Probability 21-325 **Homework 12 (due 17th April)** **TT**

1. Let $X_1, X_2, \ldots$ be random variables such that $\mathbb{P}\left(X_n = \frac{k}{n}\right) = \frac{1}{n}$, $k = 1, \ldots, n$, $n = 1, 2, \ldots$. Does the sequence (X_n) converge in distribution? If yes, find the limiting distribution.

2. Let $U_1, U_2, \ldots$ be i.i.d. random variables uniformly distributed on $[0,1]$. Let $X_n = \min\{U_1, \ldots, U_n\}$. Show that nX_n converges in distribution to an exponential random variable with parameter one.

3. Suppose that $X, X_1, X_2, \ldots$ are nonnegative integer-valued random variables. Show that $X_n \xrightarrow[n\to\infty]{d} X$, if and only if $\mathbb{P}(X_n = k) \xrightarrow[n\to\infty]{} \mathbb{P}(X = k)$, for every $k = 0, 1, 2, \ldots$.

4. For $p \in [0,1]$, let X_p be a Geometric random variable with parameter p. Show that the sequence $(\frac{1}{n} X_{1/n})$ converges in distribution to an exponential random variable with parameter 1.

5. Let $X_1, X_2, \ldots$ be i.i.d. random variables uniform on $\{1, 2, \ldots, n\}$. Let

$$N_n = \min\{l \geq 2,\ X_k = X_l \text{ for some } k < l\}.$$

In the birthday problem (HW1 Q4), we showed that $\mathbb{P}(N_n > k) = \prod_{j=1}^{k-1}\left(1 - \frac{j}{n}\right)$, for every integer $k \geq 1$. Explain why. For every $t \geq 0$ show that $\lim_{n\to\infty} \mathbb{P}\left(\frac{N_n}{\sqrt{n}} > t\right) = e^{-t^2/2}$ and show that the sequence $(\frac{N_n}{\sqrt{n}})$ converges in distribution to a random variable with density $xe^{-x^2/2}\mathbf{1}_{x\geq 0}$.

6. Let $X_1, X_2, \ldots$ be i.i.d. exponential random variables with parameter 1. Let $M_n = \max\{X_1, \ldots, X_n\}$. Show that $M_n - \log n$ converges in distribution to a random variable with the distribution function $e^{-e^{-x}}$, $x \in \mathbb{R}$.

Hint: this was essentially done in HW6 Q3.

7. Show that for positive t, $\int_t^\infty e^{-x^2/2}\mathrm{d}x \leq \frac{1}{t} e^{-t^2/2}$ and $\int_t^\infty e^{-x^2/2}\mathrm{d}x \geq \frac{t}{t^2+1} e^{-t^2/2}$. Conclude that for a standard Gaussian random variable Z and positive t,

$$\frac{1}{\sqrt{2\pi}} \frac{t}{t^2+1} e^{-t^2/2} \leq \mathbb{P}(Z > t) \leq \frac{1}{\sqrt{2\pi}} \frac{1}{t} e^{-t^2/2}$$

and

$$\lim_{t\to\infty} \frac{\mathbb{P}(Z > t)}{\frac{1}{\sqrt{2\pi}} \frac{1}{t} e^{-t^2/2}} = 1.$$

8. Let $X_1, X_2, \ldots$ be i.i.d. standard Gaussian random variables. For $n = 2, 3, \ldots$ let b_n be such that $\mathbb{P}(X_1 > b_n) = \frac{1}{n}$. Show that $\lim_{n\to\infty} \frac{b_n}{\sqrt{2\log n}} = 1$. Let $M_n = \max\{X_1, \ldots, X_n\}$. Show that $b_n(M_n - b_n)$ converges in distribution to a random variable with the distribution function $e^{-e^{-x}}$, $x \in \mathbb{R}$.

Hint: Using Q7, first show that for every $a \in \mathbb{R}$, $\lim_{t\to\infty} \frac{\mathbb{P}\left(X_1 > t + \frac{a}{t}\right)}{\mathbb{P}(X_1 > t)} = e^{-a}$.

