Problem solving seminar
IMC Preparation, Set II — Solutions

Tomasz Tkocz, Rosemberg Toala

1. Let P be a polyhedron whose edges have all the same length and are tangent to a given sphere. Suppose in addition that (at least) one face of P has an odd number of edges. Show that the vertices of P are all on a sphere.

Solution. The main idea is to conjecture that the two spheres have the same centre. Let O be the centre, r and R the radii, where $R^2 = r^2 + (d/2)^2$, d being the length of the edges. Choose an edge defined by the points A, B; we have two cases:

1. The sphere is tangent to AB at the midpoint. Then by construction $OA = OB = R$.
2. The sphere is tangent to AB at any other point. Then the three lengths OA, OB and R are all different. Moreover, a contiguous edge BC satisfies $OC = OA$ by congruence of triangles (the distance from B to the tangent points are equal).

In the second case we have that a face containing such points must have an even number of vertices, but then this property 'propagates' to the whole polyhedron, contradicting the hypothesis of a face with an odd number of vertices. Therefore all the vertices are in the desired sphere. □

2. Let $n \geq 1$ be an integer. Prove that $\sum \frac{1}{pq} = 1/2$, where the summation is taken over all integers p, q which are coprime and satisfy $0 < p < q \leq n$, $p + q > n$.

Solution. Let $f(n)$ be sum. We will prove $f(n) - f(n-1) = 0$. The summands in $f(n)$ not in $f(n-1)$ are those with $(p,q) = 1$ and $q = n$. The summands in $f(n-1)$ not in $f(n)$ are those with $(p,q) = 1$ and $p + q = n$, or equivalently, $(p,n) = 1$, $p < n - p$.

Denote by $1 = p_1 < p_2 < \ldots < p_k = n - 1$, the numbers such that $(p_i, n) = 1$. The sum $f(n)$ can be splitted into those with $p_i < n/2$ and those with $p_j > n/2$, the latter terms can be written as $\frac{1}{p_j n} = \frac{1}{(n-p_i)n}$, with now $p_i < n/2$.

Finally $\frac{1}{p_i n} + \frac{1}{(n-p_i)n} = \frac{1}{(n-p_i)p_i}$. So the two sums are equal. □

3. Let $\mathcal{F} = \{B_i\}_{i \in I}$ be a family of open Euclidean balls in $\mathbb{R}^d$, i.e. each set B_i is of the form $\{x \in \mathbb{R}^d,\ |x - a| < r\}$ for some $a \in \mathbb{R}^d$ and $r > 0$, where $|x| = \sqrt{x_1^2 + \ldots + x_d^2}$ denotes the usual Euclidean distance in $\mathbb{R}^d$. Prove that

(i) if $\mathcal{F}$ is finite, i.e. $\#I < \infty$, say $I = \{1, \ldots, n\}$, then there are $1 \leq i_1, \ldots, i_k \leq n$ such that the balls $B_{i_1}, \ldots, B_{i_k}$ are pairwise disjoint and

$$B_1 \cup \ldots \cup B_n \subset 3B_{i_1} \cup \ldots 3B_{i_k}.$$

(ii) in general, if the radii of all B_i's are bounded, then there is a subfamily $\mathcal{G} = \{B_j\}_{j \in J} \subset \mathcal{F}$, $J \subset I$ with the property that balls in $\mathcal{G}$ are pairwise disjoint and

$$\bigcup_{i \in I} B_i \subset \bigcup_{j \in J} 5B_j.$$

Here by aB we mean the ball with the same centre as B and the radius multiplied by a.

Solution.

(i) Let i_1 be such that the ball B_{i_1} has the largest radius among all B_i's. Suppose that $i_1, \ldots, i_j$ have been chosen. Let $B_{i_{j+1}}$ be the ball which is disjoint from $B_{i_1} \cup \ldots \cup B_{i_j}$ and has the largest possible radius. If there is not such a ball, then set $k := j$ and stop the procedure.

