Probability 21-325 **Midterm 2 (16th April 2018)** **TT**

1. Let X be a random variable with the distribution function

$$F(t) = \begin{cases} 0, & t < 1, \\ \frac{1}{3}(t-1)^2, & 1 \leq t < 2, \\ 1, & t \geq 2. \end{cases}$$

Find $\mathbb{P}(X \geq 1)$, $\mathbb{P}(X = 2)$ and $\mathbb{P}(X > 2)$. Is X a continuous random variable? Find the distribution function of $Y = (X-1)^2$.

2. Let g be a standard Gaussian random variable. Find $\mathbb{E}e^{g^2/4}$. Find all $c \in \mathbb{R}$ such that $\mathbb{E}e^{cg^2}$ is finite. Let $g_1, g_2, \ldots, g_n$ be independent standard Gaussian random variables. What is the distribution of $g_1 + \ldots + g_n$? Find the set of all points $a = (a_1, \ldots, a_n)$ in $\mathbb{R}^n$ for which $\mathbb{E}e^{(a_1 g_1 + \ldots + a_n g_n)^2}$ is finite.

3. Let f be a continuous function on $[0,1]$ taking values in $[0,1]$. Let $X_1, Y_1, X_2, Y_2, \ldots$ be independent random variables uniformly distributed on $[0,1]$. Define $Z_i = \mathbf{1}_{\{f(X_i) > Y_i\}}$. Show that $\frac{1}{n}\sum_{i=1}^n Z_i$ converges almost surely to $\int_0^1 f$.