Functional analysis II, Revision lecture

Term 3 2014/2015

Problems

1. Let $(X, \|\cdot\|)$ be a normed vector space. Prove that if $\|x+y\| = \|x\| + \|y\|$ for some $x, y \in X$, then for every nonnegative real numbers α, β we have $\|\alpha x + \beta y\| = \alpha\|x\| + \beta\|y\|$.

2. Let f and $f_1, f_2, \ldots, f_n$ be linear functionals defined on the same vector space. Prove that
$$\bigcap_{j=1}^{n} \ker f_j \subset \ker f$$
if and only if f is a linear combination of $f_1, \ldots, f_n$.

3. Let Y be a closed subspace of a normed vector space X. Prove that if Y and X/Y are separable, then so is X.

4. Is the quotient space ℓ_∞/c_0 separable?

5. Let Y be a closed subspace of a normed vector space X. Prove that if Y and X/Y are complete, then so is X.

6. Suppose X, Y are closed subspaces of a normed vector space. Need $X+Y$ be closed?

7. Let $1 \le p < q$. Show that the set
$$A = \left\{ f \in L_p[0,1],\ \int_0^1 |f|^q \le 1 \right\}$$
is closed with empty interior in $(L_p[0,1], \|\cdot\|_p)$. Conclude that $L_q[0,1]$ is a countable union of nowhere dense sets in $(L_p[0,1], \|\cdot\|_p)$. Why does this not contradict Baire's theorem and L_p spaces being Banach?

8. Let f be a nonzero functional on a normed vector space. Prove that the following conditions are equivalent
$$f \text{ is continuous}, \qquad (\clubsuit)$$
$$\ker f \text{ is closed}, \qquad (\spadesuit)$$
$$\ker f \text{ is nowhere dense}. \qquad (\diamondsuit)$$

9. Given a vector space X, is it always possible to define a norm $\|\cdot\|$ on X such that $(X, \|\cdot\|)$ becomes a Banach space? (In other words, is every vector space *Banach-normable*?)

10. Let X be a Banach space in which every subspace is closed. Show that X is finite dimensional.

11. Give an example of a vector space X for which there are two norms $\|\cdot\|$ and $\|\cdot\|'$ such that $(X, \|\cdot\|)$ is separable but $(X, \|\cdot\|')$ is not.

12. Define the Rademacher functions

$$r_n(t) = \operatorname{sgn}\big(\sin(2^n \pi t)\big), \quad n = 0, 1, 2, \dots.$$

Show that $\{r_n, n \geq 0\}$ is an incomplete orthogonal system in $L_2[0,1]$.

13. Show that every orthogonal subset of a separable Hilbert space is countable.

14. Let C be a nonempty closed and convex subset of a Hilbert space H. We know that for every $x \in H$ there is a unique best approximation x^* of x in C, that is $\|x^* - x\| = \inf_{a \in C} \|a - x\|$. Show that for every $x, y \in H$ we have

$$\|x^* - y^*\| \leq \|x - y\|.$$

15. Let P, Q be orthogonal projections in Hilbert space. Prove that $\|P - Q\| \leq 1$.

16. Let T be an $n \times n$ matrix with row vectors $a_1, \dots, a_n \in \mathbb{R}^n$ and column vectors $b_1, \dots, b_n \in \mathbb{R}^n$,

$$T = \begin{bmatrix} - & a_1 & - \\ - & a_2 & - \\ & \cdots & \\ - & a_n & - \end{bmatrix} = \begin{bmatrix} | & | & & | \\ b_1 & b_2 & \dots & b_n \\ | & | & & | \end{bmatrix}.$$

Show that T, as a linear operator acting on certain ℓ_p spaces, has the following norms

$$\|T\|_{\ell_p^n \to \ell_\infty^n} = \max_{j \leq n} \|a_j\|_q,$$
$$\|T\|_{\ell_1^n \to \ell_p^n} = \max_{j \leq n} \|b_j\|_p,$$

where $p \in [1, \infty]$ and $1/p + 1/q = 1$.

17. Find all $\alpha \in \mathbb{R}$ for which the linear map $T \colon \ell_3 \longrightarrow \ell_1$, $Tx = (n^\alpha x_n)_{n \geq 1}$ is bounded.

18. Give an example of a bounded linear map $S \colon c_0 \longrightarrow c_0$ for which there is *no* linear extension $\tilde{S} \colon c \longrightarrow c_0$ preserving the norm, that is $\tilde{S}|c_0 = S$ and $\|\tilde{S}\| = \|S\|$.

