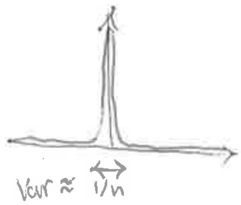
- CENTRAL LIMIT THEOREM -

Let $X_1, X_2, \dots$ be i.i.d. r.vs with $\mathbb{E}|X_i|^2 < \infty$. By LLN,



$$\frac{S_n}{n} - \mathbb{E}X_1 \approx 0 \qquad , \; S_n = X_1 + \dots + X_n$$

$$\mathrm{Var}\left(\frac{S_n}{n} - \mathbb{E}X_1\right) = \frac{1}{n^2}\mathrm{Var}(S_n) = \frac{1}{n}\mathrm{Var}(X_1)$$

If we rescale (zoom in), and consider

$$\frac{1}{\sqrt{\mathrm{Var}(X_1)}}\sqrt{n}\left(\frac{S_n}{n} - \mathbb{E}X_1\right) = \frac{S_n - \mathbb{E}S_n}{\sqrt{n\,\mathrm{Var}(X_1)}}$$

which has the same variance, one, for all n, what shape does it have as $n \to \infty$?

?

Thm (CLT) Let $X_1, X_2, \dots$ be iid r.vs with $\mathbb{E}|X_i|^2 < \infty$. Then,

$$Z_n = \frac{S_n - \mathbb{E}S_n}{\sqrt{n\,\mathrm{Var}(X_1)}},$$

converges in distribution to a standard Gaussian r.v. as $n \to \infty$.

Def. A seq. (X_n) of r.vs converges to a r.v. X in distribution if

$$F_{X_n}(t) \xrightarrow[n\to\infty]{} F_X(t)$$ for every point $t \in \mathbb{R}$ of continuity of F.

Notation: $X_n \xrightarrow{d} X$.

⚠ $X_n \xrightarrow{a.s.} X$ and $X_n \xrightarrow{L_p} X$ $\Rightarrow$ $X_n \xrightarrow{\mathbb{P}} X \underset{\text{exercise}}{\Longrightarrow} X_n \xrightarrow{d} X$.

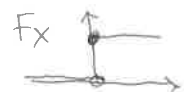
E.g. Let ε be a random sign. Consider,

$$(X_n) = (\varepsilon, -\varepsilon, \varepsilon, -\varepsilon, \dots)$$

We obviously have $X_n \xrightarrow{d} \varepsilon$ (even $F_{X_n}(t) \overset{\forall n,t}{=} F_\varepsilon(t)$),

but $X_n \overset{\mathbb{P}}{\not\to} \varepsilon$ $\left(\mathbb{P}(|X_n - \varepsilon| > \delta) \underset{n \text{ even}}{=} \mathbb{P}(2 > \delta) \underset{\delta < 2}{=} 1 \right)$.

E.g. $X_n = \frac{1}{n}$, obviously $X_n \xrightarrow[d]{a.s.,\, L_p,\, \mathbb{P}} 0$,



$$F_{X_n}(0) = \mathbb{P}(X_n \leq 0) = 0, \qquad F_X(0) = \mathbb{P}(0 \leq 0) = 1,$$

that's why we define $\xrightarrow{d}$ avoiding points of discontinuity of F_X (where F_X jumps) — only countably many such points.

Classical tool: Fourier analysis (a.k.a. characteristic functions)

Def. The characteristic function of a r.v. X is

$e^{ix} = \cos x + i \sin x$
$|e^{ix}| = 1$

$$\phi_X(t) = \mathbb{E} e^{itX}, \qquad t \in \mathbb{R}.$$

⚠ $|\mathbb{E} e^{itX}| \leq \mathbb{E}|e^{itX}| = 1$, so ϕ_X is well-defined.

E.g.

