Combinatorial Optimisation, Problems I, Solutions

Term 2 2014/2015

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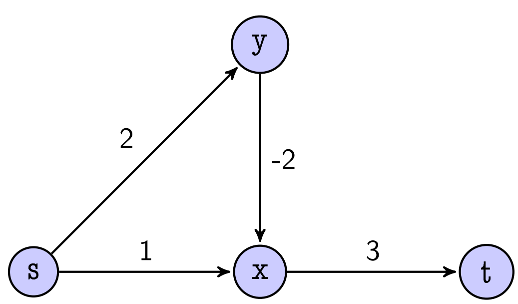
3. By Problem 2, any tree which is not a single vertex has at least 2 leaves, so we can assume that $\Delta \geq 3$. Let u be a vertex with the maximal degree Δ. Since the summands in the formula from Problem 2 are nonnegative, the number of leaves is at least $2 + \deg(u) - 2 = \deg(u) = \Delta$.

8. Suppose there are two different optimum spanning trees T_1 and T_2. Let e be the edge of the smallest weight in $E(T_1) \triangle E(T_2)$, say $e \in E(T_1)$. Adding e to T_2 creates a circuit. Take an edge f on this circuit such that f is not in T_1. Then it is easy to check that $(T_2 - f) + e$ is a spanning tree, but its total cost is smaller by $c(f) - c(e) > 0$ than the cost of T_2, which contradicts the optimality of T_2.

12. Recall that in Dijkstra's algorithm, once a vertex has been processed, the value assigned to it is the length of a shortest path leading to it. It might not be true if negative weights are allowed as the following example shows.



The order in which the algorithm will process the vertices is: s, x, y, t, returning the lengths: $\ell(s) = 0, \ell(x) = 1, \ell(y) = 2, \ell(t) = 4$, which are not correct.

Combinatorial Optimisation, Problems II, Solutions

Term 2 2014/2015

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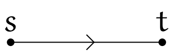
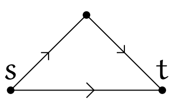
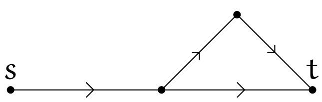
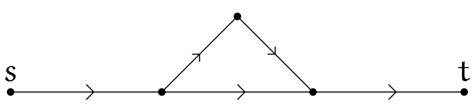
2. Suppose we take all capacities one. Then we get

- 

1 max-flow, 1 min-cut
- 

1 max-flow, 2 min-cuts
- 

many max-flows, 1 min-cut
- 

many max-flows, many min-cuts

6. Let T be a maximum spanning tree in (K_n, c). We want to show that for every two vertices i and j, their local edge connectivity in T, $\lambda_{ij}(T)$, is $\lambda_{ij} = c(\{i,j\})$. Let $v_0 - v_1 - \ldots - v_k$, $v_0 = i$, $v_k = j$, be the path in T from i to j. Since T is a tree, in order to separate i and j, it is enough to *cut through* just a single edge, so the minimum cut separating i and j in T has value

$$\begin{aligned}\lambda_{ij}(T) = \min\{c(\{v_l, vl+1\}),\ l = 0,1,\ldots,k-1\} &= \min\{\lambda_{v_l v_{l+1}},\ l = 0,1,\ldots,k-1\} \\ &\le \lambda_{v_0 v_k} = \lambda_{ij},\end{aligned}$$

where the inequality follows from the application of the assumption on λ's, $\lambda_{ik} \ge \min\{\lambda_{ij}, \lambda_{jk}\}$, recursively. If it was strict, say $\lambda_{v_3 v_4} < \lambda_{ij}$, we would swap in T the edge $\{v_3, v_4\}$ for $\{i,j\}$, obtaining a tree with a bigger value than T, which contradicts the choice of T.

10. Add a vertex w and edges $\{v_i, w\}$ for $1 \leq i \leq k$ to G, where parallel edges are added in the case of vertex multiplicity. Call the new graph G'. Then G' is k-edge connected by definition. So by Menger's Theorem there are k edge-disjoint v-w-paths. On the other hand since w is only connected by k distinct edges to v_1, v_2, $\ldots$, v_k, this gives k edge-disjoint paths connecting v and v_i in G respectively.