19.* Let $(X, \|\cdot\|)$ be an n dimensional normed vector space. Show that there are linearly independent unit vectors $x_1, \dots, x_n \in X$ and functionals $\phi_1, \dots, \phi_n \in X^*$ of norm one satisfying $\phi_j(x_i) = \delta_{ij}$ for every $i, j \leq n$. (Auerbach's lemma.)

20. Let $(X, \|\cdot\|)$ be an n dimensional normed vector space. Show that there is a basis $x_1, \dots, x_n$ of X such that for every scalars $\lambda_1, \dots, \lambda_n$ we have

$$\max_{j \leq n} |\lambda_j| \leq \left\| \sum_{j=1}^{n} \lambda_n x_n \right\| \leq \sum_{j=1}^{n} |\lambda_j|.$$

21. Let $(X, \|\cdot\|)$ be a normed vector space which is reflexive. Prove that for every bounded functional $\phi \in X^*$ there is a unit vector $x \in X$ such that $\phi(x) = \|\phi\|$.

22. Prove that the spaces: $c_0, c, \ell_1, C[0,1], L_1[0,1]$ are *not* reflexive.

23. Let X be a normed vector space. Show that every weakly convergent sequence in X is bounded.

24. Let X be a Banach space. Show that every weakly* convergent sequence in X^* is bounded.

25. Let $\{v_n,\ n \geq 1\}$ be an orthogonal bounded set in a Hilbert space. Show that the sequence (v_n) converges weakly to 0.

26. Let $T\colon X \longrightarrow Y$ be a linear map between Banach spaces X, Y. Show that T is bounded if and only if for every weakly convergent sequence (x_n) in X, the sequence (Tx_n) is weakly convergent in Y.

27. Let $p \in (1, \infty)$. Show that a sequence (x_n) is weakly convergent to x in ℓ_p if and only if it is bounded and each coordinate of x_n converges to the corresponding coordinate of x.

28.† Show that if a sequence (x_n) converges weakly in ℓ_1 to x then $\|x_n - x\|_1 \underset{n\to\infty}{\longrightarrow} 0$. (Schur's property.)

29. Let X be a normed vector space. Show that if the dual space X^* is separable, then so is X.

30.* Let $(X, \mathcal{F}, \mu)$ and $(Y, \mathcal{G}, \nu)$ be measurable spaces. Let $p \in [1, \infty)$ and suppose $f: X \times Y \longrightarrow \mathbb{R}$ is $\mathcal{F} \otimes \mathcal{G}$ measurable. Then

$$\left\| y \mapsto \int_X f(x,y)\,d\mu(x) \right\|_{L_p(\nu)} \leq \int_X \|y \mapsto f(x,y)\|_{L_p(\nu)}\,d\mu(x)$$

(Minkowski's integral inequality).

Solutions

1. Suppose that $\alpha \geq \beta$. By the triangle inequality,

$$\|\alpha x + \beta y\| = \|\alpha(x+y) - (\alpha - \beta)y\| \geq \alpha\|x+y\| - (\alpha-\beta)\|y\|$$

which combined with the assumption gives

$$\|\alpha x + \beta y\| \geq \alpha\|x\| + \beta\|y\|.$$

By the triangle inequality, the opposite inequality holds as well. □

2. If $f = \alpha_1 f_1 + \ldots + \alpha_n f_n$ for some scalars α_i then plainly

$$\bigcap_{j=1}^{n} \ker f_j \subset \ker f.$$

We show the converse inductively on n. Let $n = 1$. If $f_1 = 0$, then by $\ker f_1 \subset \ker f$ also $f = 0$, so there is nothing to prove. Take then a nonzero vector v such that $f_1(v) \neq 0$. For every vector x we have

$$x - \frac{f_1(x)}{f_1(v)} v \in \ker f_1 \subset \ker f,$$

hence

$$f\left(x - \frac{f_1(x)}{f_1(v)} v\right) = 0$$

which yields $f = \frac{f(v)}{f_1(v)} f_1$. Suppose we have $n+1$ functionals $f_1, \ldots, f_{n+1}$ and $\bigcap_{j=1}^{n+1} \ker f_j \subset \ker f$. Consider the subspace $Z = \ker f_{n+1}$ and the restricted functionals $g_i = f_i|Z$, $i \leq n$, $g = f|Z$ on Z. By the inductive assumption, $g = \alpha_1 g_1 + \ldots + \alpha_n g_n$ (on Z) for some scalars α_i. This particularly implies that