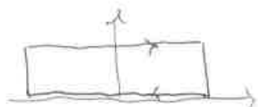
- random sign: $\phi_\varepsilon(t) = \mathbb{E} e^{it\varepsilon} = \frac{e^{it} + e^{-it}}{2} = \cos t$

- $X \sim \mathcal{E}xp(\lambda)$ $\quad \phi_X(t) = \mathbb{E} e^{itX} = \int e^{itx} f(x)\,dx$

$$= \int_0^\infty \lambda e^{itx} e^{-\lambda x}\,dx = \frac{-\lambda}{it-\lambda} = \frac{\lambda}{\lambda - it}.$$

- $X \sim \mathcal{N}(0,1)$, $\quad \phi_X(t) = \int_{\mathbb{R}} e^{itx} e^{-x^2/2} \frac{dx}{\sqrt{2\pi}}$

$$= e^{-t^2/2} \int_{\mathbb{R}} e^{-\frac{1}{2}(x-it)^2}\, dx/\sqrt{2\pi}$$



$$= e^{-t^2/2}$$

- $X \sim \mathcal{N}(\mu, \sigma^2)$ $\qquad X = \mu + \sigma Y, \quad Y \sim \mathcal{N}(0,1)$

$$\phi_X(t) = \mathbb{E} e^{it(\mu + \sigma Y)} = e^{it\mu} \phi_Y(\sigma t)$$

$$\boxed{\phi_{\mathcal{N}(\mu,\sigma^2)}(t) = e^{it\mu - \sigma^2 t^2/2}}$$

- $X \sim$ Cauchy $\qquad \phi_X(t) = e^{-|t|}$

Not conversely! $\to$ Thm $\quad \phi_{X+Y} = \phi_X \cdot \phi_Y \quad$ if X, Y are indep.

Thm
- ϕ_X is uniformly continuous
- if $\mathbb{E}|X|^n < \infty$, then $\phi_X^{(n)}$ exists,

cf. Cauchy! $\to$ $\quad \phi_X^{(n)}(t) = i^n \mathbb{E} X^n e^{itX}$, and is unif. cts.

<u>Proof</u> • $|\phi_X(t+h) - \phi_X(t)| = |\mathbb{E}\, e^{itX}(e^{ihX} - 1)|$

$$\le \mathbb{E}\, |e^{ihX} - 1| \xrightarrow[h\to 0]{} 0$$

(Leb. dom. conv.)

which implies uniform continuity

• inductively show that for $0 \le k \le n$

$$\phi_X^{(k)}(t) = \mathbb{E}\,(iX)^k e^{itX}$$

OK for $k=0$, $\quad k \rightsquigarrow k+1$:

$$\phi_X^{(k+1)}(t) = \lim_{h\to 0} \frac{\phi_X^{(k)}(t+h) - \phi_X^{(k)}(t)}{h}$$

$$= \lim_{h\to 0} \mathbb{E}\,(iX)^k e^{itX} \frac{e^{ihX} - 1}{h}$$

$$= \mathbb{E}\,(iX)^k e^{itX} \lim_{h\to 0} \frac{e^{ihX}-1}{h} = \mathbb{E}(iX)^{k+1} e^{itX}$$

Leb's dom. conv:

- $\left|\frac{e^{ihX}-1}{h}\right| \le |X|$
- $\mathbb{E}|X|^{k+1} < \infty$

Having this formula, uniform continuity follows as before. $\square$

Ch.f. determines distribution <u>Thm</u> For two r.vs X, Y

$$X \sim Y \quad (F_X \equiv F_Y) \qquad \text{iff} \qquad \phi_X \equiv \phi_Y.$$

<u>Thm</u> $X_n \xrightarrow[n\to\infty]{d} X$ iff $\forall t \in \mathbb{R}\ \ \phi_{X_n}(t) \longrightarrow \phi_X(t)$.

