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Functional analysis I, Problems for the revision lecture

Term 3 2014/2015

1. Let $(V, \|\cdot\|)$ be a normed vector space. Fix a vector $v \in V$. For every $x \in V$ define

$$\|x\|' = \big\|x + \|x\|v\big\| + \big\|x - \|x\|v\big\|.$$

Show that $\|\cdot\|'$ is a norm on V which is equivalent to $\|\cdot\|$.

Hint: The function $\mathbb{R} \ni t \mapsto \|x + tv\| + \|x - tv\| \in [0, +\infty)$ is even and convex.

2. Is it true that for every vector space V there is a function $N \colon V \longrightarrow [0, +\infty)$ which is a norm on V? (In other words, is every vector space normable?)

3. We know that any two norms on a finite dimensional vector space are equivalent. Does there exist an infinite dimensional vector space V with the property that any two norms on V are equivalent?

4. Determine whether the following functional spaces

$$C(\mathbb{R}) = \{f \colon \mathbb{R} \longrightarrow \mathbb{R},\ f \text{ is continuous}\},$$
$$C_{\text{van}}(\mathbb{R}) = \{f \in C(\mathbb{R}),\ \lim_{x \to +\infty} f(x) = \lim_{x \to -\infty} f(x) = 0\},$$
$$C_{\text{bd}}(\mathbb{R}) = \{f \in C(\mathbb{R}),\ f \text{ is bounded}\},$$
$$C_0(\mathbb{R}) = \big\{f \in C(\mathbb{R}),\ \text{cl}\{x \in \mathbb{R},\ f(x) \neq 0\} \text{ is compact}\big\},$$

equipped with the supremum norm are Banach spaces.

5. Give an example of a Banach space which is not a Hilbert space, that is whose norm is not induced by any scalar product.

Hint: Parallelogram law.

6*. Give an example of a nonseparable Hilbert space.

7. Does there exist a bounded linear operator on Hilbert space with empty point spectrum?

8. Solve the Sturm-Liouville problem for the Laplacian operator on the interval $(0, 1)$, that is find the λ for which the problem

$$\begin{cases} u''(x) = \lambda u(x), & \text{on } (0,1), \\ u(0) = u(1) = 0 \end{cases}$$

has a nontrivial solution u and show that the corresponding solutions (eigenvectors) form an orthonormal basis of $L_2(0, 1)$.