- POISSON PROCESS -

Imagine yourself working at a busy call centre of an internationall bank. Arriving calls arrive at random,

N_t = number of calls received in time interval $[0,t]$

The whole collection of random variables $N = \{N_t, t \geq 0\}$ is an example of a cts time process.

<u>A Poisson process with</u> rate $\lambda > 0$ is defined by requiring

(1) $\forall t \geq 0$ N_t is a nonneg. integer-val. r.v.

(2) $N_0 = 0$

(3) $\forall s \leq t$ $N_s \leq N_t$

(4) $\forall s < t$ $N_s \perp\!\!\!\perp N_t - N_s$
(number of calls received in $[0,s]$ is indep. of number of calls received in $(s,t]$)

(5) $\forall t, h > 0$, h small

and
$$\mathbb{P}(N_{t+h} = n+1 \mid N_t = n) = \lambda h + o(h)$$
$$\mathbb{P}(N_{t+h} = n \mid N_t = n) = 1 - \lambda h + o(h)$$

E.g. $N_t =$ • no. of calls received in $[0,t]$

• no. of buses departed in $[0,t]$

• no. births in a small town in $[0,t]$

In part. $\mathbb{E}N_t = \lambda t$

Thm If $N = \{N_t, t \geq 0\}$ is a process sat. (1)–(5), then $\forall t \geq 0$ $N_t \sim \text{Poiss}(\lambda t)$.

Proof Let $p_k(t) = \mathbb{P}(N_t = k)$, $t \geq 0$, $k = 0, 1, 2, \ldots$

Fix $t \geq 0$ and take small $h > 0$. By the partition thm,

$$\mathbb{P}(N_{t+h} = k) = \sum_{j=0}^{k} \mathbb{P}(N_{t+h} = k \mid N_t = j)\, \mathbb{P}(N_t = j)$$

$$\underset{(5)}{=} \mathbb{P}(N_{t+h} = k \mid N_t = k-1)\, \mathbb{P}(N_t = k-1) + \mathbb{P}(N_{t+h} = k \mid N_t = k)\, \mathbb{P}(N_t = k) + o(h)$$

$$= (\lambda h + o(h))\, \mathbb{P}(N_t = k-1) + (1 - \lambda h + o(h))\, \mathbb{P}(N_t = k) + o(h)$$

$$= \lambda h \cdot \mathbb{P}(N_t = k-1) + (1 - \lambda h)\, \mathbb{P}(N_t = k) + o(h)$$

$$p_k(t+h) - p_k(t) = \lambda h \big(p_{k-1}(t) - p_k(t) \big) + o(h) \quad /:h, \; h \to 0$$

$$\frac{d}{dt} p_k(t) = \lambda p_{k-1}(t) - \lambda p_k(t), \quad k = 1, 2, \ldots$$

For $k=0$: $\quad \mathbb{P}(N_{t+h}=0) = \mathbb{P}(N_{t+h}=0 \mid N_t=0)\,\mathbb{P}(N_t=0)$

$$= (1-\lambda h + o(h))\,\mathbb{P}(N_t=0)$$

$$p_0(t+h) - p_0(t) = -\lambda h\, p_0(t) + o(h)$$

$$p_0'(t) = -\lambda p_0(t)$$

so want to solve $\begin{cases} p_0'(t) = -\lambda p_0(t), & t\geq 0 \\ p_k'(t) = \lambda p_{k-1}(t) - \lambda p_k(t), & t \geq 0,\ k=1,2,\dots \end{cases}$

with the boundary condition $N_0 = 0$, $\quad p_k(0) = \begin{cases} 1 & \text{if } k=0 \\ 0 & \text{o/w}. \end{cases}$

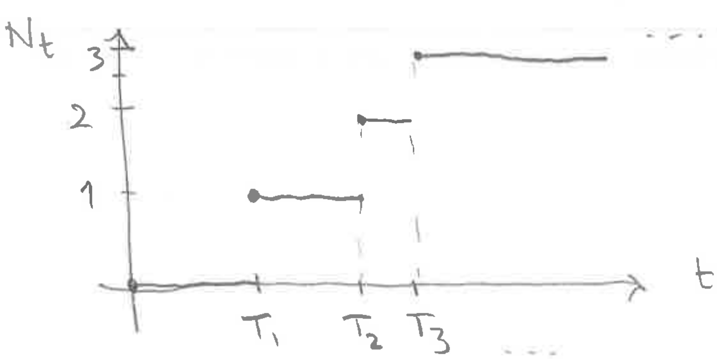
$p_0(t) = e^{-\lambda t}$ — easy

$p_1'(t) + \lambda p_1(t) = \lambda p_0(t) = \lambda e^{-\lambda t}$

$p_1(t) = \lambda t e^{-\lambda t}$

$\vdots$ ind.

$$p_k(t) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}. \quad \square$$



arrival times,

⚠ (T_n), or (X_n) completely determines (N_t)

$N_t = \max\{k \geq 0,\ T_k \leq t\}$.