Proof of CLT WTS: $Z_n = \dfrac{S_n - \mathbb{E}S_n}{\sqrt{n \operatorname{Var} X_1}} \xrightarrow{d} Z \sim N(0,1)$

$$\varphi_{Z_n}(t) = \varphi_{\sum \frac{\bar{X}_i}{\sqrt{n}}}(t) = \prod \varphi_{\bar{X}_i}\left(\frac{t}{\sqrt{n}}\right) = \varphi_{\bar{X}_1}\left(\frac{t}{\sqrt{n}}\right)^n,$$

$$\bar{X}_i = \frac{X_i - \mathbb{E}X_i}{\sqrt{\operatorname{Var}(X_1)}}, \quad \mathbb{E}\bar{X}_i = 0, \quad \operatorname{Var}(\bar{X}_i) = 1.$$

$\mathbb{E}|\bar{X}_1|^2 < \infty$
$$\varphi_{\bar{X}_1}(t) \overset{\text{Taylor}}{=} \varphi_{\bar{X}_1}(0) + t\underbrace{\varphi'_{\bar{X}_1}(0)}_{i\mathbb{E}\bar{X}_1} + \frac{1}{2}t^2 \varphi''_{\bar{X}_1}(\xi), \quad \xi \in [0,t]$$

$$= 1 + \frac{1}{2}t^2 \underbrace{\varphi''_{\bar{X}_1}(0)}_{i^2\mathbb{E}\bar{X}_1^2 = -1} + \frac{1}{2}t^2\left[\varphi''_{\bar{X}_1}(\xi) - \varphi''_{\bar{X}_1}(0)\right]$$

$$= 1 - \frac{t^2}{2} + \frac{t^2}{2}\underbrace{R(t)}_{\text{error}},$$

$|R(t)| \xrightarrow[t\to 0]{} 0$ b/c φ'' is cts

Fix $t \in \mathbb{R}$,

$$\varphi_{Z_n}(t) = \left(1 - \frac{t^2}{2n} + \frac{t^2}{2n}R\left(\frac{t}{\sqrt{n}}\right)\right)^n$$

$$= \left(1 - \frac{t^2}{2n}(1 - R_n)\right)^n \xrightarrow[n\to\infty]{} e^{-t^2/2} = \varphi_Z(t). \ \square$$

CLT: $\forall t \in \mathbb{R}$ $\quad \mathbb{P}\left(\dfrac{S_n - \mathbb{E}S_n}{\sqrt{n \operatorname{Var} X_1}} \le t\right) - \underbrace{\mathbb{P}(Z \le t)}_{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^t e^{-x^2/2}dx} \xrightarrow[n\to\infty]{} 0$

$\Downarrow$

$\forall a < b$ $\quad \mathbb{P}\left(a < \dfrac{S_n - \mathbb{E}S_n}{\sqrt{n \operatorname{Var} X_1}} \le b\right) - \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_a^b e^{-x^2/2}dx \xrightarrow[n\to\infty]{} 0$

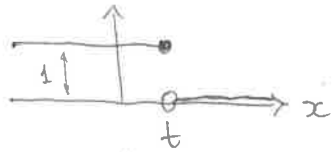
CLT does not give any error bounds !

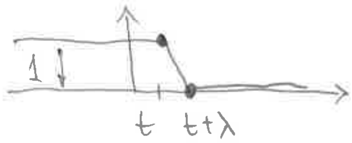
__Berry - Esseen thm__ Let $X_1, X_2, \dots$ be i.i.d. r.v. with $\mathbb{E}X_i = 0$, $\mathbb{E}X_i^2 = 1$, $\mathbb{E}|X_i|^3 < \infty$. Let $Z_n = \frac{X_1 + \dots + X_n}{\sqrt{n}}$, $Z \sim N(0,1)$. Then,

$$\forall n \geq 1 \ \forall t \in \mathbb{R} \quad \left| \mathbb{P}(Z_n \leq t) - \mathbb{P}(Z \leq t) \right| \leq \frac{C \cdot \mathbb{E}|X_1|^3}{\sqrt{n}},$$

$$C = 37\dots$$

__Proof__ (Stein method) Let , for $t, x \in \mathbb{R}$, $\lambda > 0$,

$h_t(x) =$  $= 1_{(-\infty, t]}(x)$

$h_{t,\lambda}(x) =$  $= \int_x^\infty \frac{1}{\lambda} 1_{[t, t+\lambda]}(s)\, ds.$

