perp}([0,1]^d)|$$

(the equality for the $\dim V$ *and* $d - \dim V$ *dimensional volumes of the corresponding projections).*

We start with a result from linear algebra saying that the determinants of complementary minors of an orthogonal matrix are equal.

12.8 Theorem. *Let U be a $d \times d$ real orthogonal matrix, that is $U^\top U = Id$. Let J be a nonempty subset of $\{1, \ldots, d\}$. Then*

$$|\det U_J| = |\det U_{J^c}|,$$

where $U_J = [u_{i,j}]_{i,j \in J}$ is the $|J| \times |J|$ matrix obtained from U by crossing out the rows and columns indexed by J^c.

Proof. Since permuting rows and columns changes determinant only up to a sign, without loss of generality we can assume that $J = \{1, \ldots, k\}$ for some $1 \le k \le d-1$. We divide U into blocks of first k rows and k columns etc.,

$$U = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right].$$

In particular, $A = U_J$ and $D = U_{J^c}$. Since U is orthogonal, we have

$$\begin{aligned}\left[\begin{array}{c|c} \mathrm{Id}_{k\times k} & 0 \\ \hline 0 & \mathrm{Id}_{d-k\times d-k} \end{array}\right] = \mathrm{Id}_{d-k} = U^\top U &= \left[\begin{array}{c|c} A^\top & C^\top \\ \hline B^\top & D^\top \end{array}\right]\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \\ &= \left[\begin{array}{c|c} A^\top A + C^\top C & A^\top B + C^\top D \\ \hline B^\top A + D^\top C & B^\top B + D^\top D \end{array}\right].\end{aligned}$$

In particular, $A^\top A + C^\top C = \mathrm{Id}$ and $A^\top B + C^\top D = 0$. Thus

$$\begin{aligned}\left[\begin{array}{c|c} A^\top & C^\top \\ \hline 0 & \mathrm{Id}_{d-k\times d-k} \end{array}\right] = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] &= \left[\begin{array}{c|c} A^\top A + C^\top C & A^\top B + C^\top D \\ \hline C & B^\top D \end{array}\right] \\ &= \left[\begin{array}{c|c} \mathrm{Id}_{k\times k} & 0 \\ \hline C & D \end{array}\right].\end{aligned}$$

Taking the determinants of both sides yields

$$\det A^\top \cdot \det U = \det D.$$

Since $\det U = \pm 1$, we conclude $|\det A| = |\det D|$, as desired. □

Proof of Theorem 12.7. Let $u_1, \ldots, u_k \in \mathbb{R}^d$ be an orthonormal basis of V and let $u_{k+1}, \ldots, u_d \in \mathbb{R}^d$ be an orthonormal basis of $V^\perp$. Let $v_i = P_V(e_i)$ and let $w_i = P_{V^\perp}(e_i)$, $1 \le i \le d$. The projections $Z = P_V([0,1]^d)$ and $Z' = P_{V^\perp}([0,1]^d)$ are zonotopes and

$$Z = \sum_{i=1}^d [0, v_i], \qquad Z' = \sum_{i=1}^d [0, w_i].$$

(recall $[0,1]^d = \sum_{i=1}^d [0, e_i]$). By Shephard's formula (12.1), we obtain

$$|Z| = \sum_{1 \le i_1 < \ldots < i_k \le d} |\det[\tilde{v}_{i_1} \ldots \tilde{v}_{i_k}]|$$

and

$$|Z'| = \sum_{1 \le j_1 < \ldots < j_{d-k} \le d} |\det[\tilde{w}_{j_1} \ldots \tilde{w}_{j_{d-k}}]|,$$

where the $\tilde{v}_i$ and $\tilde{w}_j$ are understood as the vectors in $\mathbb{R}^k$ and $\mathbb{R}^{d-k}$ respectively being the components of the v_i and w_j written in the orthonormal bases of V and $V^\top$ chosen earlier. Since $v_i = P_V(e_i) = \sum_{l=1}^k \langle e_i, u_l \rangle u_l$, the l-th component of v_i in the basis $\{u_1, \ldots, u_k\}$ of V is equal to $\langle e_i, u_l \rangle = u_{l,i}$, that is the i-component of u_l in the standard basis, or in other words the (l,i)-entry of U. The indices in the sum for volume of Z can be paired-up with the indices in the sum for the volume of Z': fix a set $I = \{i_1, \ldots, i_k\}$ of indices in the first sum and consider its complement, $I^c = \{j_1, \ldots, j_{d-k}\}$. If we let U be the $d \times d$ orthogonal matrix whose columns are $u_{i_1}, \ldots, u_{i_k}, u_{j_1}, \ldots, u_{j_{d-k}}$, then the minor $U_{\{1,\ldots,k\}}$ is exactly the matrix $[\tilde{v}_{i_1} \ldots \tilde{v}_{i_k}]$ and the minor $U_{\{1,\ldots,k\}}$ is the matrix $[\tilde{w}_{j_1} \ldots \tilde{w}_{j_{d-k}}]$. Thus by Theorem 12.8,

$$|\det[\tilde{v}_{i_1} \ldots \tilde{v}_{i_k}]| = |\det U_{\{1,\ldots,k\}}| = |\det U_{\{k+1,\ldots,d\}}| = |\det[\tilde{w}_{j_1} \ldots \tilde{w}_{j_{d-k}}]|,$$

consequently, $|Z| = |Z'|$. $\square$

12.3 Exercises

1. Give an example of a convex set which is not a zonotope.

2. Explain why the "in particular" part of Theorem 12.3 holds.

3. Let $v_1, \ldots, v_n \in \mathbb{R}^d$ and let $K = \sum_{i=1}^n [0, v_i]$ be a zonotope in $\mathbb{R}^d$. Show that $K = \operatorname{conv}\{\sum_{i=1}^n \sigma_i v_i,\ \sigma_1, \ldots, \sigma_n \in \{0,1\}\}$.