$$\ker f_{n+1} = Z \subset \ker(f - \alpha_1 f_1 - \ldots - \alpha_n f_n),$$

so by the case $n = 1$ we get

$$f - \alpha_1 f_1 - \ldots - \alpha_n f_n = \alpha_{n+1} f_{n+1}$$

for some scalar α_{n+1}, which completes the proof. □

3. Let $\{y_n, n \geq 1\}$ be a dense subset in Y and let $\{x_n + Y, n \geq 1\}$ be a dense subset in X/Y. For any $\epsilon > 0$ and $x \in X$ we can find n such that

$$\|(x - x_n) + Y\| = \|(x+Y) - (x_n + Y)\| < \epsilon.$$

By the definition of a quotient norm and the fact that the y_n are dense in Y we can find m such that

$$\|x - x_n - y_m\| < 2\epsilon.$$

This shows that the set $\{x_n + y_m,\ n, m \geq 1\}$ is dense in X. □

4. We know that the space c_0 is separable, whereas ℓ_∞ is not. If the quotient space ℓ_∞ / c_0 was separable, then, by Problem 3, ℓ_∞ would be separable. □

5. Suppose (x_n) is a Cauchy sequence in X. Then clearly $(x_n + Y)$ is a Cauchy sequence in X/Y. By the assumption it converges, say to $x + Y$,

$$\|(x - x_n) + Y\| \underset{n\to\infty}{\longrightarrow} 0.$$

This means that there are $y_n \in Y$ such that

$$\|x - x_n - y_n\| < \|(x - x_n) + Y\| + 1/n \underset{n\to\infty}{\longrightarrow} 0.$$

In particular, $x_n + y_n$ converges to x. It remains to show that y_n converges as well. We have

$$\begin{aligned}\|y_n - y_m\| &\leq \|y_n + x_n - x\| + \|x - x_m - y_m\| + \|x_m - x_n\| \\ &\leq \|(x - x_n) + Y\| + \|(x - x_m) + Y\| + 1/n + 1/m + \|x_m - x_n\|\end{aligned}$$

which shows that (y_n) is a Cauchy sequence in Y. □

6. The subspace $X + Y$ need not be closed. Consider for instance

$$\begin{aligned} X &= \operatorname{span}\{e_{2n},\ n \geq 1\}, \\ Y &= \operatorname{span}\left\{ e_{2n} + \frac{1}{\sqrt{n}} e_{2n+1},\ n \geq 1 \right\}, \end{aligned}$$

in ℓ_2. These are closed subspaces (why?). Moreover, $\operatorname{span}\{e_n,\ n \geq 1\} \subset X + Y$. Therefore, if $X + Y$ was closed, we would have $X + Y = \ell_2$. However,

$$\sum_{n=1}^{\infty} \frac{1}{n} e_{2n+1} \in \ell_2 = X + Y$$

would imply that

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} e_{2n} \in X$$

but this vector does not belong to ℓ_2. This contradiction shows that $X + Y$ is not closed. □

7. Suppose that $f_n \in A$ and $f_n \to f$ in $L_p[0,1]$. Convergence in L_p implies convergence in law, hence there is a subsequence n_k such that f_{n_k} converges to f a.s. By Fatou's lemma we get

$$\int_0^1 |f|^q = \int_0^1 \underline{\lim}_{k\to\infty} |f_{n_k}|^q \leq \underline{\lim}_{k\to\infty} \int_0^1 |f_{n_k}|^q \leq 1,$$

so $f \in A$ which shows that A is closed.

Suppose the interior of A in L_p is not empty, that is A contains a ball. Since $p < q$, $L_p[0,1] \supsetneq L_q[0,1]$, such a ball contains functions with infinite L_q norms. This contradicts the fact that A is bounded in the L_q norm.

Thus A is nowhere dense and so is any its dilation nA. We have

$$L_q[0,1] = \bigcup_{n\geq 1} nA$$

which shows that $L_q[0,1]$ is a countable union of nowhere dense sets in $L_p[0,1]$. This does not contradict that $(L_q[0,1], \|\cdot\|_q)$ is a Banach space because the sets nA are nowhere dense in the metric given by the norm $\|\cdot\|_p$, not $\|\cdot\|_q$. $\square$

8. (♣) $\Longrightarrow$ (♠) Obvious.