Notice $\mathbb{P}(X \leq t) = \mathbb{E} 1_{\{X \leq t\}} = \mathbb{E} h_t(X)$. Let

$\mathbb{E}X_1^2 = 1$
$\Downarrow$
$\gamma \geq 1$

$$\mathcal{L}_\gamma = \{ X \text{ r.v. s.t. } \mathbb{E}X = 0,\ \mathbb{E}X^2 = 1,\ \mathbb{E}|X|^3 = \gamma \}$$

$$B_0(\gamma, n) = \sup_{X_1, \dots, X_n \text{ iid } \in \mathcal{L}} \ \sup_{t \in \mathbb{R}} \left| \mathbb{E}h_t(Z_n) - \mathbb{E}h_t(Z) \right|$$

$$B(\lambda, \gamma, n) = \sup_{X_1, \dots, X_n \text{ iid } \in \mathcal{L}} \ \sup_{t \in \mathbb{R}} \left| \mathbb{E}h_{t,\lambda}(Z_n) - \mathbb{E}h_{t,\lambda}(Z) \right|$$

WTS $\forall n \geq 1 \ \forall \gamma \geq 1 \ \frac{\sqrt{n}}{\gamma} B_0(\gamma, n) \leq C.$

Clear for $n = 1$ with $C = 1$: $B_0(\gamma, 1) \leq 1 \leq \gamma$, so we assume $\underline{n \geq 2}$.

__Step 1__ Regularise: $B_0 \rightsquigarrow B$

$$h_{t-\lambda,\lambda} \leq h_t \leq h_{t,\lambda} \Rightarrow \mathbb{E}h_t(Z_n) - \mathbb{E}h_t(Z)$$
$$\leq \mathbb{E}h_{t,\lambda}(Z_n) - \mathbb{E}h_{t,\lambda}(Z)$$
$$+ \mathbb{E}h_{t,\lambda}(Z) - \mathbb{E}h_t(Z)$$

$$\mathbb{E}h_{t+\lambda}(Z) - \mathbb{E}h_t(Z) = \mathbb{P}(t < Z \leq t + \lambda)$$

but $\mathbb{P}(t < Z \le t+\lambda) = \int_t^{t+\lambda} e^{-x^2/2} \frac{dx}{\sqrt{2\pi}} \le \frac{\lambda}{\sqrt{2\pi}}$, so

$$\mathbb{E} h_t(Z_n) - \mathbb{E} h_t(Z) \le \mathbb{E} h_{t,\lambda}(Z_n) - \mathbb{E} h_{t,\lambda}(Z) + \frac{\lambda}{\sqrt{2\pi}}$$

$$\le B(\lambda, \gamma, n) + \frac{\lambda}{\sqrt{2\pi}}$$

... similarly, $\ge -B(\lambda,\gamma,n) - \frac{\lambda}{\sqrt{2\pi}}$

so $$B_0(\gamma, n) \le B(\lambda,\gamma,n) + \frac{\lambda}{\sqrt{2\pi}}.$$

Step 2 2 variables $\rightsquigarrow$ 1 variable (Stein method)

Fix $t \in \mathbb{R}$, $\lambda > 0$, $h = h_{t,\lambda}$. Let

$$f(x) = e^{x^2/2} \int_{-\infty}^{x} [h(u) - \mathbb{E}h(Z)]\, e^{-u^2/2}\, du.$$

$$f'(x) - x f(x) = h(x) - \mathbb{E}h(Z)$$

so $$\mathbb{E}h(Z_n) - \mathbb{E}h(Z) = \mathbb{E}\left[f'(Z_n) - Z_n f(Z_n)\right].$$

Step 3 Good estimates for f: Lm $\forall x > 0$ • $\int_x^{+\infty} e^{-u^2/2} du \le \frac{1}{x} e^{-x^2/2}$

$\Updownarrow$ • $\int_x^{+\infty} e^{-x^2/2} du \le \sqrt{\frac{\pi}{2}}\, e^{-x^2/2}$

(1) $\forall x \in \mathbb{R}$ $|f(x)| \le \sqrt{\frac{\pi}{2}}$, $|xf(x)| \le 1$, $|f'(x)| \le 2$.