23/02/2015

Combinatorial Optimisation, Problems III, Solutions

Term 2 2014/2015

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1. Let $G = (V, E)$ be the bipartite graph $K_{m,n}$, the maximally connected graph of two groups of vertices $V_1 = \{v_1, v_2, \ldots, v_m\}$ and $V_2 = \{w_1, w_2, \ldots, w_n\}$ which are independent sets but $\{v_i, w_j\} \in E$ for any i, j.

1. The independent number $\alpha(G)$ is the number of the maximum independent set of G. Since V_1 and V_2 are independent sets, $\alpha(G) \geq \max\{|V_1|, |V_2|\} = \max\{m, n\}$. On the other hand, any vertex set W of cardinality greater than $\max\{m, n\}$ has v_i and w_j for some i and j as elements, but since $\{v_i, w_j\} \in E$, W can not be an independent set, thus $\alpha(G) \leq \max\{m, n\}$. Therefore $\alpha(G) = \max\{m, n\}$.

2. The minimum cover number $\tau(G)$ is the number of the minimum vertex-cover set of G. It is covered in (no pun intended) the lecture notes that the complement of a maximum independent set is a minimum vertex-cover set. Hence $\tau(G) = |V| - \alpha(G) = \min\{m, n\}$.

3. The maximum matching number $\mu(G)$ is the number of pairs of vertices in a maximum matching. It is also covered in the lecture notes that for a bipartite graph $\mu(G) = \tau(G) = \min\{m, n\}$ by König's theorem.

4. The edge connectivity $\gamma(G)$ is the minimum number of edges to be removed for G to be disconnected. It is $\min\{m, n\}$. This is because, on the one hand, suppose $m \leq n$, then if we remove all the edges incident to w_1 then G is not connected any more. There are m such edges, so $\gamma(G) \leq m$. On the other hand if we remove $k < m$ edges from G, then since every vertex has degree at least m, the graph remains connected, which proves the other direction.

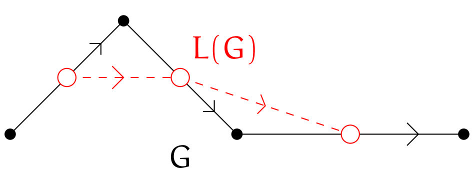
5. The vertex connectivity $\eta(G)$ of G is also $\min\{m, n\}$. We can remove V_1 or V_2, whichever has smaller cardinality, which proves $\eta(G) \leq \min\{m, n\}$, the other direction is similar to the arguments in $\gamma(G)$ and $\alpha(G)$.

6. Suppose we have an $r \times n$ brilliant rectangle $\mathcal{R}$ and $r < n$. It is enough to show how to append one row. To this end consider a bipartite graph $G(A \cup B, E)$ with $A = \{a_1, \dots, a_n\}$, $B = \{b_1, \dots, b_n\}$ and $\{a_i, b_j\} \in E$ if and only if j does *not* appear in the i^{th} column of the rectangle. If $\{a_k, b_{m_k},\ k = 1, \dots, n\} \subset E$ is a perfect matching in G, then it means that it is possible to append the row $(m_1, \dots, m_n)$ to $\mathcal{R}$.

Plainly, the degree of every vertex in A is $n - r$, as in each column of $\mathcal{R}$ there are r distinct numbers. Now fix a vertex b_j in B. If $\deg(b_j) = k$, it means that j does *not* appear in exactly k columns of $\mathcal{R}$, hence j appears in exactly $n - k$ columns of $\mathcal{R}$. Since $\mathcal{R}$ is brilliant, this implies that j appears exactly $n - k$ times in $\mathcal{R}$. Notice that every number $1, \dots, n$ appears in $\mathcal{R}$ exactly r times (exactly once in every row). Therefore, $n - k = r$, that is $\deg(b_j) = n - r$.

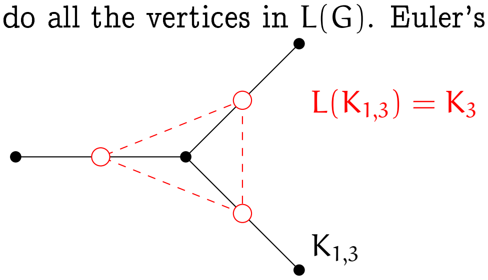
We have shown that the bipartite graph G is $n - r$ regular, hence it possesses a perfect matching (see the lecture, or do a simple double counting to check Hall's condition: for $X \subset A$ look at $\{(x, y),\ x \in X, \{x, y\} \in E\}$; on one hand the cardinality of this set equals $|X|(n - r)$, as every $x \in X$ has $n - r$ neighbours, but on the other hand, it is at most $|N(X)|(n - r)$, as every y has also $n - r$ neighbours, but not all of them might be in X, hence the bound).