4. Show that a set K in $\mathbb{R}^2$ is a zonotope if and only if K is a symmetric polygon.

5. Show that B_1^d is not a zonotope for $d \geq 3$.

6. Show that the permutohedron of order n is a zonotope for every $n \geq 1$ (see Example 5.1).

13 Minkowski's theorem in geometry of numbers

Geometry of numbers is a field belonging to number theory which employs geometric arguments to establish number theoretic results. It was initiated by Minkowski in his seminal paper [21]. The basic result is his theorem which we shall discuss in this section, along with several easy applications.

13.1 Minkowski's theorem

The so-called Minkowski's (first) theorem in geometry of numbers asserts that every symmetric convex set which is *large* enough in terms of its volume contains a nonzero lattice point. The so-called Minkowski's second theorem is more sophisticated and we do not discuss it here.

13.1 Theorem (Minkowski)**.** *Let K be a symmetric convex set in $\mathbb{R}^d$ with its volume satisfying $|K| > 2^d$. Then K contains a nonzero point $x \in \mathbb{Z}^d$.*

13.2 Remark. The example of the open cube $K = (-1,1)^d$ shows that the constant 2^d in the volume bound cannot be replaced with any smaller number.

13.3 Remark. If K is additionally compact, then Minkowski's theorem holds true if $|K| \geq 2^d$. The reason being that we can consider $K_n = (1 + 1/n)K$, $n = 1, 2, \ldots$ and then we know that there are nonzero lattice points $x_n \in K_n$. By compactness, the sequence (x_n) has a convergent subsequence, say $x_{n_k} \to x$, which has to be in fact eventually constant to be convergent (as a sequence of integral vectors). Thus the limit x is a nonzero lattice point and $x \in \bigcap_n K_n = K$.

13.4 Remark. A first guess may be that K in fact need contain a lattice point x which is *close* to the origin. That is false: given a nonzero lattice point $x \in \mathbb{Z}^d$ with relatively prime coordinates, there is a symmetric convex set K with $|K| = 2^d$ such that the only nonzero lattice points in K are $\pm x$.

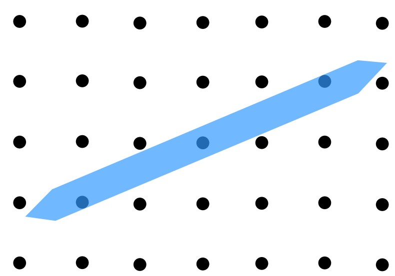

Figure 13.1: Minkowski's theorem.

Proof 1. Without loss of generality we can assume that K is bounded by considering $K \cap [-r,r]^d$ instead of K for some large $r > 0$. Let $K' = \frac{1}{2}K$. It suffices to show the following claim.

Claim. There is a nonzero point $x \in \mathbb{Z}^d$ such that $K' \cap (K' + x) \neq \varnothing$.

Indeed, if $K' \cap (K' + x) \neq \varnothing$, then $a' = b' + x$ for some $a', b' \in K$, that is $\frac{1}{2}a = \frac{1}{2}b + x$ for some $a, b \in K$ and then

$$x = \frac{a-b}{2} \in \frac{K-K}{2} = \frac{K+K}{2} = K$$

(the last equality follows from the convexity of K and the second last from the symmetry of K).

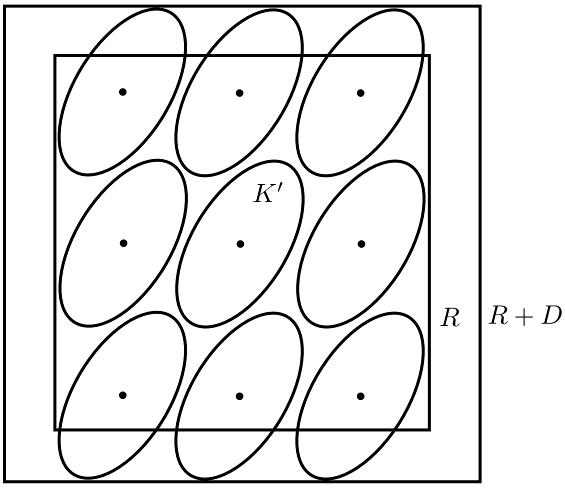

Figure 13.2: Translates of K' from $\mathcal{C}$ are pairwise disjoint, all contained in the cube $[-R-D, R+D]^d$.