Indeed, for $x<0$, $|f(x)| \le e^{x^2/2} \int_{-\infty}^{x} \underbrace{|h(u) - \mathbb{E}h(Z)|}_{\le 1} e^{-u^2/2} du$

$$\le e^{x^2/2} \int_{-\infty}^{x} e^{-u^2/2} du \underset{Lm}{\le} \sqrt{\frac{\pi}{2}},$$

for $x>0$, notice $\int_{-\infty}^{+\infty} [h(u) - \mathbb{E}h(Z)] e^{-u^2/2} \frac{du}{\sqrt{2\pi}} = 0$,

so $f(x) = -e^{x^2/2} \int_{x}^{+\infty} [h(u) - \mathbb{E}h(Z)] e^{-u^2/2} du$, so as above, $|f(x)| \le \sqrt{\frac{\pi}{2}}$.

Bounding $x f(x)$ is similar. Finally, $|f'(x)| \le |x f(x)| + |h(x) - \mathbb{E}h(Z)|$.

Good estimate for f'

$$|f'(x+y) - f'(x)| = |(x+y)f(x+y) + h(x+y) - xf(x) - h(x)|$$

$$= |y f(x+y) + x(\underbrace{f(x+y) - f(x)}_{= f'(\xi)\cdot y}) + h(x+y) - h(x)|$$

$$\le |y| \sqrt{\frac{\pi}{2}} + 2|x|\cdot|y| + |h(x+y) - h(x)|,$$

$$|h(x+y) - h(x)| = \left| \frac{1}{\lambda} \int_{x}^{x+y} 1_{[t,t+\lambda]}(u)\, du \right| \underset{u = x+vy}{=} \left| \frac{y}{\lambda} \int_0^1 1_{[t,t+\lambda]}(x+vy)\, dv \right|$$

(2) $$|f'(x+y) - f'(x)| \le |y| \left(\sqrt{\frac{\pi}{2}} + 2|x| + \frac{1}{\lambda} \int_0^1 1_{[t,t+\lambda]}(x+vy)\, dv \right).$$

<u>Step 4</u> From Step 2, to estimate $B(\lambda, \gamma, n)$, we estimate

$$\mathbb{E}h(Z_n) - \mathbb{E}h(Z) = \mathbb{E}\big(f'(Z_n) - Z_n f(Z_n)\big).$$

Notice

$$\mathbb{E}\, Z_n f(Z_n) = \mathbb{E}\, \frac{\sum X_i}{\sqrt{n}} f(Z_n) = \frac{1}{\sqrt{n}} \sum \mathbb{E} X_i f(Z_n) \underset{\text{i.d.}}{=} \sqrt{n}\, \mathbb{E} X_n f(Z_n),$$

$$\mathbb{E}\Big(f'(Z_n) - Z_n f(Z_n)\Big) = \mathbb{E}\Big(f'(Z_n) - \sqrt{n} X_n f(Z_n)\Big)$$

