29/11/2014

Functional Analysis I, Support Classes, Term 1 2014/2015

Tomasz Tkocz, t (dot) tkocz (at) warwick (dot) ac (dot) uk

Week 2

Class	1, 2a, 4
⌂	2b, 3, 6
Wishes	—

Extra Question. Let $0 < p < q < \infty$. Prove that for every sequence $x = (x_1, x_2, \ldots)$ we have

$$\|x\|_q \leq \|x\|_p.$$

Hint. Prove that for nonegative numbers a, b and $\beta \in (0,1)$ we have $(a+b)^\beta \leq a^\beta + b^\beta$.

Week 3

Class	5, 7, 11, 13
⌂	8, 9, 12
Wishes	?

Extra Question. Find all $p > 0$ for which the following holds: for every nonnegative numbers $x_{i,j}$, $i = 1, \ldots, n$, $j = 1, \ldots, m$, we have

$$\left(\sum_{i=1}^{n} \left(\sum_{j=1}^{m} x_{i,j} \right)^p \right)^{1/p} \leq \sum_{j=1}^{m} \left(\sum_{i=1}^{n} x_{i,j}^p \right)^{1/p}.$$

Extra Question. Let $\{w_i,\ i \in I\}$ be a Hamel basis of the vector space $\mathbb{R}$ over $\mathbb{Q}$ (I is the index set of the elements of the chosen basis). Then for every $x \in \mathbb{R}$ we can write $x = \sum_{i \in I} f_i(x) w_i$, where all but finitely many $f_i(x)$'s are zero.

Fix $i_0 \in I$ and prove that the function $f = f_{i_0} : \mathbb{R} \longrightarrow \mathbb{Q}$ satisfies the properties

1) $f(x+y) = f(x) + f(y)$, for every $x, y \in \mathbb{R}$,

2) the set $\mathrm{Graph}(f) = \{(x, f(x)),\ x \in \mathbb{R}\}$ is a dense subset of $\mathbb{R}^2$.

Week 4

Class	10, 14, 18, 21
⌂	15, 16, 17, 19, 20
Wishes	?

Extra Question. Prove that for a normed vector space $(V, \|\cdot\|)$ the following conditions are equivalent

(i) every absolutely convergent series is convergent, i.e. for every $v_1, v_2, \ldots \in V$ if $\sum_{n=1}^{\infty} \|v_n\| < \infty$, then the sequence $(\sum_{k=1}^{n} v_k)_{n=1}^{\infty}$ converges in$(V, \|\cdot\|)$

(ii) for every $v_1, v_2, \ldots \in V$ such that $\|v_n\| \leq 1/2^n$ the sequence $(\sum_{k=1}^{n} v_k)_{n=1}^{\infty}$ converges in $(V, \|\cdot\|)$

(iii) $(V, \|\cdot\|)$ is a Banach space.

Extra Question. Prove that $L_p([0,1], \mathbb{R})$ is a Banach space for every $p \geq 1$.

Extra Question. Let $K \subset \mathbb{R}^n$ be a symmetric compact convex set with nonempty interior. Prove that

$$\|x\| = \frac{1}{\sup\{t > 0,\ tx \in K\}}$$

defines a norm on $\mathbb{R}^n$.

We say that a subset A of $\mathbb{R}^n$ is *convex* if for every $a, b \in A$ and $t \in [0,1]$ we have $ta + (1-t)b \in A$. We say that it is *symmetric* if for every $a \in A$ we have $-a \in A$.

Week 5

Class	22, 23, 24
⌂	25
Wishes	?

Extra Question. Let $(V, \|\cdot\|)$ be a real Banach space such that the norm $\|\cdot\|$ satisfies the parallelogram identity, i.e. for every $x, y \in V$

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

Prove that there exists an inner product $\langle \cdot, \cdot \rangle$ on V for which the associated norm $\langle \cdot, \cdot \rangle^{1/2}$ is $\|\cdot\|$.

Remark. Together with Lemma 5.12 from the lecture, this proves the famous characterization of Hilbert spaces due to Jordan and von Neumann,

A Banach space is isometrically isomorphic to a Hilbert space if and only if its norm satisfies the parallelogram identity.

Week 6

Class	26, 29, 31
⌂	27, 28, 30
Wishes	?