9.* Show that for every $t \geq 0$,

$$\frac{2}{t + \sqrt{t^2 + 4}} e^{-t^2/2} < \int_t^\infty e^{-x^2/2} \mathrm{d}x < \frac{2}{t + \sqrt{t^2 + 2}} e^{-t^2/2}.$$

10.* For a random variable X,

(a) we define its **essential supremum** as

$$\text{ess sup } X = \inf\{M > 0,\ \mathbb{P}(X \leq M) = 1\}.$$

Show that

$$(\mathbb{E}|X|^p)^{1/p} \xrightarrow[p\to\infty]{} \text{ess sup } X$$

(thus it makes sense to define the ∞–moment as $\|X\|_\infty = \text{ess sup } X$).

(b) If $\mathbb{E}|X|^{p_0} < \infty$ for some $p_0 > 0$, then $\mathbb{E}\log|X|$ exists and

$$(\mathbb{E}|X|^p)^{1/p} \xrightarrow[p\to 0+]{} e^{\mathbb{E}\log|X|}.$$

(thus it makes sense to define the 0th moment as $\|X\|_0 = e^{\mathbb{E}\log|X|}$).

— Revision problems before Midterm 2 (not for grading) —

1. Let X be a random variable with the distribution function

$$F(t) = \begin{cases} \frac{1}{3}e^t, & t < 0, \\ \frac{1}{2} + \frac{1}{2}(1 - e^{-t}), & t \geq 0. \end{cases}$$

Compute $\mathbb{P}(X < -1)$, $\mathbb{P}(X < 0)$, $\mathbb{P}(X \leq 0)$, $\mathbb{P}(X = 0)$, $\mathbb{P}(X > 1)$ and $\mathbb{P}(X = 2)$. Is X a continuous random variable? Find the distribution function of $Y = e^X$.

2. For $\alpha \in \mathbb{R}$ define

$$F_\alpha(t) = \begin{cases} 0, & t < 1, \\ \alpha(t-1)^2, & 1 \leq t < 2, \\ 1, & t \geq 2. \end{cases}$$

Find all α such that F_α is a distribution function of a random variable. For those α, let X_α be a random variable with distribution function F_α. Find $\mathbb{P}(X_\alpha \geq 1)$, $\mathbb{P}(X_\alpha = 2)$ and $\mathbb{P}(X_\alpha > 2)$. Is X_α a continuous random variable? Find the distribution function of $Y = (X_0 - 1)^2$.

3. Let X be a random variable with density $f(x) = \frac{1}{2}e^{-|x|}$. Find $\mathbb{E}X$ and $\mathbb{E}|X|$. Find the distribution function of X^2.

4. Let X and Y be independent random variables such that X has the exponential distribution with parameter 2 and Y has Gaussian distribution with mean 1 and variance 2. Find $\mathbb{E}(X - 2Y)^2$.

5. Let X and Y be independent standard Gaussian random variables. What is the distribution of $Z = 3X - 4Y$? Find its density Find $\mathbb{E}e^{(Z)^2/100}$ and $\mathbb{E}\sqrt{X^2+Y^2}$. Consider the random vector $V = \left[\begin{smallmatrix} 1 & 1 \\ -3 & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} X \\ Y \end{smallmatrix}\right]$. Is V a Gaussian random vector? Find its expectation and covariance matrix. Find the density of V.

6. What is the density of a standard Gaussian random variable, that is a Gaussian random variable with mean zero and variance one? Let X and Y be independent standard Gaussian random variables. What is the distribution of $\frac{1}{2}X - \frac{\sqrt{3}}{2}Y$? Are the variables X and $X+Y$ independent? Are the variables $\frac{1}{2}X - \frac{\sqrt{3}}{2}Y$ and $\frac{\sqrt{3}}{2}X + \frac{1}{2}Y$ independent? Find the density of $\sqrt{X^2+Y^2}$.

7. Let g be a standard Gaussian random variable. Find $\mathbb{E}e^{g^2/4}$. Find all $c \in \mathbb{R}$ such that $\mathbb{E}e^{cg^2}$ is finite. Let $g_1, g_2, \ldots, g_n$ be independent standard Gaussian random variables. What is the distribution of $g_1 + \ldots + g_n$? Find the set of all points $a = (a_1, \ldots, a_n)$ in $\mathbb{R}^n$ for which $\mathbb{E}e^{(a_1g_1+\ldots+a_ng_n)^2}$ is finite.