$Z_n = \sqrt{\frac{n-1}{n}} Z_{n-1} + \frac{X_n}{\sqrt{n}}$

$$= \mathbb{E}\Big[f'(Z_n) - \sqrt{n} X_n \int_0^1 \partial_u f\Big(\sqrt{\tfrac{n-1}{n}} Z_{n-1} + u \tfrac{X_n}{\sqrt{n}}\Big) du$$

$$- \underbrace{\sqrt{n} X_n f\Big(\sqrt{\tfrac{n-1}{n}} Z_{n-1}\Big)}_{\text{in } \mathbb{E} \text{ dissapears b/c } \mathbb{E}X_n = 0}\Big]$$

$$= \mathbb{E}\Big[f'(Z_n) - X_n^2 \int_0^1 f'\Big(\sqrt{\tfrac{n-1}{n}} Z_{n-1} + u \tfrac{X_n}{\sqrt{n}}\Big) du\Big]$$

$$= \mathbb{E}\Big[f'(Z_n) - f'\Big(\sqrt{\tfrac{n-1}{n}} Z_{n-1}\Big)\Big]$$

(3)

$$+ \mathbb{E}\Big[\underset{\mathbb{E}X_n^2 = 1}{-X_n^2} \int_0^1 \Big(f'\Big(\sqrt{\tfrac{n-1}{n}} Z_{n-1} + u \tfrac{X_n}{\sqrt{n}}\Big) - f'\Big(\sqrt{\tfrac{n-1}{n}} Z_{n-1}\Big)$$

<u>Step 5</u> $\Big|\mathbb{E}\Big[f'(Z_n) - f'\Big(\sqrt{\tfrac{n-1}{n}} Z_{n-1}\Big)\Big]\Big| \underset{(2)}{\leq} \mathbb{E}\, \Big|\frac{X_n}{\sqrt{n}}\Big| \Big(\sqrt{\tfrac{\pi}{2}} + 2 \overset{<1}{\sqrt{\tfrac{n-1}{n}}} |Z_{n-1}|$

$$+ \frac{1}{\lambda} \int_0^1 1_{[t,t+\lambda]}\Big(\sqrt{\tfrac{n-1}{n}} S_{n-1} + u \tfrac{X_n}{\sqrt{n}}\Big) du\Big)$$

$$\leq \frac{\overbrace{\mathbb{E}|X_n|}^{\leq 1}}{\sqrt{n}} \sqrt{\frac{\pi}{2}} + 2 \frac{\overbrace{\mathbb{E}|X_n||Z_{n-1}|}^{\leq 1}}{\sqrt{n}} + \frac{1}{\lambda\sqrt{n}}\, \mathbb{E}_{X_n} |X_n| \int_0^1 \mathbb{E}_{S_{n-1}} 1_{[t,t+\lambda]}\Big(\sqrt{\tfrac{n-1}{n}} Z_{n-1} + u \tfrac{X_n}{\sqrt{n}}\Big)$$

$$\int_0^1 \cdots = \int_0^1 \mathbb{P}\Big(\overbrace{\Big(t - u\tfrac{X_n}{\sqrt{n}}\Big)\sqrt{\tfrac{n}{n-1}}}^{a} \leq Z_{n-1} \leq \Big(t - u\tfrac{X_n}{\sqrt{n}}\Big)\sqrt{\tfrac{n}{n-1}} + \lambda \underset{\leq \sqrt{2}}{\sqrt{\tfrac{n}{n-1}}}\Big)\, du$$

$$\mathbb{P}(a \leq Z_{n-1} \leq a + \lambda\sqrt{2}) = \mathbb{P}(Z_{n-1} \leq a + \lambda\sqrt{2}) - \mathbb{P}(Z \leq a + \lambda\sqrt{2})$$

$$+ \mathbb{P}(Z \leq a) - \mathbb{P}(Z_{n-1} \leq a)$$

$$+ \mathbb{P}(a \leq Z \leq a + \lambda\sqrt{2}) \leq 2B_0(\gamma, n-1) + \frac{\lambda\sqrt{2}}{\sqrt{2\pi}}$$

$$☺ \quad \le \quad \frac{1}{\sqrt{n}}\sqrt{\frac{\pi}{2}} + \frac{2}{\sqrt{n}} + \frac{1}{\lambda\sqrt{n}}\left(2B_0(\gamma, n-1) + \frac{\lambda}{\sqrt{\pi}}\right)$$

