-10-

– LAWS OF LARGE NUMBERS –

Suppose we roll a die n times and the outcomes are $X_1, X_2, \ldots, X_n$. We expect $S_n = \dfrac{X_1 + \ldots + X_n}{n} \approx 3.5 = \mathbb{E}X_1$ as $n \to \infty$.

Laws of large numbers (LLN) establish that rigorously, in a fairly general situation,

- weak LLN : $S_n \xrightarrow{\mathbb{P}} \mathbb{E}X_1$
  $\Uparrow$
- strong LLN : $S_n \xrightarrow{a.s.} \mathbb{E}X_1$

as $\underbrace{n \to \infty}$,

"large n" = "large number of trials"

E.g. $X_1, X_2, \ldots$ iid Cauchy (density $\frac{1}{\pi(1+x^2)}$). Then

$$S_n = \frac{X_1 + \ldots + X_n}{n} \sim X_1$$

(S_n behaves like a random number drawn acc. to Cauchy dist.)

so in no reasonable sense $S_n \approx \mathbb{E}X_1$.

Reason: $\mathbb{E}X_1$ does not exist! $\left(\mathbb{E}X_1^+ = \int_0^\infty \frac{x}{\pi(1+x^2)}\,dx = +\infty, \quad \mathbb{E}X_1^- = +\infty \right)$.

E.g. $\varepsilon_1, \varepsilon_2, \ldots$ iid random signs, $S_n = \dfrac{\varepsilon_1 + \ldots + \varepsilon_n}{n}$

By Bernstein's ineq $\mathbb{P}\left(\left|\frac{S_n}{n}\right| > t\right) \le 2e^{-nt^2/2}$, so

$$\sum_{n=1}^{\infty} \mathbb{P}\left(\left|\frac{S_n}{n}\right| > t\right) < \infty, \text{ so } \quad \frac{S_n}{n} \xrightarrow{a.s.} 0 = \mathbb{E}\varepsilon_1, \text{ so}$$

SLLN holds for random signs.

Weak LLN

L_2 weak law

Thm Let $X_1, X_2, \ldots$ be r.vs s.t. $\forall i \; \mathbb{E}|X_i|^2 < \infty$. If

$S_n = X_1 + \ldots + X_n$

$\frac{1}{n^2} \operatorname{Var}(X_1 + \ldots + X_n) \underset{n\to\infty}{\to} 0$, then $\frac{S_n}{n} \xrightarrow{L_2} \mathbb{E}\frac{S_n}{n}$.

In part., say the X_i are uncorrelated with bdd variance ($\forall i \; \operatorname{Var} X_i \le M$). Then $\frac{S_n}{n} \xrightarrow{L_2} \mathbb{E}\frac{S_n}{n}$, in part., the X_i satisfy the WLLN.

Proof

$$\mathbb{E}\left|\frac{S_n}{n} - \mathbb{E}\frac{S_n}{n}\right|^2 = \frac{1}{n^2}\mathbb{E}|S_n - \mathbb{E}S_n|^2$$

$$= \frac{1}{n^2}\operatorname{Var}(X_1 + \ldots + X_n) \underset{n\to\infty}{\to} 0.$$

If the X_i are uncorr with bdd var,

$$\operatorname{Var}(X_1 + \ldots + X_n) = \operatorname{Var}(X_1) + \ldots + \operatorname{Var}(X_n) \le n \cdot M,$$

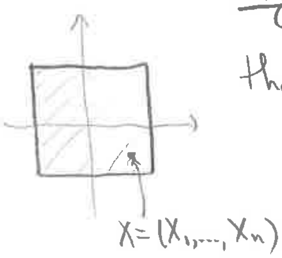
so $$\frac{1}{n^2}\operatorname{Var}(X_1 + \ldots + X_n) \le \frac{M}{n} \underset{n\to\infty}{\to} 0. \quad \square$$

gen. weak law

Thm If $X_1, X_2, \ldots$ are i.i.d. s.t. $t\mathbb{P}(|X_1| > t) \underset{t\to\infty}{\to} 0$, then

$$\frac{S_n}{n} - \mu_n \xrightarrow{\mathbb{P}} 0, \qquad \mu_n = \mathbb{E}X_1 \mathbb{1}_{\{|X_1| \le n\}}.$$

E.g. Let $X \sim \text{Unif}([-1,1]^n)$ (random point in the cube)

