Problem solving seminar
IMC Preparation, the match — Solutions

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1. Let A be a real 4×2 matrix and B be a real 2×4 matrix such that

$$AB = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}.$$

Find BA.

Solution. Let

$$A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}, \qquad B = \begin{bmatrix} B_1 & B_2 \end{bmatrix},$$

where A_1, A_2, B_1, B_2 are 2×2 matrices. Then

$$AB = \begin{bmatrix} A_1B_1 & A_1B_2 \\ A_2B_1 & A_2B_2 \end{bmatrix} = \begin{bmatrix} I & -I \\ -I & I \end{bmatrix}.$$

Thus, $A_1B_1 = I = A_2B_2$ and $A_1B_2 = -I = A_2B_1$. As a result,

$$BA = B_1A_1 + B_2A_2 = 2I.$$

□

2. Let A, B be two matrices over a field K of size $m \times n$, $n \times m$ respectively. Show that for every $x \in K$ we have

$$x^n \det(xI_m - AB) = x^m \det(xI_n - BA).$$

Solution. Let us define block matrices

$$C = \begin{bmatrix} xI_m & A \\ B & I_n \end{bmatrix}, \qquad D = \begin{bmatrix} I_m & 0 \\ -B & xI_n \end{bmatrix}.$$

Then

$$\det(CD) = \det \begin{bmatrix} xI_m - AB & xA \\ 0 & xI_n \end{bmatrix} = x^n \det(xI_m - AB)$$

and

$$\det(DC) = \det \begin{bmatrix} xI_m & A \\ 0 & xI_n - BA \end{bmatrix} = x^m \det(xI_n - BA).$$

Since $\det(CD) = \det(DC)$, the proof is complete. □

3. Let $2 \le k \le n$ and $v_1, \ldots, v_k$ be unit vectors in $\mathbb{R}^n$. Prove that there are $1 \le i < j \le k$ such that $\langle v_i, v_j \rangle \ge -\frac{1}{k-1}$.

Solution. Notice that

$$\begin{aligned} 0 \le |v_1 + \ldots + v_k|^2 &= \sum_{i=1}^k |v_i|^2 + \sum_{i \ne j} \langle v_i, v_j \rangle \\ &\le k + k(k-1) \cdot \max_{i \ne j} \langle v_i, v_j \rangle, \end{aligned}$$

hence

$$\max_{1 \le i < j \le k} \langle v_i, v_j \rangle \ge -\frac{1}{k-1}.$$

□

4. Prove that

$$\det \begin{bmatrix} \frac{1}{1} & \frac{1}{2} & \cdots & \frac{1}{2014} \\ \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{2015} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{1}{2014} & \frac{1}{2015} & \cdots & \frac{1}{4027} \end{bmatrix} > \frac{1}{2^{2^{3^3}}}.$$

Solution. The formula for a Cauchy determinant yields (see Question 4, Set 3)

$$\begin{aligned} \det \left[\frac{1}{k+l-1} \right]_{1 \le k,l \le n} &= \frac{\prod_{1 \le k < l \le n} (l-k)^2}{\prod_{1 \le k,l \le n} (k+l-1)} \\ &> \frac{1}{(2n-1)^{n^2}}. \end{aligned}$$

For $n = 2014$ we easily check that

$$4027^{2014^2} < \left(2^{12}\right)^{\left(2^{11}\right)^2} = 2^{12 \cdot 2^{22}} < 2^{2^{26}} < 2^{2^{3^3}}.$$

□

5. Given $m, n \ge 1$, count the number of $m \times n$ matrices with $0, 1$ entries such that in every row and in every column there is an even number of 1's.

Solution. Answer: $2^{(m-1)(n-1)}$.

Notice that for a matrix with an even number of 1's in each row and in each column, the last row is determined by the first $m-1$ rows by putting a 1 into the columns containing an odd number of 1's. Moreover, having chosen $m-1$ rows, each containing an even number of 1's, the number of columns containing an odd number of 1's is even (otherwise the total number of 1's would be odd). So the number of matrices with m rows equals the number of choices for the first $m-1$ rows which is $\left(2^{n-1}\right)^{m-1}$.

□

6. Given a matrix with $0,1$ entries, containing an even number of 1's, is it always possible to find a submatrix (by crossing-out certain rows and certain columns, not necessarily consecutive) containing exactly a half of the 1's?

Solution. (Prof. K. Ball) Answer: no!

Consider the 5×9 matrix containing 44 ones and one zero at the upper-left corner. $\square$

7. Suppose that G is a simple graph with n vertices and with more than $n^2/4$ edges. Prove that G contains a triangle.

Show that for an even number n there exists a graph G with n vertices and $n^2/4$ edges containing no triangle.

Here by a simple graph we mean an undirected graph with neither loops nor multiple edges.