$X_i = T_i - T_{i-1}$, $T_0 = 0$ — inter-arrival times.

Thm For a Poisson process with rate λ, the inter-arrival times $X_1, X_2, \ldots$ are iid $\mathcal{E}xp(\lambda)$ r.vs.

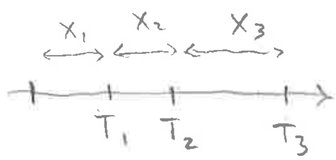
Proof (sketch)

- $$\mathbb{P}(X_1 > u) = \mathbb{P}(N_u = 0) \underset{\substack{\uparrow \\ N_u \sim \text{Poiss}(\lambda u)}}{=} e^{-\lambda u} \rightsquigarrow X_1 \sim \mathcal{E}xp(\lambda).$$

- $$\mathbb{P}(X_k > u) = \mathbb{P}\big(N_{T_{k-1}+u} = \overset{N_{T_{k-1}}}{(k-1)}\big)$$
$$= \mathbb{P}\big(N_{T_{k-1}+u} - N_{T_{k-1}} = 0\big)$$

$N_{u+v} - N_u \sim \text{Poiss}(\lambda v)$

$$= e^{-\lambda u}.$$



X_k "depends only" on $(N_{t+T_{k-1}} - N_{T_{k-1}})_t$ which is "independent of" what happened in $[0, T_{k-1}]$

so X_k is indep. of $X_1, \ldots, X_{k-1}$. $\square$

Construction of a Poisson process

1. $X_1, X_2, \ldots$ iid $\mathcal{E}xp(\lambda)$

2. $T_0 = 0$, $T_k = X_1 + \ldots + X_k$ (time of k^{th} arrival)

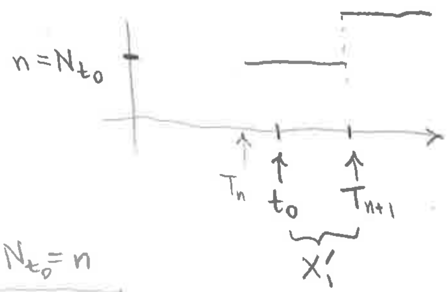
3. $N_t = \max\{k \geq 0,\ T_k \leq t\}$

E.g. Buses arrive at a bus stop randomly, according to a Poisson process.

idea: you wait $\sim Exp(\lambda)$ indep. of the past b/c $N'_t = (N_{t+t_0} - N_{t_0}) \perp\!\!\!\perp N_{t_0}$ Markov property

You get to the bus stop at time t_0. How long do you have to wait for the next bus?

$$X'_1 = T'_1 = T_{N_{t_0}+1} - t_0$$



$$\mathbb{P}(X'_1 > s \mid N_{t_0} = n)$$

$$= \underbrace{\frac{1}{\mathbb{P}(N_{t_0}=n)}}_{e^{-\lambda t_0}\frac{(\lambda t_0)^n}{n!}} \underbrace{\mathbb{P}\big(T_{N_{t_0}+1} - t_0 > s, \overbrace{T_n \le t_0}^{N_{t_0}=n}\big)}_{\mathbb{P}(T_{n+1} > s+t_0,\ T_n \le t_0)}$$

$$= \int_0^{t_0} \mathbb{P}(\underbrace{T_{n+1}}_{T_n + X_{n+1}} > s+t_0 \mid T_n = x)\, f_{T_n}(x)\,dx$$

$$= \int_0^{t_0} \mathbb{P}(\underbrace{X_{n+1} > s+t_0-x}_{} \mid \underbrace{T_n = x}_{})\, \underbrace{f_{T_n}(x)}_{}\,dx$$

indep. $\quad\Uparrow$ Gamma$_n$

$$= \int_0^{t_0} e^{-\lambda(s+t_0-x)} \cdot \lambda^n \frac{x^{n-1}}{(n-1)!} e^{-\lambda x}\,dx$$

$$= \lambda^n \frac{t_0^n}{n!} e^{-\lambda(s+t_0)}$$

$$= e^{-\lambda s}$$

so $X'_1 \sim Exp(\lambda)$, $\quad X'_1 \perp\!\!\!\perp N_{t_0}$

Suppose someone at the stop tells you, they have been waiting 20 mins. This means $X'_1 > 20$ mins, which is neither good nor bad news b/c X'_1 as Exp is memory-less,

$$\mathbb{P}(X'_1 > t+s \mid X'_1 > s) = \mathbb{P}(X'_1 > t).$$