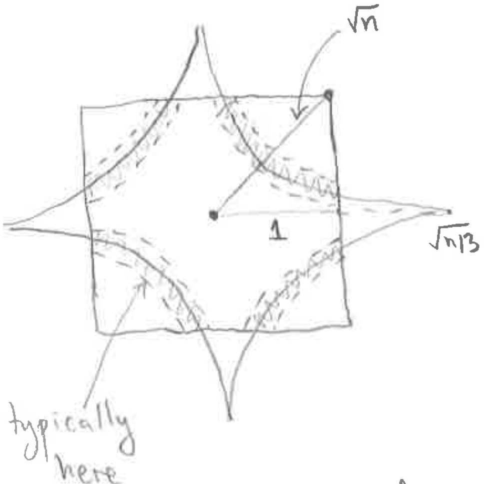


that is $X=(X_1,\dots,X_n)$, X_i iid $\text{Unif}([-1,1])$.

Then by the L_2 weak law

$$\frac{X_1^2+\dots+X_n^2}{n} \xrightarrow{\mathbb{P}} \mathbb{E}X_1^2 = \frac{1}{3},$$

that is $\forall \varepsilon$
$$\mathbb{P}\left(\left| \frac{X_1^2+\dots+X_n^2}{n} - \frac{1}{3} \right| > \varepsilon \right) \xrightarrow[n\to\infty]{} 0$$



$$\mathbb{P}\left(\left| \frac{X_1^2+\dots+X_n^2}{n} - \frac{1}{3} \right| < \varepsilon \right) \longrightarrow 1$$

$$\mathbb{P}\left(\frac{1}{3}-\varepsilon < \frac{X_1^2+\dots+X_n^2}{n} < \frac{1}{3}+\varepsilon \right) \longrightarrow 1$$

$$\mathbb{P}\left(\sqrt{n\left(\tfrac{1}{3}-\varepsilon\right)} < \sqrt{X_1^2+\dots+X_n^2} < \sqrt{n\left(\tfrac{1}{3}+\varepsilon\right)} \right) \to 1$$

so a random point in a high dim. cube is typically near the boundary of the ball of radius $\sqrt{\frac{n}{3}}$.

Strong LLN

Thm (Kolmogorov) If $X_1, X_2, \dots$ are iid s.t. $\mathbb{E}|X_1| < \infty$, then

$$\frac{S_n}{n} \xrightarrow{a.s.} \mathbb{E}X_1.$$

To prove this we need to prepare some tools.

<u>Lm</u> (Kronecker) Let (a_n) be a seq. of reals.
If $\sum_{n=1}^{\infty} \frac{a_n}{n}$ converges, then, $\frac{a_1+\dots+a_n}{n} \xrightarrow[n\to\infty]{} 0$.

<u>Proof</u> Let $s_n = \sum_{k=1}^{n} \frac{a_k}{k}$. Then $s_1 = a_1$, $s_n - s_{n-1} = \frac{a_n}{n}$, $n>1$, so

$$\frac{a_1+\dots+a_n}{n} = \frac{s_1 + 2(s_2-s_1) + 3(s_3-s_2) + \dots + n(s_n - s_{n-1})}{n}$$

$$= \frac{n s_n - s_1 - s_2 - \dots - s_{n-1}}{n}.$$

Fix $\varepsilon > 0$. If (s_n) converges, then by the Cauchy condition

$$\exists N \ \forall n,m > N \qquad |s_n - s_m| < \varepsilon,$$

and s_n is bdd, say $\forall n \ |s_n| \le M$, so for $n > N$,

$$\left| \frac{n s_n - s_1 - \dots - s_{n-1}}{n} \right| = \left| \frac{(N+1)s_n - s_1 - \dots - s_N}{n} + \frac{s_n - s_{N+1}}{n} + \dots + \frac{s_n - s_{n-1}}{n} \right|$$

$$\le \frac{(2N+1)M}{n} + \frac{(n-N-1)\cdot\varepsilon}{n} < 2\varepsilon,$$

for n large enough. $\square$

<u>Thm</u> (Kolmogorov's maximal ineq.) If $X_1, \dots, X_n$ are <u>indep.</u> r.vs s.t.

$\forall i \ \mathbb{E}|X_i|^2 < \infty$ and $\mathbb{E}X_i = 0$, then for $t > 0$,

$$\mathbb{P}\left(\max_{1 \le k \le n} |X_1 + \dots + X_k| \ge t \right) \le \frac{1}{t^2} \mathrm{Var}(X_1 + \dots + X_n).$$

Proof Let $S_k = X_1 + \dots + X_k$, $k = 1, \dots, n$, $S_0 = 0$,

$A_k = \{ |S_j| < t$ for $j < k$, and $|S_k| \geq t \}$

Then A_j are disjoint events , $\bigcup A_j = \{ \max_{1 \le j \le n} |S_j| \geq t \}$.