10.



It is readily checked that an Eulerian circuit in G gives a Hamiltonian circuit in $L(G)$.

To check that $L(G)$ is also Eulerian notice that the degree of a vertex $e = \{v, w\} \in E(G)$ in $L(G)$ equals $\deg(v) + \deg(w) - 2$. Thus, if all vertices in G have even degrees, then so do all the vertices in $L(G)$. Euler's theorem finishes the proof.



The converse does not hold. For instance, the graph $K_3 = L(K_{1,3})$ is both Eulerian and Hamiltonian, but $K_{1,3}$ is not Eulerian (nor is it Hamiltonian).

19/03/2015

Combinatorial Optimisation, Problems IV, Solutions

Term 2 2014/2015

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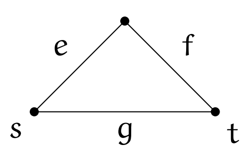
1. (a) The output matching has to contain (M, W). Suppose not, say the output contains (M, W') and (M', W). Since M ranks W higher than W' and W ranks M higher than M', the matching is not stable.

(b) Yes, it is possible. Consider the situation in which man M_i's first choice is woman W_i and vice versa for $i \leq n-1$, man M_n's last choice is woman W_n and M_n is second on every woman's list. Then after $n-1$ steps the Gale-Shapley algorithm has matched M_i to W_i and when it comes to M_n in the n^{th} step, he gets rejected by every woman from his list but the last one — W_n, since every woman prefers her current match M_i, which is her first choice, to M_n, which is her second choice.

(c) Answer: $N + (N-1) + \ldots + 1$ and it does not depend on women's preferences. Indeed, after the first N steps, W_1 has been asked N times, so she has been matched to her best candidate. She will not be asked any more and in the next $N-1$ subsequent steps, W_2 will be asked $N-1$ times, and so on.

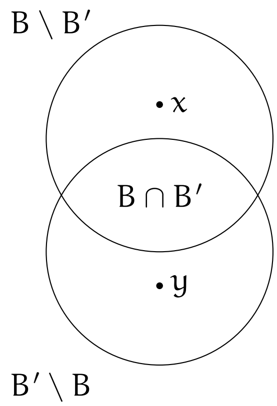
4. Since s, t are connected, $\mathcal{F}$ is nonempty. It is clear that if X belongs to $\mathcal{F}$ then so does any subset of X. Particularly, $\varnothing \in \mathcal{F}$. This shows that $\mathcal{F}$ is an independence system.



To show that in general $\mathcal{F}$ is not a matroid, consider the graph in the picture. Plainly, $\mathcal{F} = \big\{\varnothing, \{e\}, \{f\}, \{g\}, \{e, f\}\big\}$. Thus, if $X = \{e, f\}$ and $Y = \{g\}$, there is no $x \in X \setminus Y$ for which $Y \cup \{x\} \in \mathcal{F}$. Therefore, $\mathcal{F}$ is not a matroid.
Notice however, that if the graph is a tree, then $\mathcal{F}$ is a (uniform) matroid.

7. First, suppose that $\mathcal{B} \subset \mathcal{F}$ is the set of the bases of a matroid $(E, \mathcal{F})$. We want to show that

$$\text{for every two bases } B, B' \text{ and every } x \in B \setminus B' \text{ there exists } y \in B' \setminus B \text{ such that } (B' \setminus \{y\}) \cup \{x\} \in \mathcal{B}. \tag{$\star$}$$



Solution I (J. W. Turner). Fix two bases B, B' and $x \in B \setminus B'$. We know that $|B| = |B'|$ because $\mathcal{F}$ is a matroid. If $|B \cap B'| = |B| - 1$, then there is only one element in $B' \setminus B$; call it y. Plainly, $(B' \setminus \{y\}) \cup \{x\} = B$, so we are done in this case. If $|B \cap B'| < |B| - 1$, set $X = (B \cap B') \cup \{x\}$ and $Y = B'$. Then X, Y are both independent, $|X| < |Y|$, so by the matroid property proved in the lecture, there is a subset Z of $Y \setminus X$ of size $|Y| - |X|$ such that $X \cup Z$ is independent. Since $|Z| = |B'| - |B \cap B'| - 1$ and $Z \subset B' \setminus (B \cap B') = B' \setminus B$, there is a unique element $y \in B' \setminus B$ such that $Z = (B' \setminus B) \setminus \{y\}$. Therefore $X \cup Z = (B' \setminus \{y\}) \cup \{x\}$ and we know this set is independent. Because this set has the same size as B', it is a basis.