<u>Step 6</u> $\quad \left| \mathbb{E}\left[-X_n^2 \int_0^1 \left(f'\left(\sqrt{\tfrac{n-1}{n}} Z_{n-1} + u\frac{X_n}{\sqrt{n}}\right) - f'\left(\sqrt{\tfrac{n-1}{n}} Z_{n-1}\right)\right) du\right]\right|$

$$\le \mathbb{E}\, X_n^2 \cdot \frac{|X_n|}{\sqrt{n}} \int_0^1 u \left(\sqrt{\frac{\pi}{2}} + 2\sqrt{\frac{n-1}{n}}|Z_{n-1}| + \frac{1}{\lambda}\int_0^1 1_{[t,t+\lambda]}\left(\sqrt{\tfrac{n-1}{n}} Z_{n-1} + uv\frac{X_n}{\sqrt{n}}\right) dv\right) du$$

$$\le \mathbb{E}_{X_n} \frac{|X_n|^3}{\sqrt{n}} \int_0^1 \left(\sqrt{\frac{\pi}{2}} + 2 + \frac{1}{\lambda}\int_0^1 \mathbb{P}_{Z_{n-1}}\left(t \le \sqrt{\tfrac{n-1}{n}} Z_{n-1} + uv\frac{X_n}{\sqrt{n}} \le t+\lambda\right) dv\right) du$$

$$\le \frac{\gamma}{\sqrt{n}}\left(\sqrt{\frac{\pi}{2}} + 2 + \frac{1}{\lambda}\left(2B_0(\gamma, n-1) + \frac{\lambda}{\sqrt{\pi}}\right)\right)$$

Step 5 & 6 $\underset{(3)}{\rightsquigarrow}$ $\quad \left|\mathbb{E}[f'(Z_n) - Z_n f(Z_n)]\right| \le$

$$\le 2\frac{\gamma}{\sqrt{n}}\left(\left(2 + \sqrt{\frac{\pi}{2}} + \frac{1}{\sqrt{\pi}}\right) + \frac{2}{\lambda} B_0(\gamma, n-1)\right)$$

By Step 1,2, this gives,

$$B_0(\gamma, n) \le B(\lambda, \gamma, n) + \frac{\lambda}{\sqrt{2\pi}} \le 2\frac{\gamma}{\sqrt{n}}\left(2 + \sqrt{\frac{\pi}{2}} + \frac{1}{\sqrt{\pi}} + \frac{2}{\lambda}B_0(\gamma, n-1)\right) + \frac{\lambda}{\sqrt{2\pi}}$$

Choose $\lambda = \frac{8\gamma}{\sqrt{n}}$, to get

$$B_0(\gamma, n) \le \frac{1}{2} B_0(\gamma, n-1) + 2\left(2 + \sqrt{\frac{\pi}{2}} + \frac{1}{\sqrt{\pi}}\right)\frac{\gamma}{\sqrt{n}} + \frac{8}{\sqrt{2\pi}}\frac{\gamma}{\sqrt{n}}.$$

Multiplying by $\frac{\sqrt{n}}{\gamma}$ gives that $B = \sup\limits_{\gamma, n} B_0(\gamma, n) \cdot \frac{\sqrt{n}}{\gamma}$ satisfies

$\sqrt{n} = \sqrt{n-1}\sqrt{\frac{n}{n-1}}$, $\le \sqrt{2}$

$$B \le \frac{1}{2}\sqrt{2} \cdot B + 2\left(2 + \sqrt{\frac{\pi}{2}} + \frac{1}{\sqrt{\pi}}\right) + \frac{8}{\sqrt{2\pi}}, \quad \text{hence}$$

$$B \le \frac{1}{1 - \frac{1}{\sqrt{2}}}\left(4 + \sqrt{2\pi} + \frac{4\sqrt{2} + 2}{\sqrt{\pi}}\right) \approx 36.96 < 37.$$