Moreover, A_k depends only on $X_1, \dots, X_k$ (not on $X_{k+1}, \dots, X_n$).

$\mathbb{E}X_i = 0$
$\mathbb{E}S_n = 0$

$$\operatorname{Var} S_n = \mathbb{E} S_n^2 \geq \mathbb{E} S_n^2 \mathbb{1}_{\{\bigcup_{k=1}^n A_k\}} \underset{\substack{\uparrow \\ A_k \text{ disjoint}}}{=} \sum_{k=1}^n \mathbb{E} S_n^2 \mathbb{1}_{A_k}$$

$$= \sum_k \mathbb{E} (S_n - S_k + S_k)^2 \mathbb{1}_{A_k}$$

$$= \sum_k \Big[\mathbb{E} (S_n - S_k)^2 \mathbb{1}_{A_k} + 2\, \mathbb{E} (S_n - S_k) \cdot S_k \mathbb{1}_{A_k} + \mathbb{E} S_k^2 \mathbb{1}_{A_k} \Big]$$

$$\geq \sum_k \Big[2\, \underbrace{\mathbb{E}(S_n - S_k) \cdot S_k \mathbb{1}_{A_k}}_{\|\ \text{indep.}} + \mathbb{E} S_k^2 \mathbb{1}_{A_k} \Big]$$

$$\underbrace{\mathbb{E}(S_n - S_k)}_{0}, \ \mathbb{E} S_k \mathbb{1}_{A_k} = 0$$

$$= \sum_k \mathbb{E} S_k^2 \mathbb{1}_{A_k} \underset{\substack{\text{on } A_k \\ |S_k| \geq t}}{\geq} \sum \mathbb{E} t^2 \mathbb{1}_{A_k}$$

$$= t^2 \sum_k \mathbb{P}(A_k) \underset{\substack{\uparrow \\ A_k \text{ disjoint}}}{=} t^2\, \mathbb{P}\Big(\bigcup A_k \Big)$$

$$= t^2\, \mathbb{P}\Big(\max_{1 \le k \le n} |S_n| \geq t \Big). \ \square$$

Thm. Let $X_1, X_2, \ldots$ be indep. r.vs s.t. $\forall i$ $\mathbb{E}|X_i|^2 < \infty$ and $\mathbb{E}X_i = 0$. If $\sum \mathrm{Var}\, X_n$ converges, then $\sum X_n$ converges a.s.

$\boxed{\sum X_n \text{ converges}}$ $\overset{\text{Cauchy}}{\Updownarrow}$ $\boxed{\forall \varepsilon\ \exists N\ \sup_{n>N} |X_N + \ldots + X_n| < \varepsilon}$

Proof WTS $\mathbb{P}(\sum X_n \text{ diverges}) = 0$. We have,

$$\mathbb{P}\left(\sum X_n \text{ diverges}\right) = \mathbb{P}\left(\exists \ell\ \forall N \sup_{n>N} |X_N + \ldots + X_n| \geq \tfrac{1}{\ell}\right)$$

$$= \mathbb{P}\left(\bigcup_{\ell} \bigcap_{N} \left\{\sup_{n>N} |X_N + \ldots + X_n| > \tfrac{1}{\ell}\right\}\right)$$

$$\leq \sum_{\ell} \mathbb{P}\left(\bigcap_{N} \left\{\sup_{n>N} |X_N + \ldots + X_n| > \tfrac{1}{\ell}\right\}\right)$$

so it suffices to show that each term is zero,

$$\mathbb{P}\left(\bigcap_{N}\left\{\sup_{n>N} |X_N + \ldots + X_n| > \tfrac{1}{\ell}\right\}\right) = \lim_{N\to\infty} \mathbb{P}\left(\sup_{n>N} |X_N + \ldots + X_n| > \tfrac{1}{\ell}\right)$$

$$= \lim_{N\to\infty} \underbrace{\mathbb{P}\left(\bigcup_{n>N} \left\{\max_{N<k\leq n} |X_N + \ldots + X_k| > \tfrac{1}{\ell}\right\}\right)}$$

$$\| $$

$$\lim_{n\to\infty} \underbrace{\mathbb{P}\left(\max_{N<k\leq n} |X_N + \ldots + X_k| > \tfrac{1}{\ell}\right)}$$