For the proof of the claim, suppose it does not hold. Then take a positive integer R and consider the family of translates of K' by integer vectors from the box $[-R,R]^d$,

$$\mathcal{C} = \{K' + v,\ v \in [-R,R]^d \cap \mathbb{Z}^d\}.$$

Note that every two distinct translates are disjoint (if, say $(K' + v_1) \cap (K' + v_2) \neq \varnothing$, then $a_1 + v_1 = a_2 + v_2$ for some $a_1, a_2 \in K'$, so $a_1 = a_2 + v_2 - v_1$, that is $K' \cap (K' + v) \neq \varnothing$ for $v = v_2 - v_1 \in \mathbb{Z}^d$). All the translates are contained in $[-R-D, R+D]^d$, where $D = \operatorname{diam}(K')$. Taking the volume yields

$$(2R+2D)^d = |[-R-D, R+D]^d| \geq |\mathcal{C}| \cdot |K'| = (2R+1)^d \cdot |K'|,$$

thus

$$1 < \frac{1}{2^d}|K| = |K'| \leq \left(\frac{2R+2D}{2R+1}\right)^d.$$

Letting $R \to \infty$ gives a contradiction. □

Proof 2. We have the following lemma (which only relies on volume estimates, not convexity).

13.5 Lemma (Blichfeld, 1914)**.** *Let k be a positive integer and let A be a measurable subset of $\mathbb{R}^d$ with its volume satisfying $|A| > k$. Then there is a point $x \in \mathbb{R}^d$ such that $A + x$ contains at least $k+1$ points from $\mathbb{Z}^d$.*

Applying this lemma to $A = \frac{1}{2}K$ with $k = 1$ gives a translate $A + x$ which contains two (distinct) lattice points $p, q \in \mathbb{Z}^d$. Then $p - q$ is a nonzero lattice point and

$$p - q \in \frac{K + x}{2} - \frac{K + x}{2} = \frac{K - K}{2} = K,$$

where, again, the last equality holds because K is symmetric and convex. □

Proof of Lemma 13.5. Without loss of generality we can assume that A is bounded (consider $A \cap RB_2^d$ for large $R > 0$). Define the function

$$f(x) = \text{number of points from } \mathbb{Z}^d \text{ in } A + x = \sum_{y \in \mathbb{Z}^d} \mathbf{1}_{A+x}(y).$$

We show that its integral over $[0,1]^d$ is large,

$$\begin{aligned}\int_{[0,1]^d} f(x)\mathrm{d}x = \sum_{y \in \mathbb{Z}^d} \int_{[0,1]^d} \mathbf{1}_{A+x}(y)\mathrm{d}x &= \sum_{y \in \mathbb{Z}^d} \int_{-[0,1]^d + y} \mathbf{1}_A(t)\mathrm{d}t \\ &= \int_{\mathbb{R}^d} \mathbf{1}_A(t)\mathrm{d}t = |A| > k,\end{aligned}$$

so there is a point $x \in [0,1]^d$ with $f(x) > k$ and since f is integer valued, in fact, $f(x) \geq k + 1$. □

By a simple linear transformation, Theorem 13.1 can be generalised to arbitrary lattices. Given linearly independent vectors $z_1, \ldots, z_d$ in $\mathbb{R}^d$ we define the **lattice generated by** $z_1, \ldots, z_d$ as the following set (a discrete subgroup of $\mathbb{R}^d$)

$$\Lambda = \Lambda(z_1, \ldots, z_d) = \left\{ \sum_{i=1}^d x_i z_i, \ x_1, \ldots, x_d \in \mathbb{Z} \right\}.$$

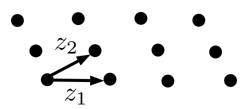

Figure 13.3: An example of a lattice.

The vectors $z_1, \ldots, z_d$ are called the basis of Λ. Of course, a particular lattice can be generated by many sets of vectors, that is it has many bases. For instance,

$$\mathbb{Z}^2 = \Lambda(\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]) = \Lambda(\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}3\\1\end{smallmatrix}\right])$$

For a lattice Λ in $\mathbb{R}^d$ generated by, say $z_1, \ldots, z_d$, we define its determinant, $\det \Lambda$ as the volume of the parallelopiped spanned by the generating vectors,

$$\det \Lambda = |\{\lambda_1 z_1 + \cdots + \lambda_d z_d,\ \lambda_1, \ldots, \lambda_d \in [0,1]\}| = |\det[z_1 \ldots z_d]|.$$

This is a well-defined quantity of the set Λ, that is it does not depend on the choice of the basis (z_i) (exercise). This quantity appears in a natural generalisation of Minkowski's theorem to arbitrary lattices.

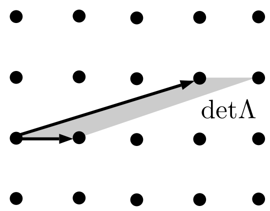

Figure 13.4: The determinant of a lattice Λ is the volume of the parallelopiped spanned by its basis.