(♠) $\Longrightarrow$ (♢) The only subspace with nonempty interior is the whole space; since f is nonzero, its kernel is a proper subspace, so it has empty interior and as being closed, it is nowhere dense.

(♢) $\Longrightarrow$ (♣) Suppose f is not bounded. Then there are unit vectors x_n for which $|f(x_n)| \geq n$. For any vector x and n we have

$$y_n = x - \frac{f(x)}{f(x_n)} x_n \in \ker f$$

Moreover,

$$\|y_n - x\| \leq \frac{|f(x)|}{n},$$

so $y_n \to x$. Therefore $x \in \operatorname{cl} \ker f$. Since x is arbitrary, $\operatorname{cl} \ker f$ is the whole space, but this contradicts its interior being empty. $\square$

In the next several questions we will use the following nice consequence of Baire's theorem proved in class:

If a Banach space is infinite dimensional, then its Hamel basis is uncountable. $(\star)$

Recall also the following fact concerning separability:

If a normed vector space contains an uncoutable set of points any two of which are distance 1 apart, then it is not *separable.* $(\star\star)$

9. Consider the vector space c_{00} of all sequences eventually zero. For instance the set $\{e_n,\ n \geq 1\}$ is a Hamel basis for this space, which is countable. In view of $(\star)$, the space c_{00} is not Banach-normable. □

10. Suppose $\dim X = \infty$. Then there are countably many linearly independent vectors $x_1, x_2, \ldots$. Consider the subspace $Y = \text{span}\{x_n,\ n \geq 1\}$. As a closed subspace of a Banach space, Y is a Banach space, but this contradicts $(\star)$. □

11. Take $X = \ell_2$ and set $\|\cdot\|$ to be the standard ℓ_2 norm. Fix a Hamel basis $\{b_t,\ t \in T\}$ in ℓ_2 and define for every vector $x = \sum_{t\in T} \beta_t b_t$ (almost all β_t are zero)

$$\|x\|' = \sum_{t\in T} |\beta_t|.$$

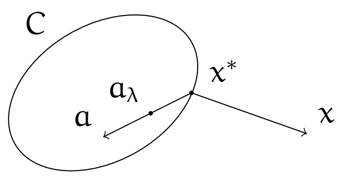
It is readily checked that this defines a norm on ℓ_2. For every pair of distinct $s, t \in T$ the vectors b_s, b_t are distance 2-apart, $\|b_s - b_t\|' = 2$, and T is uncountable. By $(\star\star)$ the space $(\ell_2, \|\cdot\|')$ is not separable. □

12. Checking that $\langle r_k, r_l\rangle = 0$ for $k \neq l$ is straightforward. The system $\{r_n\}$ is incomplete as it is readily verified that $\langle r_k, \mathbf{1}_{[0,1/4]} - \mathbf{1}_{[1/4,3/4]} + \mathbf{1}_{[3/4,1]}\rangle = 0$ for every k. □

13. Any orthogonal set can be made orthonormal. If u, v are orthogonal unit vectors in a Hilbert space, then $\|u - v\|^2 = 2$. If an orthonormal set was uncountable, we would have uncountably many pairs of points which are distance $\sqrt{2}$-apart, which would contradict separability by $(\star\star)$. □

14. Fix $x \in H$. First we show that for every $a \in C$ we have

$$\mathfrak{Re}\langle x - x^*, a - x^*\rangle \leq 0.$$



Fix $a \in C$ and set $a_\lambda = (1-\lambda)x^* + \lambda a$, $\lambda \in [0,1]$. By convexity, $a_\lambda \in C$. In view of the fact that x^* is the best approximation of x in C we have

$$\begin{aligned}\|x - x^*\|^2 \le \|x - a_\lambda\|^2 &= \|(x - x^*) + (x^* - a_\lambda)\|^2 \\ &= \|x - x^*\|^2 + 2\mathfrak{Re}\langle x - x^*, x^* - a_\lambda\rangle + \|x^* - a_\lambda\|^2,\end{aligned}$$

hence

$$-2\mathfrak{Re}\langle x - x^*, x^* - a_\lambda\rangle \le \|x^* - a_\lambda\|^2.$$

Note that $x^* - a_\lambda = \lambda(x - a)$. Plugging this back, dividing by λ and then letting $\lambda \to 0$ yield the result.