Kolmogorov's ineq. $\mathbin{|\!\wedge}$

$$\frac{1}{(1/\ell)^2} \mathrm{Var}\,(X_N + \ldots + X_n) = \ell^2 \sum_{k=N}^{n} \mathrm{Var}\, X_k$$

$$\leq \ell^2 \lim_{N\to\infty} \sum_{k=N}^{\infty} \mathrm{Var}\, X_k = 0$$

$\sum_{k=1}^{\infty} \mathrm{Var}\, X_k < \infty$ $\square$

Lm (Borel-Cantelli) If $A_1, A_2, \dots$ are events s.t.

$$\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty \quad \text{then} \quad \mathbb{P}(\text{infinitely many } A_n \text{ occur}) = 0$$

Proof $\mathbb{P}(\infty\text{-many } A_n \text{ occur}) = \mathbb{P}(\forall N\ \exists n>N\ \ A_n \text{ occur})$

$$= \mathbb{P}\Big(\bigcap_N \bigcup_{n>N} A_n\Big) = \lim_{N\to\infty} \mathbb{P}\Big(\bigcup_{n>N} A_n\Big)$$

$$\le \lim_{N\to\infty} \sum_{n>N} \mathbb{P}(A_n) = 0 \quad \square$$

Proof of Kolmogorov's SLLN

⚠ If we assumed $\mathbb{E}|X_1|^2 < \infty$, we would finish quickly:

WTS $\quad \frac{S_n}{n} \xrightarrow{a.s.} \mathbb{E}X_1 \iff \underbrace{\frac{S_n}{n} - \mathbb{E}X_1}_{\frac{(X_1-\mathbb{E}X_1)+\dots+(X_n-\mathbb{E}X_n)}{n}} \xrightarrow{a.s.} 0$

$$\frac{\bar{X}_1 + \dots + \bar{X}_n}{n} \xrightarrow{a.s.} 0 \quad \underset{\text{Kronecker's Lm}}{\Longleftarrow} \quad \sum \frac{\bar{X}_n}{n} \text{ converges a.s.}$$

$$\Uparrow \text{ OTT Lm}$$

$$\text{Var}(X_1) \sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=1}^{\infty} \frac{\text{Var}(\bar{X}_n)}{n^2} = \sum_n \text{Var}\Big(\frac{\bar{X}_n}{n}\Big) < \infty$$

finite b/c $\mathbb{E}|X_1|^2 < \infty$ $\qquad$ iid $\text{Var}(\bar{X}_n) = \text{Var}(\bar{X}_1) = \text{Var}\, X_1$

Let's only assume $\mathbb{E}|X_1| < \infty$. Consider truncations

$$Y_n = X_n \mathbb{1}_{\{|X_n| \le n\}}, \quad n = 1, 2, \dots$$

Y_n are <u>indep</u>

We have

$$\frac{X_1 + \dots + X_n}{n} - \mathbb{E}X_1 = R_n + S_n + T_n ,$$

$$R_n = \frac{X_1 + \dots + X_n - (Y_1 + \dots + Y_n)}{n}$$

$$S_n = \frac{Y_1 + \dots + Y_n - (\mathbb{E}Y_1 + \dots + \mathbb{E}Y_n)}{n}$$

$$T_n = \frac{\mathbb{E}Y_1 + \dots + \mathbb{E}Y_n}{n} - \mathbb{E}X_1 .$$

WTS $\quad R_n \xrightarrow{a.s.} 0 , \quad S_n \xrightarrow{a.s.} 0 , \quad T_n \longrightarrow 0 .$

T_n :

$$a_n = \mathbb{E}Y_n = \mathbb{E}X_n \mathbb{1}_{\{|X_n| \le n\}} = \mathbb{E}X_1 \mathbb{1}_{\{|X_1| \le n\}} \xrightarrow[\text{Leb. dom.}]{n \to \infty} \mathbb{E}X_1$$

so $\quad \lim a_n = \mathbb{E}X_1$, so $\quad \lim \frac{a_1 + \dots + a_n}{n} = a .$

R_n :

$$\sum_{n=1}^{\infty} \mathbb{P}(\overbrace{X_n \ne Y_n}^{A_n}) = \sum_{n=1}^{\infty} \mathbb{P}(|X_n| > n)$$

$$= \sum_{n=1}^{\infty} \mathbb{P}(|X_1| > n) \underset{\mathbb{E}|X_1| < \infty}{<} \infty$$

so by B-C Lm : $\quad \mathbb{P}(\text{ finitely many } \{X_n \ne Y_n\} \text{ occur}) = 1$,

that is $\quad \mathbb{P}(\{\omega,$ the sequences $(X_n(\omega))$ and $(Y_n(\omega))$ are eventually the same $) = 1$

$\shortparallel$

$\mathbb{P}(\lim R_n = 0)$.

