Problem solving seminar
IMC Preparation, Set III — Solutions

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1. Complex numbers a, b, c satisfy $a|bc| + b|ca| + c|ab| = 0$. Prove that
$$|(a-b)(b-c)(c-a)| \geq 3\sqrt{3}|abc|$$

Solution. Rewrite the hypothesis as

$$\frac{a}{|a|} + \frac{b}{|b|} + \frac{c}{|c|} = 0$$

so these unit vectors are the vertices of an equilateral triangles, and the angle between any two of them is $120°$. Hence by the cosine rule

$$\begin{aligned} |a-b|^2 &= |a|^2 + |b|^2 + |ab| \\ &\geq 2|ab| + |ab| = 3|ab| \end{aligned}$$

Similarly, for $|b-c|$ and $|c-a|$. The result follows by multiplying them and taking the square root. □

2. Determine whether the series $\sum_{n=0}^{\infty} \arctan\left(\frac{1}{1+n+n^2}\right)$ converges. If so, compute its value.

Solution. It converges.

One way to see it is by comparing with $\sum 1/n^2$, since $\arctan(x) < x$ for x positive.

To find the value we write it as a telescopic series, note that

$$\begin{aligned} \tan(a_n) &= \frac{1}{1+n+n^2} = \frac{(n+1)-n}{1+n(n+1)} \\ &= \frac{\tan\big(\arctan(n+1)\big) - \tan\big(\arctan(n)\big)}{1+\tan\big(\arctan(n)\big)\tan\big(\arctan(n+1)\big)} \\ &= \tan\big(\arctan(n+1) - \arctan(n)\big) \end{aligned}$$

hence, $a_n = \arctan(n+1) - \arctan(n)$ and $\sum a_n = \lim \arctan(n+1) = \pi/2$ □

3. The edges of a complete graph are painted with two colours, in such a way that for any four vertices there is a monochromatic triangle. Prove that it is possible to split the vertices into two groups such that each group is a complete monochromatic graph.

Solution. We will use induction. For $n = 4$ is clear.

Assume it holds for n vertices and consider a new point P. The n points are divided into $A_1, ..., A_k$ and $B_1, ..., B_{n-k}$, say amber and blue, respectively. Consider two cases

1. The segments PA_i are all amber or the segments PB_j are all blue. Then we are done by including P into the corresponding group.

2. There exists a blue segment PA_1 and an amber segment PB_1. We will prove that we can swap A_1 with B_1 and preserve the two monochromatic graphs. Indeed, take first an A_i and consider $\{P, A_1, B_1, A_i\}$, since PB_1 and A_1A_i are amber there cannot a blue triangle, moreover B_1A_i must be amber. This is for all $i \neq 1$.

   An analogous argument shows that A_1B_j, $j \neq 1$, is blue. Therefore $B_1, A_2, ..., A_k$ and $A_1B_2, ...B_{n-k}$ are monochromatic and we have reduced the number of 'not matching edges' from P. This procedure can be applied finitely many times to reduce the configuration to the previous case.

□

4. Let complex numbers $z_1, \ldots, z_n$, $w_1, \ldots, w_n$ be such that $z_k - w_l \neq 0$ for every k, l. Prove that

$$\det\left[\frac{1}{z_k - w_l}\right]_{k,l=1,\ldots,n} = \frac{\prod_{1\leq k<l\leq n}(z_l - z_k)(w_k - w_l)}{\prod_{1\leq k,l\leq n}(z_k - w_l)}.$$

Solution. Take the first row, multiply it by $\frac{z_1-w_1}{z_k-w_1}$ and subtract from the k^{th} one, $k \geq 2$. We obtain the matrix

$$\left[\begin{array}{c|c} \frac{1}{z_1-w_1} & \frac{1}{z_1-w_2} \quad \cdots \quad \frac{1}{z_1-w_n} \\ \hline 0 & \\ \vdots & \left[\frac{z_k-z_1}{z_k-w_1}\frac{w_1-w_j}{z_1-w_j}\frac{1}{z_k-w_j}\right]_{k,j\geq 2} \\ 0 & \end{array}\right]$$