Fix $x, y \in H$. Using what we just showed gives

$$\mathfrak{Re}\langle x - x^*, y^* - x^*\rangle \le 0,$$
$$\mathfrak{Re}\langle y - y^*, x^* - y^*\rangle \le 0.$$

Adding these we obtain

$$0 \ge \mathfrak{Re}\langle y - y^* - x + x^*, x^* - y^*\rangle = \|x^* - y^*\|^2 + \mathfrak{Re}\langle y - x, x^* - y^*\rangle.$$

To finish, move the inner product over and apply the Cauchy-Schwarz inequality,

$$\|x^* - y^*\|^2 \le \mathfrak{Re}\langle x - y, x^* - y^*\rangle \le \|x - y\| \cdot \|x^* - y^*\|. \quad \square$$

15. Observe that for every vector x by orthogonality of $x - Px$ and Px we have

$$\|x - 2Px\|^2 = \|(x - Px) - Px\|^2 = \|x - Px\|^2 + \|Px\|^2 = \|x\|^2.$$

The same holds for Q as well. Therefore

$$2\|Px - Qx\| \le \|2Px - x\| + \|x - 2Qx\| = 2\|x\|. \quad \square$$

16. By $x \cdot y = \sum_{j \le n} x_j y_j$ we denote the standard inner product on $\mathbb{R}^n$. Fix a vector x in $\mathbb{R}^n$ with $\|x\|_p = 1$. Then by Hölder's inequality

$$\|Tx\|_\infty = \max_{j \le n} |a_j \cdot x| \le \max_{j \le n} \|a_j\|_q \cdot \|x\|_p = \max_{j \le n} \|a_j\|_q$$

and if $\|a_{j_0}\|_q = \max_{j \le n} \|a_j\|_q$, in order to to get equality we choose x for which $|a_{j_0} \cdot x| = \|a_{j_0}\|_q$. This establishes that

$$\|T\|_{\ell_p^n \to \ell_\infty^n} = \max_{j \le n} \|a_j\|_q.$$

Now fix a vector $x = (x_1, \dots, x_n)$ in $\mathbb{R}^n$ with $\|x\|_1 = 1$. We get

$$\|Tx\|_p = \left\| \sum_{j \le n} x_j b_j \right\|_p \le \sum_{j \le n} |x_j| \cdot \|b_j\|_p \le \max_{j \le n} \|b_j\|_p.$$

If $\|b_{j_0}\|_p = \max_{j \le n} \|b_j\|_p$, then to get equality we choose simply $x = e_{j_0}$. This establishes that

$$\|T\|_{\ell_1^n \to \ell_p^n} = \max_{j \le n} \|b_j\|_p. \quad \square$$

17. Using Hölder's inequality,

$$\|Tx\|_1 = \sum_{n=1}^{\infty} |n^\alpha x_n| \le \left(\sum_{n=1}^{\infty} n^{\frac{3}{2}\alpha} \right)^{2/3} \left(\sum_{n=1}^{\infty} |x_n|^3 \right)^{1/3} = \left(\sum_{n=1}^{\infty} n^{\frac{3}{2}\alpha} \right)^{2/3} \cdot \|x\|_3,$$

so if $\alpha < -2/3$, the series $\sum n^{3\alpha/2}$ converges and T is bounded.

Suppose now that T is bounded. Then for every $x \in \ell_3$ the series $\sum n^\alpha x_n$ is absolutely convergent and bounded by $\|T\| \cdot \|x\|_3$. This means that

$$\left(x \mapsto \sum_{n=1}^{\infty} n^\alpha x_n \right) \in \ell_3^*,$$

so by the duality $\ell_3^* \simeq \ell_{3/2}$ we get $(n^\alpha) \in \ell_{3/2}$ which holds if and only if $\alpha < -2/3$. $\square$

18. Take simply $S = \mathrm{Id}\colon c_0 \to c_0$ and suppose that it can be extended to $\tilde{S}\colon c \to c_0$ without increasing the norm. Denote the constant sequence $(1, 1, \dots)$ by e. Let $y = \tilde{S}e$. We have $\|e - 2e_n\|_\infty = 1$ and $\tilde{S}e_n = e_n$, so

$$|y_n - 2| \le \|y - 2e_n\|_\infty = \|\tilde{S}e - 2\tilde{S}e_n\|_\infty \le \|\tilde{S}\| \cdot \|e - 2e_n\|_\infty = 1.$$