13.6 Theorem (Minkowski's theorem for general lattices)**.** *Let Λ be a lattice in $\mathbb{R}^d$ and let K be a symmetric convex set in $\mathbb{R}^d$ such that its volume satisfies $|K| > 2^d \det \Lambda$. Then there is a nonzero point from Λ in K.*

Proof. Let vectors $z_1, \ldots, z_d$ generate Λ. Consider the linear bijection $f(x_1, \ldots, x_d) = \sum_{i=1}^d x_i z_i$, that is $f(e_j) = z_i$. We have $\Lambda = f(\mathbb{Z}^d)$ and note that $\det \Lambda = \det f$. Let $K' = f^{-1}(K)$. Then $|K'| = \frac{|K|}{|\det f|} = \frac{|K|}{\det \Lambda} > 2^d$. Thus, by Minkowski's theorem (Theorem 13.1), there is a nonzero point $x \in K' \cap \mathbb{Z}^d$. Then $f(x)$ is a nonzero point in K from Λ $\square$

13.2 Application: approximations by rationals

13.7 Theorem. *Let $\alpha \in (0,1)$ and let N be a positive integer. There are integers m, n with $0 < n \le N$ and $\left|\alpha - \frac{m}{n}\right| < \frac{1}{nN}$.*

Proof. Consider the set

$$K = \left\{(x,y) \in \mathbb{R}^2,\ -N - \frac{1}{2} \le x \le N + \frac{1}{2}, |\alpha x - y| < \frac{1}{N}\right\}.$$

This is a convex symmetric set (it is a parallelogram) with area

$$|K| = (2N+1)\frac{2}{N} > 4 = 2^2,$$

so by vanilla Minkowski's theorem (Theorem 13.1), there is a nonzero point $(n,m) \in K$ with integer coordinates m, n. By the definition of K, $n \neq 0$ and by symmetry, we can assume $n > 0$. Moreover, $n \le N + \frac{1}{2}$, so $n \le N$ and $|\alpha n - m| < \frac{1}{N}$. $\square$

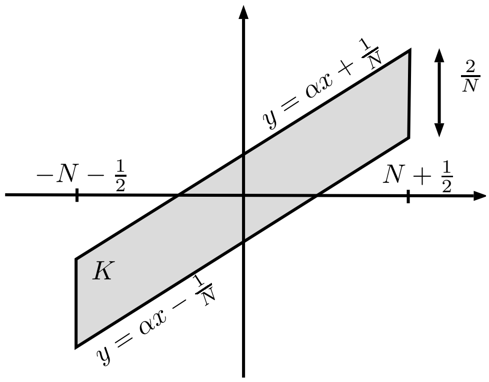

Figure 13.5: The parallelogram K used in the proof of Theorem 13.7.

13.3 Application: The sum of two squares theorem

As an application of Minkowski's theorem for general lattices, we give a proof of the sum of two squares theorem.

13.8 Theorem. *Let p be a prime number with $p \equiv 1 \pmod 4$. Then $p = x^2 + y^2$ for some integers x and y.*

Recall that for a prime p, $F_p = \{0, 1, \ldots, p-1\}$ is the finite field of residues modulo p with the usual addition and multiplication modulo p. We recall two elementary facts.

13.9 Lemma. *For every prime number p, we have $(p-1)! \equiv -1 \pmod p$.*

Proof. First remark that the equation $x^2 \equiv 1 \pmod p$, $x \in F_p$ has only two solutions: $x = 1$ and $x = p - 1 \equiv -1$ (these obviously solve the equation and there cannot be more solutions because it is a polynomial equation of degree 2). Consequently, for every $x \in F_p$, $x \neq 1, p-1$, there is a unique $y \in F_p$, $y \neq x$ such that $xy \equiv 1 \pmod p$, namely the inverse of x in F_p. So all the elements $2, 3, \ldots, p-2$ are joined in pairs with the product of each pair being 1 modulo p. Thus

$$(p-1)! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (p-2) \cdot (p-1) \equiv 1 \cdot (p-1) \equiv -1 \pmod p.$$

$\square$

13.10 Lemma. *Let p be a prime number with $p \equiv 1 \pmod 4$. Then $x^2 \equiv -1 \pmod p$ for some integer x.*

Proof. If the equation $x^2 \equiv -1 \pmod p$ has no solution in F_p, then for every $x \in F_p$, there is a unique $y \in F_p$, $y \neq x$ such that $xy \equiv -1 \pmod p$, namely "minus the inverse of x" in F_p. Then, pairing up the elements $1, \ldots, p-1$ accordingly,

$$(p-1)! \equiv (-1)^{\frac{p-1}{2}} \equiv 1 \pmod p,$$

where the last equivalence holds because $\frac{p-1}{2}$ is even (by the assumption). This however contradicts Lemma 13.9. $\square$

Proof of Theorem 13.8. By Lemma 13.10, choose an integer q with $q^2 \equiv -1 \pmod{p}$. Consider the lattice Λ generated by the vectors $z_1 = \left[\begin{smallmatrix} 1 \\ q \end{smallmatrix}\right]$ and $z_2 = \left[\begin{smallmatrix} 0 \\ p \end{smallmatrix}\right]$. We have $\det \Lambda = p$. Let K be a centred open disk of radius $\sqrt{2p}$,

$$K = \{(x, y) \in \mathbb{R}^2,\ x^2 + y^2 < 2p\}.$$

Since

$$|K| = 2p\pi > 4p = 2^2 \det \Lambda,$$

by Theorem 13.6, there is a nonzero point (a, b) in $K \cap \Lambda$. In particular, $(a, b) = k_1 z_1 + k_2 z_2 = (k_1, k_1 q + k_2 p)$ for some integers k_1, k_2 and we get

$$a^2 + b^2 = k_1^2 + (k_1 q + k_2 p)^2 \equiv k_1^2 + k_1^2 q^2 \equiv k_1^2 - k_1^2 = 0 \pmod{p},$$

that is $a^2 + b^2$ is divisible by p. Since (a, b) is nonzero and is in K, we have $0 < a^2 + b^2 < 2p$. Thus, $a^2 + b^2 = p$. $\square$

13.4 Exercises

1. Let m be a positive integer. Show that every symmetric convex set K in $\mathbb{R}^d$ with its volume satisfying $|K| > m \cdot 2^d$ contains at least m nonzero distinct points from $\mathbb{Z}^d$.