Solution II (B. Madley). Let C be the unique circuit in $B' \cup \{x\}$ (B' is a basis, so this set is dependent). Now we can remove any other element of this circuit to make the leftover set independent. At least one of these choices must be in $B' \setminus B$ as otherwise B would contain the circuit and hence not be independent. So there is y in $B' \setminus B$ such that $(B' \setminus \{y\}) \cup \{x\}$ is a basis.

Solution III. The set $B' \cup \{x\}$ is dependent. Let U be a maximal subset of B' such that $U \cup \{x\}$ is independent. We claim that $|U| = |B'| - 1$. Otherwise, $|U \cup \{y\}| < |B'|$ and by the augmentation property there exists an element z in B' such that $(U \cup \{y\}) \cup \{y\}$ is still independent which contradicts the maximality of U. Denote $B' \setminus U = \{y\}$. We have that $(B' \setminus \{y\}) \cup \{x\} = U \cup \{x\}$ is independent and a basis as it is of size $|B'|$.

Solution IV. We use induction on $k = |B' \setminus B| = |B \setminus B'|$. If $k = 0$, the statement is trivial. The case $k = 1$ is dealt with as in Solution I. We assume $k \geq 2$. Then there is $x' \in B \setminus B'$ with $x' \neq x$. By the matroid property, there exists $y' \in B' \setminus B$ such that $B'' = (B \setminus \{x'\}) \cup \{y'\}$ is a basis. We have $x \in B'' \setminus B'$. Since $|B'' \setminus B'| = k - 1$, by the inductive assumption we get that $B' \setminus \{y\} \cup \{x\}$ is a basis for some $y \in B' \setminus B''$. Since $y \neq x'$ ($x' \in B \setminus B'$ but $y \in B'$), in fact $y \in B' \setminus B$.

Second, suppose that $\mathcal{B} \subset 2^E$ satisfies ($\star$). We would like to show that $\mathcal{B}$ is the set of the bases of some matroid $(E, \mathcal{F})$. There is only one reasonable way to define $\mathcal{F}$, namely

$$\mathcal{F} = \{X \subset E, \text{there is some } B \text{ in } \mathcal{B} \text{ for which } X \subset B\}.$$

With this definition, it is clear that $\mathcal{F}$ is an independence set whose bases are $\mathcal{B}$. To show that $\mathcal{F}$ is a matroid, we first check that all B in $\mathcal{B}$ have the same cardinality. Suppose not and among the pairs $(B_1, B_2) \in \mathcal{B} \times \mathcal{B}$ with $|B_1| \neq |B_2|$ choose the one for which $|B_1 \cap B_2|$ is the largest possible. Say $|B_2| > |B_1|$. Then picking $y \in B_2 \setminus B_1$, there is $x \in B_1 \setminus B_2$ such that $B_1' = (B_1 \setminus \{x\}) \cup \{y\} \in \mathcal{B}$. But $|B_1' \cap B_2|$ is greater than $|B_1 \cap B_2|$, which is a contradiction.

Now suppose that there are two elements X and Y in $\mathcal{F}$ with $|Y| > |X|$, which violate the matroid property and choose such a pair for which $|X \cap Y|$ is maximal. Since $X, Y \in \mathcal{F}$, we have some $B_X, B_Y \in \mathcal{B}$ which contain X and Y respectively. Choose them so that $|B_X \cap B_Y|$ is maximal as well. We have that $B_X \cap (Y \setminus X) = \varnothing$ as otherwise we could extend X by one element from Y to have a member of $\mathcal{F}$ (so the matroid property would hold for X and Y). Therefore

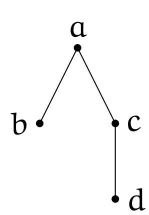
$$|B_X \cap B_Y| + |X \setminus B_Y| + |(B_X \setminus B_Y) \setminus X| \geq |B_X| = |B_Y| \geq |B_X \cap B_Y| + |Y \setminus X|.$$