Since y is in c_0 (as the image of e under $\tilde{S}$), the left-hand-side converges to 2, which gives a contradiction. $\square$

19. Take any basis in X of unit vectors (y_j) and its dual (y_j^*), meaning $y_j^*(y_i) = \delta_{ij}$ for all i, j. The problem is that the y_j^* may not have norm one. To fix it we define the function

$$V(z_1, \dots, z_n) = \det \left[y_j^*(z_i) \right]_{i,j=1,\dots,n}$$

on $X \times \dots \times X$. It is continuous, hence it attains its supremum on the compact set $S_X \times \dots \times S_X$ at, say $(x_1, \dots, x_n)$ (the set S_X denotes the unit sphere in X). For a fixed index j let us define the functional

$$\phi_j(x) = \frac{V(x_1, \dots, x_{j-1}, x, x_{j+1}, \dots, x_n)}{V(x_1, \dots, x_j)}, \qquad x \in X.$$

Then $\phi_j(x_i) = 0$, if $i \neq j$, as the determinant of a matrix with two identical columns equals 0. Clearly $\phi_j(x_j) = 1$. Moreover, since V on the set $S_X \times \ldots \times S_X$ attains its maximum at $(x_1, \ldots, x_n)$, we have $\sup_{x \in S_X} \phi_j(x) = 1$, so $\|\phi\| = 1$. □

20. Let $x_1, \ldots, x_n$ be a basis in X provided by Auerbach's lemma and let $\phi_1, \ldots, \phi_n$ be the corresponding functionals of norm one such that $\phi_j(x_i) = \delta_{ij}$ for all i, j (see Question 19). Since the vectors x_j are unit the right inequality follows simply by the triangle inequality. Notice that

$$|\lambda_j| = \left|\phi_j\left(\sum_{i=1}^{n} \lambda_i x_i\right)\right| \leq \|\phi_j\| \cdot \left\|\sum_{i=1}^{n} \lambda_i x_i\right\|.$$

This shows the left inequality as $\|\phi_j\| = 1$. □

21. Application of the Hahn-Banach theorem to the vector $\phi \in X^*$ yields a unit functional $p \in X^{**}$ on X^* for which $p(\phi) = \|\phi\|$. By reflexivity, the canonical isometric embedding $X \overset{\iota}{\hookrightarrow} X^{**}$ is onto, hence there is $x \in X$ such that $p = \iota(x)$ and $1 = \|p\| = \|x\|$. Then

$$\|\phi\| = p(\phi) = \iota(x)(\phi) = \phi(x),$$

so x is the unit vector we want to find. □

22. By Question 21, to show that the spaces $c_0, c, \ell_1, C[0,1], L_1[0,1]$ are not reflexive, for each of them it is enough to find a bounded functional ϕ which does not attain its norm. It can be readily checked that we can take

- $\phi(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} x_n$ on c_0,
- $\phi(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} x_n$ on c,
- $\phi(x) = \sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right) x_n$ on ℓ_1,
- $\phi(f) = \int_0^{1/2} f - \int_{1/2}^{1} f$ on $C[0,1]$,
- $\phi(f) = \int_0^1 x f(x) dx$ on $L_1[0,1]$. □

23. Suppose $x_n \rightharpoonup x$ (weakly in X). The sequence x_n is bounded if and only if its image under the canonical embedding ι of X into X^{**} is bounded. Let $x_n^{**} = \iota(x_n)$. For a fixed $\phi \in X^*$ we have

$$\sup_n |x_n^{**}(\phi)| = \sup_n |\phi(x_n)| < \infty$$

as the sequence $\phi(x_n)$ is convergent. Therefore by the Banach-Steinhaus theorem, the family of functionals x_n^{**} (acting on X^* which is a Banach space) is norm-bounded, that is

$$\sup_n \|x_n\| = \sup_n \|x_n^{**}\| < \infty. \quad \square$$

24. Follows directly by applying the Banach-Steinhaus theorem as in Question 23. $\square$

25. Let $u_n = v_n/\|v_n\|$ be the normalised sequence and set $M = \sup_n \|v_n\|^2$. Fix a vector v. By Bessel's inequality

$$\sum_{n=1}^{\infty} |\langle v, v_n\rangle|^2 = \sum_{n=1}^{\infty} |\langle v, u_n\rangle|^2 \cdot \|v_n\|^2 \leq M \sum_{n=1}^{\infty} |\langle v, u_n\rangle|^2 \leq M\|v\|^2$$

so the series $\sum |\langle v, v_n\rangle|^2$ converges and particularly $\langle v, v_n\rangle \to 0$. This shows that the sequence (v_n) converges weakly to 0. $\square$