2. Let $\alpha \in (0, 1)$ be irrational. Using Theorem 13.7, show that there are infinitely many integers m, n such that $|\alpha - \frac{m}{n}| < \frac{1}{n^2}$.

3. Let $\alpha_1, \ldots, \alpha_k \in (0, 1)$ and let N be a positive integer. Show that there are integers $m_1, n_1 \ldots, m_k, n_k$ with $0 < n_j \leq N$ and $\left|\alpha - \frac{m_j}{n_j}\right| < \frac{1}{n_j N^{1/k}}$ for every $j \leq k$.

4. If Z, Z' are two $d \times d$ matrices with column vectors being bases (z_i) and (z_i') of the same lattice Λ in $\mathbb{R}^d$, then $Z = UZ'$ for a $d \times d$ integral matrix U with $\det U = \pm 1$. In particular, $\det Z = \pm \det Z'$ and thus $\det \Lambda$ is well-defined.

5. Let $A = [a_{ij}]$ be a $d \times d$ matrix with $\det A = 1$. Let $b_1, \ldots, b_d$ be positive numbers with $\prod_{i=1}^d b_i = 1$. Show that there is nonzero integral solution x to the system of inequalities $|\sum_{j=1}^d a_{ij} x_j| \leq b_i$, $i = 1, \ldots, d$.

14 The plank problem

How many planks of width 1 foot each do you need to cover a circular tabletop of diameter k feet? Obviously k planks suffice, but can we do any better?

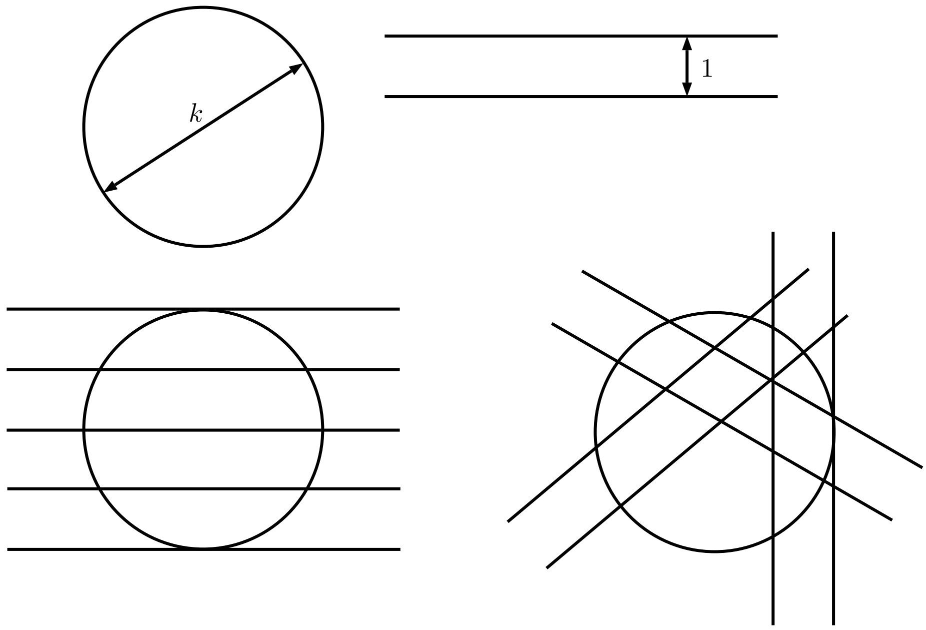

Figure 14.1: Obviously k planks suffice to cover the disk of diameter k, but can we do better?.

Tarski asked in [31] a general question of this sort about covering convex sets by strips and solved the planar case (the so-called plank problem). About 20 years later Bang in [3] resolved Tarski's question in the affirmative.

A **strip** S in $\mathbb{R}^d$ is a set of the form $S = \{x \in \mathbb{R}^d,\ s \leq \langle x, a\rangle \leq t\}$ for some vector $a \in \mathbb{R}^d$ and reals $s < t$. That is, S is the set of all points between two parallel hyperplanes perpendicular to a. The **width** $w(S)$ of S is simply the distance between those hyperplanes, that is $w(S) = \frac{t-s}{|a|}$. Recall the definition of the **width** of a convex set K, denoted $w(K)$ as the minimal width of a strip containing K (Chapter 3.1).

14.1 Theorem (Bang, [3])**.** *Let K be a convex set in $\mathbb{R}^d$ of width w. If K is covered by finitely many strips, then the sum of their widths is at least w.*

14.1 Archimedes' Hat-Box Theorem and a solution on the plane

Archimedes' hat-box theorem asserts that if we intersect a Euclidean sphere in $\mathbb{R}^3$ with a strip in $\mathbb{R}^3$, then the surface measure of the intersection depends only on the width of the strip (see Figure 14.2). This unique feature of dimension 3 is explained in the following lemma.