Extra Question. Let $p \in (1, 2]$ and $f, g \in L_p([0, 1])$. Prove that

$$\left\|\frac{f+g}{2}\right\|_p^q + \left\|\frac{f-g}{2}\right\|_p^q \le \left(\frac{\|f\|_p^p + \|g\|_p^p}{2}\right)^{q-1},$$

where $1/p + 1/q = 1$.

Week 7

Class	30, 31
⌂	37
Wishes	?

Extra Question. Prove that there exists a constant C such that for every polynomial p of degree less than or equal to 2014 we have $|p(17)| \le C \sup_{x\in[0,1]} |p(x)|$.

Hints. What is the dimension of the space of polynomials of degree less than or equal to 2014? What can be said about continuity of functionals acting on a finite dimensional space?

Week 8

Class	33, 34, 32/39, 40
⌂	35, 36, 38, 41
Wishes	?

Extra Question. Define the following *shift* operator $T\colon \ell_2 \longrightarrow \ell_2$,

$$T(x_1, x_2, \ldots) = (x_2, x_3, \ldots).$$

Prove that it is bounded, find its norm and find its adjoint $T^\star$.

Extra Question. Define the following operator $T\colon \ell_\infty \longrightarrow \ell_\infty$,

$$T(x_1, x_2, x_3, \ldots) = \left(x_1, \frac{x_1 + x_2}{2}, \frac{x_1 + x_2 + x_3}{3}, \ldots \right).$$

Obviously T is bounded. Prove that T is also bounded as an operator acting on ℓ_2 and find the norm $\|T\|_{\ell_2 \to \ell_2}$. Find its adjoint $T^\star$ and derive the inequality

$$\left\| \left(\sum_{k=1}^{\infty} \frac{y_k}{k}, \sum_{k=2}^{\infty} \frac{y_k}{k}, \ldots \right) \right\|_2 \leq 2 \|(y_1, y_2, \ldots)\|_2.$$

Hints. For a sequence of real numbers $a_1, a_2, \ldots, a_N$ define $A_n = a_1 + \ldots + a_n$. Using e.g. summation by parts show the identity

$$\sum_{n=1}^{N} \left(\frac{A_n}{n} \right)^2 - 2 \sum_{n=1}^{N} a_n \frac{A_n}{n} = -\frac{A_N^2}{N} - \sum_{n=2}^{N} (n-1) \left(\frac{A_{n-1}}{n-1} - \frac{A_n}{n} \right)^2.$$

Then applying the Cauchy-Schwarz inequality conclude that

$$\sqrt{\sum_{n=1}^{N} \left(\frac{A_n}{n} \right)^2} \leq 2 \sqrt{\sum_{n=1}^{N} a_n^2},$$

in fact proving that $\|T\|_{\ell_2 \to \ell_2} \leq 2$.

Week 9

Class	42, 45, 48/50
⌂	44, 46, 47, 49, 51
Wishes	?

Extra Question. Let γ be the standard Gaussian measure on $\mathbb{R}$, i.e. the measure with density $\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$. Consider the Hilbert space $L_2(\mathbb{R}, \gamma)$. Recall that the Hermite polynomials $\{h_n, n = 0, 1, 2, \ldots\}$,

$$h_n(x) = \frac{(-1)^n}{\sqrt{n!}} e^{x^2/2} \frac{\mathrm{d}^n}{\mathrm{d}x^n} e^{-x^2/2},$$

form an orthonormal set in $L_2(\gamma)$ (Question 28). Prove that it is complete, i.e. every $f \in L_2(\gamma)$ can be written uniquely as

$$f = \sum_{n=0}^{\infty} \langle f, h_n \rangle h_n.$$

Prove that for a smooth function $f \in L_2(\gamma)$ we have the following inequality

$$\mathrm{Var}_\gamma(f) = \int \left(f - \int f \mathrm{d}\gamma \right)^2 \mathrm{d}\gamma \leq \int (f')^2 \mathrm{d}\gamma$$

sometimes referred to as the *Poincaré inequality*.

Week 10

Class	52, 53, 54, 55
⌂	—
Wishes	?

Extra Question. Take $a \in \ell_\infty$ and define the following *multiplication operator*,

$$T_a \colon \ell_2 \longrightarrow \ell_2,$$
$$T_a(x_1, x_2, \ldots) = (a_1 x_1, a_2 x_2, \ldots).$$

Show that T_a is compact if and only if $a \in c_0$.