26. If T is bounded, then clearly every weakly convergent sequence gets mapped to a weakly convergent sequence. Conversely, suppose a sequence (x_n) converges (in norm) to 0. We want to show that $Tx_n \to 0$ (continuity at 0 implies by linearity the boundedness of T). Since $x_n \to 0$, also $x_n \rightharpoonup 0$, so Ax_n converges weakly. By Question 23, the sequence (Ax_n) is bounded. Fix $\epsilon > 0$. We want to show that eventually $\|Ax_n\| \leq \epsilon$. If $\|Ax_n\| > \epsilon$ for infinitely many n, then considering the sequence $y_n = x_n/\sqrt{\|x_n\|}$ which converges to 0 as $\|y_n\| = \sqrt{\|x_n\|}$, we get similarly that the sequence Ay_n is bounded, but for infinitely many n, $\|Ay_n\| > \frac{\epsilon}{\sqrt{\|x_n\|}} \to \infty$. This contradiction finishes the proof. $\square$

27. Let e_n be the standard unit vectors in ℓ_p and by $e_n^* \in \ell_p^*$ we denote their duals, $e_n^*(x) = x_n$. Since $\ell_p^* \simeq \ell_q$ and $q = p/(p-1) \in (1, \infty)$, the sequence (e_n^*) is dense in ℓ_p^*.

If a sequence (x_n) converges weakly in ℓ_p to x, then it is bounded by Question 23 and the convergence of coordinates follows by testing with e_n^*. Conversely, suppose a sequence (x_n) is bounded by, say $a > 0$ in ℓ_p and for some sequence x we have that for every n, $e_n^*(x_m) \underset{m\to\infty}{\longrightarrow} e_n^*(x)$. Since

$$\sum_{n=1}^{N} |e_n^*(x)|^p = \overline{\lim_{m\to\infty}} \sum_{n=1}^{N} |e_n^*(x_m)|^p \leq \overline{\lim_{m\to\infty}} \|x_m\|_p^p \leq a^p,$$

the sequence x is in ℓ_p and $\|x\|_p \leq a$. Fix $\phi \in \ell_p^*$. We want to show that $\phi(x_m) \underset{m\to\infty}{\longrightarrow} \phi(x)$. Fix $\epsilon > 0$. By density, there is a finite linear combination ψ of the e_n^* such that $\|\phi - \psi\| < \epsilon/(4a)$. By the assumption, $\psi(x_m) \underset{m\to\infty}{\longrightarrow} \psi(x)$, so there is M such that $|\psi(x_m) - \psi(x)| < \epsilon/2$ for all $m > M$. Then for those

m we obtain

$$\begin{aligned}|\phi(x_m)-\phi(x)| &\le |\psi(x_m)-\psi(x)|+|\psi(x)-\phi(x)|+|\psi(x_m)-\phi(x_m)|\\ &\le \frac{\epsilon}{2}+\|\psi-\phi\|\cdot(\|x\|_p+\|x_m\|_p)\le \frac{\epsilon}{2}+\frac{\epsilon}{4a}\cdot 2a=\epsilon. \quad \square\end{aligned}$$

28. Left for the dedicated student.

29. Let $\{\phi_n\}\subset S_{X^*}$ be a countable dense subset in the unit sphere of the dual space. For every n choose a unit vector $x_n\in X$ such that $|\phi_n(x_n)|>\frac{1}{2}\|\phi_n\|=\frac{1}{2}$. We want to show that $Y=\text{cl span}\{x_n,\ n\ge 1\}$ is X. Suppose that $Y\subsetneq X$. Then by the Hahn-Banach theorem there is a functional ϕ of norm one such that $\phi|Y=0$. Choose k so that $\|\phi-\phi_k\|<1/3$. We have

$$\frac{1}{2}<|\phi_k(x_k)|=|\phi_k(x_k)-\phi(x_k)|\le\|\phi_k-\phi\|\cdot\|x_k\|<\frac{1}{3}. \quad \square$$

30. Left for the dedicated student.