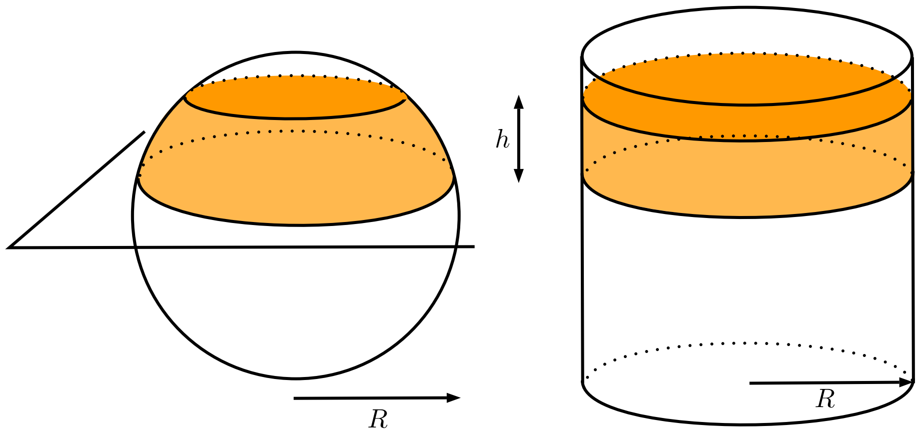

Figure 14.2: Archimedes' hat-box theorem.

14.2 Lemma. *Let S^{d-1} be the centred unit sphere in $\mathbb{R}^d$. Then for every unit vector u in $\mathbb{R}^d$ and $-1 \le a \le b \le 1$, we have*

$$|\{x \in S^{d-1},\ a \le \langle x, u\rangle \le b\}| = |S^{d-2}| \int_a^b (1-t^2)^{\frac{d-3}{2}} \mathrm{d}t.$$

In particular, when $d = 3$, the integral becomes simply $b - a$.

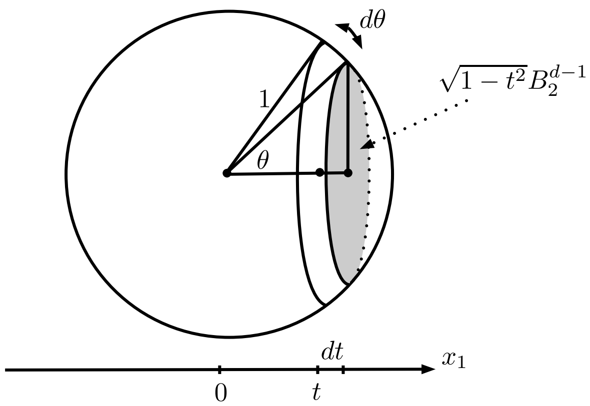

Figure 14.3: The surface measure of a thin strip equals perimeter × height.

Proof. By symmetry, we can assume $u = e_1$. The surface measure of a thin strip $t \le x_1 \le t + \mathrm{d}t$ equals $|\partial(\sqrt{1-t^2}B_2^{d-1})|\mathrm{d}\theta = \sqrt{1-t^2}^{\,d-2}|\partial B_2^{d-1}|\mathrm{d}\theta$. Since $t = \cos\theta$, we have $\mathrm{d}\theta = \frac{\mathrm{d}t}{\sin\theta} = \frac{\mathrm{d}t}{\sqrt{1-t^2}}$, so

$$\begin{aligned}|\{x \in S^{d-1},\ a \le \langle x, u\rangle \le b\}| &= |\partial B_2^{d-1}| \int_a^b \sqrt{1-t^2}^{\,d-2} \frac{\mathrm{d}t}{\sqrt{1-t^2}} \\ &= |S^{d-2}| \int_a^b (1-t^2)^{\frac{d-3}{2}} \mathrm{d}t\end{aligned}$$

which finishes the proof. $\square$

We leave it as an exercise to deduce the solution to Tarski's question for a disk on the plane.

14.2 Bang's solution

Let K be a convex set in $\mathbb{R}^d$ of width $w(K)$. We begin with 3 lemmas.

14.3 Lemma. *Given an arbitrary direction, there is a chord in K in that direction of length at least $w(K)$.*

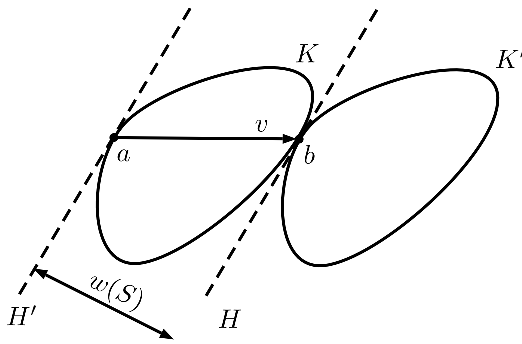

Figure 14.4: Proof of Lemaa 14.3.