Solution. Let V be the set of vertices of G and E be the set of its edges. The degree (the number of neighbours) of a vertex $v \in G$ is denoted by $d(v)$. Suppose that G contains no triangle. Then, for every edge $\{u,v\} \in E$, we have $d(u)+d(v) \le n$. Therefore,

$$n \cdot \#E \ge \sum_{\{u,v\}\in E} \Big(d(u)+d(v)\Big) = \sum_{v\in V} d(v)^2$$
$$\ge \frac{1}{\#V}\left(\sum_{v\in V} d(v)\right)^2 = \frac{1}{n}\big(2\#E\big)^2.$$

It follows that $\#E \le n^2/4$.

For the *example* part, consider the complete $\frac{n}{2} \times \frac{n}{2}$ bipartite graph. $\square$

8. Given positive numbers $a_1, b_1, \dots, a_n, b_n$ prove that the function $f\colon \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = \sum_{k=1}^n a_k \cos(b_k x)$ has a zero.

Solution. We start with proving a simple lemma.

Lemma. *If a function $g\colon \mathbb{R} \longrightarrow \mathbb{R}$ satisfies for all reals x, y*

$$g\left(\frac{x+y}{2}\right) < \frac{g(x)+g(y)}{2},$$

then g is not bounded above.

Proof. Suppose g is bounded above and let $M := \sup_{\mathbb{R}} g$. Because of the strict inequality in the assumption, f is not constant, thus there is z with $g(z) < M$. Then $\frac{g(z)+M}{2} < M$. By the definition of M, there is x with $\frac{g(z)+M}{2} < g(x)$. Then

$$\frac{g(z)+M}{2} < g(x) < \frac{g(z)+g(2x-z)}{2} \le \frac{g(z)+M}{2},$$

a contradiction. $\square$

Let $g(x) = \sum_{k=1}^n \left(-\frac{a_k}{b_k^2}\right)\cos(b_k x)$. Then $g'' = f$. Since $f(0) = \sum_k a_k > 0$, if f had no zeros, then by continuity f would be positive everywhere implying that $g'' > 0$ on $\mathbb{R}$, i.e. g would be strictly convex which is not possible as g is bounded above. $\square$

9. Prove that

$$\lim_{n\to\infty} \sum_{j=1}^n \left(\frac{j}{n}\right)^n = \frac{e}{e-1}.$$

Solution. First notice that with the aid of the inequality $1 - x \le e^{-x}$, $x \in \mathbb{R}$, trivially

$$\sum_{j=1}^n \left(\frac{j}{n}\right)^n = \sum_{j=0}^{n-1}\left(1-\frac{j}{n}\right)^n < \sum_{j=0}^{\infty} e^{-j} = \frac{e}{e-1}.$$

To get a lower bound, we shall use the inequality

$$1 - x \ge e^{-x-x^2}, \quad x \in [0, 1/2]$$

which can be easily verified by taking the logarithm and looking at the derivatives of both sides. We obtain

$$\sum_{j=0}^{n-1}\left(1-\frac{j}{n}\right)^n \ge \sum_{j=0}^{\lfloor \ln n\rfloor} e^{-j}e^{-j^2/n}$$
$$\ge e^{-\lfloor \ln n\rfloor^2/n}\sum_{j=0}^{\lfloor \ln n\rfloor} e^{-j}$$
$$= e^{-\lfloor \ln n\rfloor^2/n}\frac{1-e^{-\lfloor \ln n\rfloor}}{1-e^{-1}}$$

and the right hand side clearly tends to $\frac{e}{e-1}$. $\square$

10. Let $f\colon [0,1] \longrightarrow \mathbb{R}$ be a convex function of class $C^\infty(0,1)$. Prove that

$$\int_0^1 (f(x))^2 \mathrm{d}x - \left(\int_0^1 f(x)\mathrm{d}x\right)^2 \le \int_0^1 (f'(x))^2 \mathrm{d}x.$$

Solution. Since both sides do not chance when we add a constant to f, we can assume that $\int_0^1 f = 0$. Notice that

$$2\left(\int_0^1 f(x)\mathrm{d}x\right)^2 = \int_0^1\int_0^1 \big(f(x)-f(y)\big)^2 \mathrm{d}x\mathrm{d}y$$
$$\le \int_0^1\int_0^1 \left|\frac{f(x)-f(y)}{x-y}\right|^2 \mathrm{d}x\mathrm{d}y$$

From convexity we get

$$\left|\frac{f(x)-f(y)}{x-y}\right| \le \max\{|f'(x)|, |f'(y)|\},$$

thus

$$2\left(\int_0^1 f(x)\mathrm{d}x\right)^2 \le \int_0^1\int_0^1 \Big(|f'(x)|^2 + |f'(y)|^2\Big)\mathrm{d}x\mathrm{d}y$$
$$= 2\int_0^1 |f'(x)|^2 \mathrm{d}x$$

which finishes the proof. $\square$