Probability 21-325 **Homework 9 (due 27th March)** **TT**

1. Let X be a nonnegative random variable. Show that for $p > 0$, we have

$$\mathbb{E}X^p = \int_0^\infty p t^{p-1} \mathbb{P}\left(X > t\right) \mathrm{d}t.$$

2. Let X be a random variable such that $\mathbb{E}|X|^p < \infty$ for some $p > 0$. Show that $\lim_{t\to\infty} t^p \mathbb{P}\left(|X| > t\right) = 0$.

3. Show that the probability that in n throws of a fair die the number of sixes lies between $\frac{1}{6}n - \sqrt{n}$ and $\frac{1}{6}n + \sqrt{n}$ is at least $\frac{31}{36}$.

4. Let X be a random variable with values in an interval $[0, a]$. Show that for every t in this interval, we have

$$\mathbb{P}\left(X \geq t\right) \geq \frac{\mathbb{E}X - t}{a - t}.$$

5. Prove the Payley-Zygmund inequality: for a nonnegative random variable X and every $\theta \in [0, 1]$, we have

$$\mathbb{P}\left(X > \theta \mathbb{E}X\right) \geq (1-\theta)^2 \frac{(\mathbb{E}X)^2}{\mathbb{E}X^2}.$$

6. Let $\varepsilon_1, \ldots, \varepsilon_n$ be independent random signs. Prove that there is a positive constant c such that for every $n \geq 1$ and real numbers $a_1, \ldots, a_n$, we have

$$\mathbb{P}\left(\left|\sum_{i=1}^n a_i \varepsilon_i\right| > \frac{1}{2}\sqrt{\sum_{i=1}^n a_i^2}\right) \geq c.$$

Hint. Use the Paley-Zygmund inequality and Q4 HW5.

7. Prove that for nonnegative random variables X and Y, we have

$$\mathbb{E}\frac{X}{Y} \geq \frac{(\mathbb{E}\sqrt{X})^2}{\mathbb{E}Y}.$$

8. Let $X, X_1, X_2, \ldots$ be identically distributed random variables such that $\mathbb{P}\left(X > t\right) > 0$ for every $t > 0$. Suppose that for every $\eta > 1$, we have $\lim_{t\to\infty} \frac{\mathbb{P}(X > \eta t)}{\mathbb{P}(X > t)} = 0$. For $n \geq 1$, let a_n be the smallest number a such that $n\mathbb{P}\left(X > a\right) \leq 1$. Show that for every $\varepsilon > 0$, we have $\max_{i\leq n} X_i \leq (1+\varepsilon)a_n$ with high probability as $n \to \infty$, i.e. $\mathbb{P}\left(\max_{i\leq n} X_i \leq (1+\varepsilon)a_n\right) \xrightarrow[n\to\infty]{} 1$.

9. Let $n \geq 1$, $p \in (0,1)$ and let $X_{i,j}$, $1 \leq i < j \leq n$ be i.i.d. Ber(p) random variables. Let $G = (V, E)$ be an undirected simple graph with the vertex set $V = \{1, \ldots, n\}$ and the (random) edge set $E = \{\{i,j\}, X_{i,j} = 1, 1 \leq i < j \leq n\}$ (the so-called Erdös-Rényi, a.k.a. $G_{n,p}$ model). Show that for every $\varepsilon > 0$, if $p > (1+\varepsilon)\frac{\log n}{n}$, then G has no isolated vertices with high probability as $n \to \infty$, i.e. $\mathbb{P}(G \text{ has no isolated vertices}) \xrightarrow[n\to\infty]{} 1$.

10. Let X be an integrable random variable and define

$$X_n = \begin{cases} -n, & X < -n \\ X, & |X| \leq n \\ n, & X > n. \end{cases}$$

Does the sequence X_n converge a.s., in L_1, in probability?

11.* Let $\varepsilon_1, \varepsilon_2, \ldots$ be i.i.d. symmetric random signs. Show that there is a constant $c > 0$ such that for every $n \geq 1$ and reals $a_1, \ldots, a_n$, we have

$$\mathbb{P}\left(\left|\sum_{i=1}^n a_i \varepsilon_i\right| \leq \sqrt{\sum_{i=1}^n a_i^2}\right) \geq c.$$