Proof. Let a segment $[a, b]$ be a longest chord in K in the given direction. Translate K by $v = b - a$ to obtain K'. Note that K and K' have no interior points in common (otherwise, there is a longer cord, for if a ball $q+\epsilon B_2^n$ is in $\text{int}(K)\cap\text{int}(K')$, by translating "back", the ball $q - v + \epsilon B_2^n$ is in $\text{int}(K)$ and the chord $[q - v, q]$ can be extended within K to a longer one). Thus, K and K' can be separated by a hyperplane, say H. The translate of H by $-v$, say H' is a supporting hyperplane for K (because it is for K'). Then the strip S formed by H and H' contains K, so

$$w(K) \le w(S) \le |b - a|,$$

so $[a, b]$ is the desired chord. $\square$

14.4 Lemma. *If v is a vector in $\mathbb{R}^d$ with $|v| < \frac{1}{2}w(K)$, then*

$$w\Big((K - v) \cap (K + v)\Big) \ge w(K) - 2|v|.$$

Proof. By Lemma 14.3, there is a chord $[a, b]$ parallel to v of length $l = |b-a|$ satisfying $l \ge w(K) > 2|v|$. Thus

$$[a + v, b - v] \subset (K - v) \cap (K + v).$$

Without loss of generality we can assume that the origin is the midpoint of the chord $[a,b]$. We consider the dilate $\tilde{K} = \left(1 - \frac{2|v|}{l}\right) K$. Plainly,

$$w(\tilde{K}) = \left(1 - \frac{2|v|}{l}\right) w(K) \geq \left(1 - \frac{2|v|}{w(K)}\right) w(K) = w(K) - 2|v|.$$

Consequently, showing $\tilde{K} \subset (K - v) \cap (K + v)$ will finish the proof. Observe that for every $x \in K$, we have

$$\left(1 - \frac{2|v|}{l}\right) x \pm v = \left(1 - \frac{2|v|}{l}\right) x + \frac{2|v|}{l} \left(\pm \frac{l}{2} \frac{v}{|v|}\right).$$

The left hand side is an arbitrary point of $\tilde{K} \pm v$. Since $\pm \frac{l}{2} \frac{v}{|v|}$ is either a or b, which are both in K, by convexity, we see that the right hand side is a point in K, hence $\tilde{K} \pm v \subset K$, that is $\tilde{K} \subset (K - v) \cap (K + v)$. □

14.5 Lemma. *If $v_1, \ldots, v_m$ are vectors in $\mathbb{R}^d$ such that $2|v_1| + \cdots + 2|v_m| < w(K)$, then*

$$w\left(\bigcap_{\varepsilon = (\varepsilon_1, \ldots, \varepsilon_m) \in \{-1,1\}^m} K - \sum_{i=1}^m \varepsilon_i v_i \right) > 0.$$

Proof. Considering

$$\begin{aligned}
K_1 &= (K - v_1) \cap (K + v_1), \\
K_2 &= (K_1 - v_2) \cap (K_1 + v_2) \\
&= (K - v_1 - v_2) \cap (K + v_1 - v_2) \cap (K - v_1 + v_2) \cap (K + v_1 + v_2), \\
&\vdots \\
K_m &= (K_{m-1} - v_m) \cap (K_{m-1} + v_m) \\
&= \bigcap_{\varepsilon \in \{-1,1\}^m} \left(K - \sum_{i=1}^m \varepsilon_i v_i \right),
\end{aligned}$$

and repetitively using Lemma 14.4 gives assertion. We leave the details as an exercise. □

We are ready to prove the main theorem.

Proof of Theorem 14.1. Suppose we are given strips

$$S_i = \{x \in \mathbb{R}^d, \ |\langle x, v_i \rangle - c_i| \leq |v_i|^2\}, \qquad i = 1, \ldots, m$$

($v_i \in \mathbb{R}^d$ are normal vectors and $c_i \in \mathbb{R}$ specify positions: $\frac{c_i v_i}{|v_i|^2}$ is a centre of symmetry of S_i). Note that the width of S_i is $2|v_i|$. We show the contrapositive: suppose $2|v_1| + \cdots + 2|v_m| < w(K)$ and we shall argue that $S_1 \cup \cdots \cup S_m \not\supset K$ (the strips do not cover K). By Lemma 14.5, the set $\bigcap_\varepsilon (K - \sum \varepsilon_i v_i)$ has positive width, in particular, it has nonempty interior. By translating if needed, suppose that 0 is in its interior. Then for every $\varepsilon \in \{-1,1\}^m$, we have that $\sum \varepsilon_i v_i$ is an interior point of K, so there is $\lambda > 1$ such that for every $\varepsilon \in \{-1,1\}^m$, we have

$$\lambda \sum_{i=1}^{m} \varepsilon_i v_i \in K. \tag{14.1}$$

We consider the function $f\colon \{-1,1\}^m \to \mathbb{R}$, defined by

$$f(\varepsilon) = \lambda \left| \sum_{i=1}^{m} \varepsilon_i v_i \right|^2 - 2\sum_{i=1}^{m} \varepsilon_i c_i.$$

It achieves its maximal value as ε varies over $\{-1,1\}^m$ at some point $\eta \in \{-1,1\}^m$. For $i = 1, \ldots, m$ denote

$$\sigma_i \eta = (\eta_1, \ldots, \eta_{i-1}, \eta_i, \eta_{i+1}, \ldots, \eta_d)$$

(the vector η with i coordinate flipped). Fix $1 \le j \le d$. Using $|a|^2 - |b|^2 = \langle a-b, a+b \rangle$, we have,

$$\begin{aligned}
0 &\le f(\eta) - f(\sigma_j \eta) \\
&= \lambda \left(\left| \sum_{i=1}^{m} \eta_i v_i \right|^2 - \left| \sum_{i=1}^{m} (\sigma_j \eta)_i v_i \right|^2 \right) - 2 \left(\sum_{i=1}^{m} \eta_i c_i - \sum_{i=1}^{m} (\sigma_j \eta)_i c_i \right) \\
&= \lambda \left\langle 2\eta_j v_j, 2\sum_{i \neq j} \eta_i v_i \right\rangle - 4\eta_j c_j \\
&= -4\lambda |v_j|^2 + 4\lambda \left\langle \eta_j v_j, \sum_{i} \eta_i v_i \right\rangle - 4\eta_j c_j.
\end{aligned}$$