Extra Question (Hilbert-Schmidt operators)**.** Let $(\phi_n)_{n=1}^\infty$, $(\psi_n)_{n=1}^\infty$ be orthonormal bases of a Hilbert space H. Let $T \colon H \longrightarrow H$ be a bounded linear operator.

1. Rewriting $\sum_{m,n} |\langle T\phi_m, \psi_n \rangle|^2$ show that $\sum_n \|T\phi_n\|^2 = \sum_n \|T^*\psi_n\|^2$.
2. Conclude that the quantity $\sqrt{\sum_n \|T\psi_n\|^2} \in [0, \infty]$ is well-defined, i.e. it does not depend on the choice of the basis. It is denoted by $\|T\|_{HS}$ and called the Hilbert-Schmidt norm of T. If it is finite we say that T is a *Hilbert-Schmidt operator*. Show that $\|T\|_{HS} = \|T^*\|_{HS}$.
3. Observe that
$$\|Tx\|^2 = \sum_n \langle T\phi_n, Tx \rangle \langle x, \phi_n \rangle.$$
Then, by using the Cauchy-Schwarz inequality twice, prove that $\|T\| \leq \|T\|_{HS}$.
4. Suppose that $H = \mathbb{C}^n$ and let T be given by a matrix $T = [t_{ij}]_{i,j=1}^n$. Find $\|T\|_{HS}$ in terms of t_{ij}'s.

5. By approximating a HS operator T with finite rank operators $T_N(x) = \sum_{n=1}^{N} \langle x, \phi_n \rangle T\phi_n$ show that T is compact.

6. (Key example of HS operators) Let $k \in L_2([0,1]^2)$ and define the following *convolution operator*, $K\colon L_2([0,1]) \longrightarrow L_2([0,1])$,

$$(Kf)(x) = \int_0^1 k(x,y)f(y)\mathrm{d}y.$$

Note that $(Kf)(x) = \langle k(x,\cdot), \bar{f}(\cdot)\rangle$, and hence

$$\langle Kf, Kf\rangle = \int_0^1 \left|\langle k(x,\cdot), \bar{f}(\cdot)\rangle\right|^2 \mathrm{d}x.$$

Using this show that K is a Hilbert-Schmidt operator.

Extra Question (Square root operator)**.** Let H be a complex Hilbert space and consider a bounded operator $T\colon H \longrightarrow H$ which is positive semi-definite (positive in short), i.e. $\langle Tx, x\rangle \geq 0$ for every $x \in H$.

1. Establish the following *polarisation identity*,

$$\langle Tx, y\rangle = \frac{1}{4}\Big(\langle T(x+y), x+y\rangle - \langle T(x-y), x-y\rangle + i\langle T(x+iy), x+iy\rangle - i\langle T(x-iy), x-iy\rangle\Big).$$

Show that T begin positive is also self-adjoint.

2. Assuming that in addition T is compact, construct an operator $Q\colon H \longrightarrow H$ such that $T = Q^2$.

3. The Taylor expansion of the function $z \to \sqrt{1-z}$ about $z = 0$ reads

$$\sqrt{1-z} = \sum_{n=0}^{\infty} \frac{-1}{2n-1} 4^{-n} \binom{2n}{n} z^n.$$

Show that the series converges absolutely on the whole unit disc $\{z \in \mathbb{C},\ |z| \leq 1\}$.

4. By the result of 1. we get that for a positive operator T,

$$\|T\| = \sup_{\|x\|=1} \langle Tx, x\rangle.$$

Fix a positive operator T with norm at most one. Using the above formula and 3. show that $Q = \sum_{n\geq 0} \frac{-1}{2n-1} 4^{-n} \binom{2n}{n} T^n$ defines a bounded operator. Check that $Q^2 = I - T$.

5. Conclude that for a positive bounded operator T acting on a complex Hilbert space there exists a bounded operator Q such that $Q^2 = T$. Show that there is only one operator Q with this property. Moreover, show that Q is also positive and commutes with any operator that T commutes with.

Rules

1. There will be extra questions each week.
2. You can submit your solution to any extra question at any time during the term (no deadlines ;)).
3. Please submit your work by email or into my pigeon hole which is located on the first floor (opposite B1.38).
4. The author of the first correct solution of each question will receive a small prize (e.g. a chocolate bar, subject to my limited resources).
5. By Friday in Week n you can email your wishes for the Monday class in Week $n+1$, $n \in \{2, 3, \dots, 9\}$.