⚠ Under the assumption $\mathbb{E}|X_1|^3 < \infty$, the rate $\frac{1}{\sqrt{n}}$ for the error cannot be improved (Berry-Esseen thm is optimal)

Consider, $Z_n = \frac{\sum_1^n \varepsilon_i}{\sqrt{n}}$,

$$\mathbb{P}(Z_{2n} \le 0) = \frac{1 + \mathbb{P}(Z_{2n} = 0)}{2} = \frac{1 + \frac{1}{2^{2n}}\binom{2n}{n}}{2}$$

$$\underset{\text{Stirling}}{\approx} \frac{1}{2} + \underbrace{\frac{1}{2\sqrt{\pi}}\frac{1}{\sqrt{n}}}$$

$$\mathbb{P}(Z \le 0) = \frac{1}{2}.$$

E.g. $S_n \sim Bin(n, p)$, $S_n = X_1 + \dots + X_n$, $X_i \underset{iid}{\sim} Ber(p)$

Error in the B-E thm: $\frac{\mathbb{E}|X_1 - \mathbb{E}X_1|^3}{\sqrt{n}\sqrt{Var X_1}^3}$, let's compute it,

$Var X_1 = p(1-p)$, $\mathbb{E}|X_1 - \mathbb{E}X_1|^3 = \mathbb{E}|X_1 - p|^3 = (1-p)^3 p + p^3(1-p)$

$$= p(1-p)\big(\underbrace{(1-p)^2 + p^2}_{2p^2 - 2p + 1}\big)$$

$$\text{Error} = \frac{1}{\sqrt{n}} \frac{p(1-p)(-2p(1-p) + 1)}{(p(1-p))^{3/2}} = \frac{1}{\sqrt{n}}\left(\frac{1}{\sqrt{p(1-p)}} - 2\sqrt{p(1-p)}\right)$$

$$\geq \frac{1}{\sqrt{np(1-p)}}$$

If, e.g. $np \approx$ const for large n, then the B-E thm doesn't give anything. What is S_n like for large n in such cases?

Thm (Poisson limit thm) If $p_n \in [0,1]$ is such that $n \cdot p_n \xrightarrow[n\to\infty]{} \lambda$, then $\mathrm{Bin}(n,p_n) \sim S_n \xrightarrow{d} X \sim \mathrm{Poiss}(\lambda)$.

Proof By ⌂ ; $S_n \xrightarrow{d} X \iff \forall k = 0,1,\dots \ \mathbb{P}(S_n = k) \to \mathbb{P}(X=k)$.

$$\mathbb{P}(S_n = k) = \binom{n}{k} p_n^k (1-p_n)^{n-k} \underset{\text{Stirling}}{\overset{\substack{k\text{-fixed}\\ n\text{-large}}}{\approx}} \frac{1}{k!}\sqrt{\frac{n}{n-k}}\left(\frac{n}{e}\right)^n \left(\frac{e}{n-k}\right)^{n-k} p_n^k (1-p_n)^{n-k}$$

$$= \frac{1}{k!}\sqrt{\frac{n}{n-k}}\, e^{-k} (np_n)^k \left(\frac{1-p_n}{1-\frac{k}{n}}\right)^{n-k}$$

$$\underset{1+x\approx e^x}{\approx} \frac{1}{k!}\sqrt{\frac{n}{n-k}}\, e^{-k} (np_n)^k\, e^{-p_n n + p_n k + k - \frac{k^2}{n}}$$

$$\xrightarrow[n\to\infty]{} \frac{1}{k!}\lambda^k e^{-\lambda} = \mathbb{P}(X=k). \quad \square$$

Sanity check: $\mathbb{E} S_n = np_n \to \lambda$

$\mathrm{Var}(S_n) = np_n(1-p_n) \to \lambda$.

Then, $Z_n = \dfrac{S_n - \mathbb{E}S_n}{\sqrt{\mathrm{Var}(S_n)}}$ for large n is not Gaussian, but "shifted -rescaled Poisson"