If we let $x = \lambda \sum_i \eta_i v_i$, then the right hand side becomes $4\eta_j(\langle x, v_j \rangle - c_j) - 4\lambda |v_j|^2$. Since it is nonnegative, we get

$$\eta_j(\langle x, v_j \rangle - c_j) \ge \lambda |v_j|^2$$

which means that $x \notin S_j$. This holds for every j, thus the point x, which is in K (recall (14.1)), is not covered by the union of the S_j, as desired. $\square$

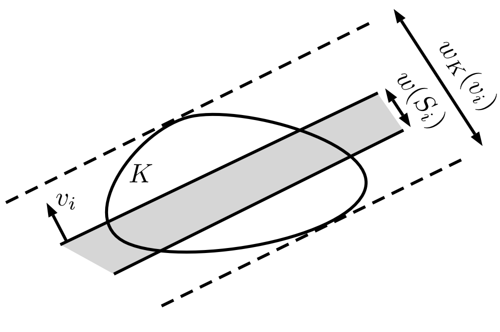

Figure 14.5: Relative width.

Recall the definition of a width of K in a given direction v, denoted $w_K(v)$: it is the minimal width of a strip with normal v containing K (Chapter 3.1). Bang in his paper posed the following question: suppose a convex set K in $\mathbb{R}^d$ is covered by a union of strips $S_1, \dots, S_m$ with normals $v_1, \dots, v_m$; does this imply that

$$\frac{w(S_1)}{w_K(v_1)} + \ldots + \frac{w(S_m)}{w_K(v_m)} \geq 1?$$

The ratios $\frac{w(S_i)}{w_K(v_i)}$ can be thought of as relative widths in directions v_i. Since we have $w_K(v_i) \geq w(K)$, the above immediately implies that $w(S_1) + \cdots + w(S_m) \geq w(K)$, which is Bang's theorem. Ball in [5] gave an affirmative answer to Bang's question for symmetric sets. The nonsymmetric case remains unsolved (even on the plane!).

14.3 Exercises

1. Using Archimedes' hat-box theorem argue that if a disk of width 1 is covered by strips of widths $w_1, \ldots, w_l$, then $w_1 + \cdots + w_l \geq 1$.

2. Fill out the details in the proof of Lemma 14.5.

A Appendix: Euler's formula

Using graphs, we shall prove the following useful relationship between the number of vertices, edges and facets of polytopes in $\mathbb{R}^3$.

A.1 Theorem (Euler's formula)**.** *Let P be a 3-dimensional polytope in $\mathbb{R}^3$ with v vertices, e edges and f facets. Then*

$$v - e + f = 2. \tag{A.1}$$

A standard and quick proof starts with constructing a *graph* of P which is planar: take an interior point of P and radially project from it the skeleton of P (the union of its edges) onto a large sphere S containing P. This gives a graph of P on S which is simple and has nonintersecting edges. It remains to choose a point in the interior of one of the faces of this graph as the north pole and do a stereographical projection to obtain a simple connected planar graph of P with the number of vertices, edges and faces (including the outer one corresponding to the face from which we projected) being respectively v, e, f. Then (A.1) follows from a corresponding theorem for planar graphs, which can be shown by induction. The faces of a planar graph are the connected components of $\mathbb{R}^2 \setminus$ (a drawing of G).

Figure A.1: The faces are the connected components of the plane with the graph removed. This planar graph has 4 faces.

A.2 Theorem (Euler's formula for graphs)**.** *Let G be a connected planar graph with v vertices, e edges and f faces. Then*

$$v - e + f = 2. \tag{A.2}$$

Proof. We proceed by induction on the number of faces f. If $f = 1$, then G has no cycles, so G is a tree, so $v = e + 1$ and thus $v - e + f = (e + 1) - e + 1 = 2$. Suppose $f \geq 2$. Then G has a cycle C. The cycle C separates the plane into two connected components. Choose an edge of the cycle C. By removing this edge from G, we decrease the number of edges and faces by 1 without changing the number of vertices, that is the new graph has v vertices, $e - 1$ edges and $f - 1$ faces. By induction, $v - (e - 1) + (f - 1) = 2$, which gives $v - e + f = 2$. $\square$

A.3 Corollary. *Let G be a connected planar graph with v vertices, e edges and f facets. Then*

$$\begin{aligned} e &\leq 3v - 6, \\ f &\leq 4v - 2. \end{aligned} \tag{A.3}$$

Consequently, if P is a 3-dimensional polytope in $\mathbb{R}^3$ with v vertices, e edges and f facets, then the same inequalities hold.

Proof. Suppose the graph G has v vertices, e edges and f faces. By double-counting the pairs (edge,face) in G (every edge belongs to 2 faces and every face contains at least 3 edges), we get $2e \geq 3f$. Since, by Euler's formula (A.2), $f = 2 + e - v$, we get $2e \geq 6 + 3e - 3v$, that is $e \leq 3v - 6$. Similarly for the number of facets f. The statement for polytopes follows by considering their graphs as before. $\square$

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