- PROBABILITY GENERATING FUNCTIONS -

The <u>generating function</u> $G(t)$ of a sequence $u_0, u_1, u_2, \dots$ is

$$G(t) = \sum_{k=0}^{\infty} u_k t^k \quad , \qquad |t| < R = \text{radius of convergence of this series}$$

<u>E.g.</u> $1, 2, 2^2, \dots$ has gen. fun. $1 + 2t + 2^2 t^2 + \dots = \frac{1}{1-2t}$, $|t| < \frac{1}{2}$.

Generating functions determine the sequence uniquely: if a function $G(t)$ has a convergent Taylor series, then expanding

$$G(t) = u_0 + u_1 t + u_2 t^2 + \dots \quad , \ t \text{ small}$$

generates the sequence $u_0, u_1, \dots$ uniquely.

<u>E.g.</u> $(1+t)^N = \sum_{n=0}^{N} \binom{N}{n} t^n$ is the gen. fun. of $\binom{N}{0}, \dots, \binom{N}{N}, 0, 0, \dots$.

Let X be a discrete r.v. with $\operatorname{Im} X \subset \{0, 1, 2, \dots\}$.

The <u>probability generating function</u> of X is

$$G_X(t) = \sum_{k=0}^{\infty} t^k \, \mathbb{P}(X = k) = \mathbb{E} t^X \quad , \qquad |t| < \text{radius of convergence}$$

(the gen. fun. of the sequence given by the pmf of X: $p_k = \mathbb{P}(X = k)$)

⚠ $$\left|\sum_{k=0}^{\infty} t^k \,\mathbb{P}(X=k)\right| \le \sum_{k=0}^{\infty} |t|^k \,\mathbb{P}(X=k) \underset{|t|\le 1}{\le} \sum_{k=0}^{\infty} \mathbb{P}(X=k) = 1$$

so G_X is well defined for $|t| \le 1$.

⚠ $G_X(1) = 1$.

Prob. gen. fun. determines distribution

<u>Thm</u> Let X and Y have prob gen fun's G_X, G_Y. Then

$$G_X(t) = G_Y(t), \quad |t| \le 1 \quad \text{iff} \quad \forall k = 0,1,\ldots \quad \mathbb{P}(X=k) = \mathbb{P}(Y=k).$$

($X \sim Y$ ← X, Y have the same distribution)

<u>Proof</u> Follows from the uniquess of generating functions.

<u>E.g.</u>

- $X \sim \mathrm{Ber}(p)$ $\quad G_X(t) = 1-p+pt$
- $X \sim \mathrm{Bin}(n,p)$ $\quad G_X(t) = \sum_{k=0}^{n} t^k \binom{n}{k} p^k (1-p)^{n-k}$

$$= (1-p+tp)^n = \left(G_{Ber}(t)\right)^n$$

accident ?

- $X \sim \mathrm{Poiss}(\lambda)$ $\quad G_X(t) = \sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} t^k$

$$= e^{-\lambda} e^{\lambda t} = e^{\lambda(t-1)}$$

- $X \sim \mathrm{Geom}(p)$ $\quad G_X(t) = \sum_{k=1}^{\infty} t^k (1-p)^{k-1} p = \frac{pt}{1-(1-p)t}, \quad |t| < \frac{1}{1-p}.$

Thm Let X be a r.v. with prob. gen. fun. G_X. Then

$$\frac{d^{(r)}}{dt^{(r)}} G_X \Big|_{t=1} = \mathbb{E}\, X(X-1)\cdot\ldots\cdot(X-r+1), \quad r=1,2,\ldots$$

(provided that this expectation exists)

Proof • $r=1$: for $|t|<1$

$$\frac{d}{dt} G_X(t) = \sum_{k=0}^{\infty} k t^{k-1}\, \mathbb{P}(X=k),$$

moreover, $\sum_{k=0}^{\infty} k\, \mathbb{P}(X=k) = \mathbb{E}X$, so taking $t \to 1-$

$$\frac{d}{dt} G_X \Big|_{t=1} \underset{\text{Abel's thm}}{=} \sum_{k=0}^{\infty} k\, \mathbb{P}(X=k) = \mathbb{E}X.$$

• $r=2,3,\ldots$ similar. □

E.g. $X \sim \text{Poiss}(\lambda)$, $G_X(t) = e^{\lambda(t-1)}$

$$\mathbb{E}X = G_X'(1) = \lambda e^{\lambda(t-1)}\Big|_{t=1} = \lambda$$

$$\mathbb{E}X(X-1) = G_X''(1) = \lambda^2 e^{\lambda(t-1)}\Big|_{t=1} = \lambda^2,$$

so $\operatorname{Var} X = \mathbb{E}X^2 - (\mathbb{E}X)^2 = \mathbb{E}X(X-1) + \mathbb{E}X - (\mathbb{E}X)^2$

$$= \lambda^2 + \lambda - \lambda^2 = \lambda.$$

Thm If X, Y are indep r.vs with values in $\{0,1,2,\ldots\}$, then

$$G_{X+Y} = G_X \cdot G_Y.$$

<u>Proof</u> $G_{X+Y}(t) = \mathbb{E}t^{X+Y} = \mathbb{E}t^X t^Y \underset{\text{indep}}{=} \mathbb{E}t^X\, \mathbb{E}t^Y$
$= G_X(t)\, G_Y(t)$. $\square$

<u>Eg.</u> Let $X_1, X_2, \dots$ be indep. identically distributed (i.i.d.) r.v.s with values in $\{0,1,2,\dots\}$. Let N be an indep. r.v. with values in $\{1,2,\dots\}$. Consider $S = X_1 + \dots + X_N$. What is $\mathbb{E}S$?

If N was a fixed number, we'd have $\mathbb{E}S = \mathbb{E}X_1 + \dots + \mathbb{E}X_N = N \cdot \mathbb{E}X_1$.

When N random,

$$G_S(t) = \mathbb{E}\, t^{X_1+\dots+X_N} = \sum_{n=1}^{\infty} \mathbb{E}\left(t^{X_1+\dots+X_n} \mid N=n\right) \cdot \mathbb{P}(N=n)$$

$$= \sum_{n=1}^{\infty} \underbrace{\mathbb{E}t^{X_1} \cdot \dots \cdot \mathbb{E}t^{X_n}}_{(\mathbb{E}t^{X_1})^n = (G_{X_1}(t))^n} \cdot \mathbb{P}(N=n)$$

$$= \sum_{n=1}^{\infty} \left[G_{X_1}(t)\right]^n \cdot \mathbb{P}(N=n) = \mathbb{E}\left(G_{X_1}(t)\right)^N$$

$$= G_N\left(G_{X_1}(t)\right)$$

$$\mathbb{E}S = G_S'(1) = G_N'\big(\underbrace{G_{X_1}(1)}_{1}\big) \cdot G_{X_1}'(1) = G_N'(1)\, G_{X_1}'(1)$$

$$= \mathbb{E}N \cdot \mathbb{E}X_1 \quad \square$$