Since $|Y \setminus X| > |X \setminus Y| \geq |X \setminus B_Y|$, we obtain that $|(B_X \setminus B_Y) \setminus X| > 0$, so this set is nonempty and choose an x in it. By the property of $\mathcal{B}$, there is $y \in B_Y \setminus B_X$ such that $B_Y' = (B_Y \setminus \{y\}) \cup \{x\} \in \mathcal{B}$. Notice that this y must be in Y as otherwise the sets B_X, B_Y', containing X, Y respectively, contradict the choice of B_X, B_Y having the largest intersection. Since $y \in Y$ and $Y' = (Y \setminus \{y\}) \cup \{x\}$ is contained in B_Y', the pair X and Y' contradict the choice for X and Y.

Remark. The matroid property of an independence system $(\mathcal{F}, E)$ (as shown in the lecture) is equivalent to

for *each* $X \subset E$ all the bases of X have the same size.

To show that $\mathcal{F}$ is a matroid, it is *not* enough to show that the bases have the same size.



To see this, consider the following example. Let $E = \{a, b, c, d\}$ and $\mathcal{F} = \big\{\varnothing, \{a\}, \{b\}, \{c\}, \{d\}, \{a, d\}, \{b, c\}, \{b, d\}\big\}$ (the independent sets of the graph shown in the picture). Clearly, this is an independent system, but is not a matroid as $X = \{a\}$ cannot be extended through $Y = \{b, c\}$. However, the bases of $\mathcal{F}$ are $\{a, d\}, \{b, c\}, \{b, d\}$; they have the same size. (The point being that the bases of $\{a, b, c\}$ are *not* of the same size.)

Remark. (For the adventurous student) We can actually show slightly more for a matroid, what is called *the strong basis exchange property*

$$\text{for every two bases } B, B' \text{ and every } x \in B \setminus B' \text{ there exists } y \in B' \setminus B \text{ such that } (B' \setminus \{y\}) \cup \{x\} \text{ and } (B \setminus \{x\}) \cup \{y\} \text{ are in } \mathcal{B}. \qquad (\star\star)$$

To this end we shall use two facts known from the lecture

1. rank axiom: $r(X)+r(Y)\geq r(X\cup Y)+r(X\cap Y)$, for every $X, Y\subset E$
2. circuit axiom: a dependent set contains a unique circuit.

Since B' is a basis, $B'\cup\{x\}$ is a dependent set, so by 2. it contains a unique circuit C. Since $C\not\subset B'$, necessarily $x\notin C$. Since C is a circuit, $C\setminus\{x\}$ is an independent set, thus

$$r(C\setminus\{x\})=|C\setminus\{x\}|=|C|-1=r(C).$$

By 1.,

$$r(C)+r\big((B\cup C)\setminus\{x\}\big)\geq r(B\cup C)+r(C\setminus\{x\}),$$

so $r(C)=r(C\setminus\{x\})$ cancels and we have

$$r\big((B\cup C)\setminus\{x\}\big)\geq r(B\cup C).$$

The opposite inequality holds trivially, so we have equality. Since $r\big((B\cup C)\setminus\{x\}\big)=r(B\cup C)=|B|$ (B is a basis, hence the last equality), by the definition of rank the set $(B\cup C)\setminus\{x\}$ contains a basis, say B''. By the matroid property applied to $B\setminus\{x\}$ and B'' there is $y\in B''\setminus(B\setminus\{x\})$ such that $(B\setminus\{x\})\cup\{y\}$ is a basis. We will show that in addition, $(B'\setminus\{y\})\cup\{x\}$ is a basis as well. Then we will be done as this, along with $x\notin B'$, comparing cardinality, gives that $y\in B'$. We have

$$B''\setminus(B\setminus\{x\})\ \subset\ \big((B\cup C)\setminus\{x\}\big)\setminus(B\setminus\{x\})\ \subset\ C\setminus\{x\},$$

so $y\in C\setminus\{x\}$ Since C is the unique circuit in $B'\cup\{x\}$, we get that $(B'\setminus\{y\})\cup\{x\}$ is a basis (it does not contain any circuit as it does not contain C and if it contained another circuit C', then C' would be contained in $B'\cup\{x\}$, so $C'=C$; obviously, a set is independent if and only if it does not contain a circuit).