$\textcircled{S_n}$: As in $\triangle$, By Kronecker's Lm and OTT Lm it suffices to show

$$\sum_n \frac{\mathrm{Var}(Y_n - \mathbb{E}Y_n)}{n^2} < \infty.$$

- $\mathrm{Var}(Y_n - \mathbb{E}Y_n) = \mathrm{Var}\, Y_n = \mathbb{E}Y_n^2 - (\mathbb{E}Y_n)^2$

$$\leq \mathbb{E}Y_n^2 = \sum_{k=1}^{\infty} \mathbb{E}\, Y_n^2 \mathbb{1}_{\{k-1 \leq |Y_n| \leq k\}}$$

$|Y_n| \leq n$ and $Y_n = 0$ if $|X_n| > n$

$$= \sum_{k=1}^{n} \mathbb{E}\, X_n^2 \mathbb{1}_{\{k-1 < |X_n| \leq k\}}$$

$$= \sum_{k=1}^{n} \mathbb{E}\, X_1^2 \mathbb{1}_{\{k-1 < |X_1| \leq k\}}$$

$$\leq \sum_{k=1}^{n} k\, \mathbb{E}|X_1| \mathbb{1}_{\{k-1 < |X_1| \leq k\}}$$

- $$\sum_{n=1}^{\infty} \frac{\mathrm{Var}(Y_n - \mathbb{E}Y_n)}{n^2} = \sum_{n=1}^{\infty}\sum_{k=1}^{n} \frac{1}{n^2}\, k\, \mathbb{E}|X_1| \mathbb{1}_{\{k-1 < |X_1| \leq k\}}$$

$$= \sum_{k=1}^{\infty} \underbrace{\left(\sum_{n=k}^{\infty} \frac{1}{n^2}\right)}_{\underset{\text{exercise}}{\leq} \frac{2}{k}} \cdot k\, \mathbb{E}|X_1| \mathbb{1}_{\{k-1 < |X_1| \leq k\}}$$

$$\leq \sum_{k=1}^{\infty} \frac{2}{k} \cdot k \cdot \mathbb{E}|X_1| \mathbb{1}_{\{k-1 < |X_1| \leq k\}}$$

$$= 2\mathbb{E}|X_1| < \infty. \quad \square$$

E.g. Find $\lim_{n\to\infty} I_n$, $I_n = \int_0^1 \cdots \int_0^1 \frac{x_1^3 + \cdots + x_n^3}{x_1 + \cdots + x_n} \, dx_1 \ldots dx_n$.

Let $X_1, \ldots, X_n$ iid Unif $[0,1]$. Density of $(X_1, \ldots, X_n)$ is $\prod_{i=1}^{n} \mathbb{1}_{[0,1]}(x_i)$, so

$$\mathbb{E} f(X_1, \ldots, X_n) = \int_0^1 \cdots \int_0^1 f(x_1, \ldots, x_n) \, dx_1 \ldots dx_n,$$

$$I_n = \mathbb{E} \underbrace{\boxed{\frac{X_1^3 + \cdots + X_n^3}{X_1 + \cdots + X_n}}}_{Y_n} = \mathbb{E} \frac{\frac{X_1^3 + \cdots + X_n^3}{n}}{\frac{X_1 + \cdots + X_n}{n}},$$

by strong LLN,

$$\frac{X_1^3 + \cdots + X_n^3}{n} \xrightarrow{a.s.} \mathbb{E}X_1^3 = \frac{1}{4},$$

$$\frac{X_1 + \cdots + X_n}{n} \xrightarrow{a.s.} \mathbb{E}X_1 = \frac{1}{2},$$

so $Y_n \xrightarrow{a.s.} \frac{1/4}{1/2} = \frac{1}{2}$, moreover $|Y_n| \le 1$,

so by Lebesgue's dom. convergence thm,

$$I_n = \mathbb{E} Y_n \longrightarrow \frac{1}{2}. \quad \square$$