Its determinant equals

$$\frac{1}{z_1-w_1}\prod_{k\geq 2}\frac{z_k-z_1}{z_k-w_1}\prod_{j\geq 2}\frac{w_1-w_j}{z_1-w_j}\cdot\det\left[\frac{1}{z_k-w_j}\right]_{k,j\geq 2}.$$

Iterating yields

$$\prod_{j\geq 1}\frac{1}{z_j-w_j}\prod_{k>l\geq 1}\frac{z_k-z_l}{z_k-w_l}\prod_{j>k\geq 1}\frac{w_k-w_j}{z_k-w_j}$$
$$=\frac{\prod_{1\leq k<l\leq n}(z_l-z_k)(w_k-w_l)}{\prod_{1\leq k,l\leq n}(z_k-w_l)}.$$

□

5. Let $d \geq 2$ and let A be a bounded open subset of $\mathbb{R}^d$. Prove that there exist a finite or countable family $\mathcal{F}$ of pairwise disjoint closed balls such that $\bigcup_{B\in\mathcal{F}} B \subset A$ and $A \setminus \bigcup_{B\in\mathcal{F}} B$ is of measure zero.

A set $E \subset \mathbb{R}^d$ is of measure zero if for every $\epsilon > 0$ there are closed balls $B_1, B_2, \ldots$ such that $\bigcup_{i=1}^{\infty} B_i \supset E$ and $\sum_{i=1}^{\infty} |B_i| < \epsilon$, where $|B|$ denotes the volume of B.

Solution. Here by aB we mean the ball with the same centre as B and the radius multiplied by a.

Let $\mathcal{F}$ be the family of all closed balls, intersecting A, centred at points from $\mathbb{Q}^d$ and with rational radii less than 1. By Question 3 (ii) from Set II, there is a subfamily $\mathcal{G} \subset \mathcal{F}$ of pairwise disjoint balls for which

$$\forall B \in \mathcal{F}\ \exists C \in \mathcal{G}\quad B \cap C \neq \varnothing,\ B \subset 5C. \qquad (\star)$$

Let $Z = A \setminus \bigcup_{B\in\mathcal{G}} B$. Fix $\epsilon > 0$. We want to cover Z with balls of total volume less than ϵ. Let $\mathcal{G}_n \subset \mathcal{G}$ be the subfamily of balls with radii in $(2^{-n-1}, 2^{-n}]$, $n = 0, 1, \ldots$. Observe that

$$\sum_{n=0}^{\infty}\sum_{B\in\mathcal{G}_n}|B| = \sum_{B\in\mathcal{G}}|B| < \infty$$

as the balls from $\mathcal{G}$ are disjoint and intersect A which is bounded. Therefore there is N such that

$$\sum_{n>N}\sum_{B\in\mathcal{G}_n}|B| < \epsilon.$$

Fix $z \in Z$. Since z does not belong to the closed set $K = \bigcup_{n\leq N}\bigcup_{B\in\mathcal{G}_n} B$, by the definition of $\mathcal{F}$, there is $B_0 \in \mathcal{F}$ such that $z \in B_0$ and $B_0 \cap K \neq \varnothing$. Then by ($\star$) there is $B_1 \in \mathcal{G}$ which intersects B_0 and $B_0 \subset 5B_1$. Moreover, $B_1 \in \bigcup_{n>N}\mathcal{G}_n$ as otherwise $\varnothing \neq B_0 \cap B_1 \subset B_0 \cap K$. Thus,

$$Z \subset \bigcup_{n>N}\bigcup_{B\in\mathcal{G}_n} 5B,$$

and

$$|Z| \leq \sum_{n>N}\sum_{B\in\mathcal{G}_n}|5B| \leq 5^d \cdot \epsilon.$$

□