Convex and discrete geometry 21-366 **TT**

Final take-home exam

Work on all question but submit your solutions to only 6 of them. Due the 6th of December.

1. Consider the unit disk $D = \{x \in \mathbb{R}^2,\ x_1^2 + x_2^2 \le 1\}$. Give an example of a set A different than $\frac{1}{2}D$ such that $A - A = D$.

2. If sets $K_1, \dots, K_n$ in $\mathbb{R}^d$ are convex, then so is their Minkowski sum $K_1 + \cdots + K_n$.

3. Let P be a symmetric convex polytope in $\mathbb{R}^d$. Show that for some n, there is a d-dimensional subspace F in $\mathbb{R}^n$ and a linear injective map $T\colon \mathbb{R}^d \to \mathbb{R}^n$ such that $[-1,1]^n \cap F = T(P)$ (in words, every symmetric polytope is a central section of the cube in sufficently high dimension).

4. Show that a permutohedron of order n has $n!$ vertices.

5. (a) Let X be subset of $\mathbb{R}^d$. If every $d+1$ points from X can be covered by a (closed) ball of radius r, then X can be covered by such a ball.

   (b) Every set of $d+1$ points in $\mathbb{R}^d$ of diameter at most 2 can be covered by a closed ball of radius $r \le \sqrt{\frac{2d}{d+1}}$ (which is sharp for a regular simplex).

   (c) If X is a subset of $\mathbb{R}^d$ with diameter at most 2, then X can be covered by a closed ball of radius at most $\sqrt{\frac{2d}{d+1}}$ (Jung's theorem).

   (d)* If such X does not lie in any smaller ball, then the closure of X contains the vertices of a regular d-dimensional simplex of edge-length 2.

6. Let K be a compact convex set in $\mathbb{R}^2$ with support function h_K.

   (a) If K is a polygon, then $|\partial K| = \int_0^{2\pi} h_P(\cos\theta, \sin\theta)\mathrm{d}\theta$ ($|\partial K|$ is the perimeter of K).

   (b) If K_1 and K_2 are two convex polygons in $\mathbb{R}^2$ such that $K_1 \subset K_2$, then we have $|\partial K_1| \le |\partial K_2|$, with equality if and only if $K_1 = K_2$.

   (c)* By approximation arguments, show that (a) is valid for all compact convex planar sets K. Deduce Barbier's theorem: *all plane convex sets of constant width b have the same perimeter πb.*

7. Prove that for every nonnegative numbers $\alpha_1, \dots, \alpha_d$ and $\beta_1, \dots, \beta_d$, we have

$$\left(\prod_{i=1}^d (\alpha_i + \beta_i)\right)^{1/d} \ge \left(\prod_{i=1}^d \alpha_i\right)^{1/d} + \left(\prod_{i=1}^d \beta_i\right)^{1/d}.$$

8. Show the following analogue of the Brunn-Minkowski inequality: for $d \times d$ positive semi-definite real matrices A and B, we have

$$\big[\det(A+B)\big]^{1/d} \geq \big[\det(A)\big]^{1/d} + \big[\det(B)\big]^{1/d}.$$

9. Show that all norms on $\mathbb{R}^d$ are equivalent, that is if $\|\cdot\|$ and $\|\cdot\|'$ are two norms on $\mathbb{R}^d$, then there are positive finite constants α, β such that for every x in $\mathbb{R}^d$, we have

$$\alpha\|x\| \leq \|x\|' \leq \beta\|x\|.$$

10. Let $\|\cdot\|$ be a norm on $\mathbb{R}^d$ and let $K = \{x \in \mathbb{R}^d,\ \|x\| \leq 1\}$ be its unit ball. Show that K is symmetric, convex, compact, with nonempty interior (K is a symmetric convex body).

11. Let K be symmetric convex body in $\mathbb{R}^d$. Define for $x \in \mathbb{R}^d$, $p_K(x) = \inf\{t > 0,\ x \in tK\}$ (the so-called Minkowski's functional of K). Show that p_K is a norm on $\mathbb{R}^d$ and its unit ball is K.

12. Let $p \in (1, \infty)$. Find an ℓ_p-equilateral set in $\mathbb{R}^d$ of size $d+1$.

13.* Show that n points on the plane determine at most

(a) $O(n^{7/3})$ triangles with a given angle α,

(b) $O(n^{7/3})$ triangles with area 1,

(c) $O(n^{7/3})$ isosceles triangles.