Now we prove that for every i we have $B_i \subset \bigcup_{s=1}^k 3B_{i_s}$. It it obvious when i is one of the i_j's. If not, take the smallest s such that B_i is disjoint from B_{j_s}. By the construction such s exists and B_{j_s} has its radius greater than or equal to the radius of B_i, hence $B_i \subset 3B_{j_s}$ which easily follows from the triangle inequality.

(ii) Let R be the supremum of the radii of B_i's and let $\mathcal{F}_n$ be the subfamily of balls with radius from the interval $(2^{-n-1}R, 2^{-n}R]$, $n = 0, 1, \ldots$. Let $\mathcal{H}_0 = \mathcal{F}_0$, $\mathcal{G}_0$ be the maximal subfamily of $\mathcal{H}_0$ consisting of pairwise disjoint balls. Suppose that $\mathcal{G}_0, \ldots, \mathcal{G}_k$ have been chosen. Then we set $\mathcal{H}_{k+1}$ to be the collection of the balls from $\mathcal{F}_{k+1}$ which are disjoint from $\mathcal{G}_0 \cup \ldots \cup \mathcal{G}_k$ and we define $\mathcal{G}_{k+1}$ as the maximal subfamily of $\mathcal{H}_{k+1}$ consisting of pairwise disjoint balls. Let $\mathcal{G} = \bigcup_{n \geq 0} \mathcal{G}_n$.

Now we show that for every $B \in \mathcal{F}$ we have $B \subset \bigcup_{U \in \mathcal{G}} 5U$. Let n be such that $B \in \mathcal{F}_n$. We can assume that $B \notin \mathcal{G}$. Either $B \notin \mathcal{H}_n$, so $n > 0$ and B intersects a ball from $\mathcal{G}_0 \cup \ldots \cup \mathcal{G}_{n-1}$, or $B \in \mathcal{H}_n$, so B intersects a ball from $\mathcal{G}_n$. In any case, B intersects a ball $U \in \mathcal{G}_0 \cup \ldots \cup \mathcal{G}_n$. Since the radius of B is greater than $2^{-n-1}R$ and the radius of U is less than or equal to $2^{-n}R$, the triangle inequality yields $B \subset 5U$.

□

4. Given a positive number c prove the inequalities

$$\frac{1}{c^2 + 1/2} < \sum_{n=1}^{\infty} \frac{2n}{(n^2 + c^2)^2} < \frac{1}{c^2}.$$

Solution. First notice that for $n \geq 1$

$$\frac{1}{\left(n - \frac{1}{2}\right)^2 + c^2 - \frac{1}{4}} - \frac{1}{\left(n + \frac{1}{2}\right)^2 + c^2 - \frac{1}{4}} = \frac{2n}{(n^2 + c^2 - n)(n^2 + c^2 + n)} > \frac{2n}{(n^2 + c^2)^2}.$$

Adding up these inequalities and performing the telescoping summation which occurs on the right hand side yields the desired upper bound.

Now observe that we have the inequalities

$$\frac{1}{\left(n - \frac{1}{2}\right)^2 + c^2 + \frac{1}{4}} - \frac{1}{\left(n + \frac{1}{2}\right)^2 + c^2 + \frac{1}{4}} = \frac{2n}{(n^2 + c^2)^2 + c^2 + \frac{1}{4}} < \frac{2n}{(n^2 + c^2)^2}, \qquad n \geq 1$$

and add them up to get the desired lower bound. □

5. Using two colours, is it possible to colour the set of nonnegative real numbers (assign to each nonnegative number one of two colours) so that whenever $a + b = 2c$ for some $a, b, c \geq 0$, then a, b, c will *not* be of the same colour?

Solution. We shall show that such a colouring does not exist. Suppose that we coloured each nonnegative number white or red and the property that *whenever $a + b = 2c$ then a, b, c are* not *of the same colour* holds. Let us say that 6 is white. One of the numbers $8, 10, 12$ has to be white as well. Call it x. Then the numbers $2x - 6$ and $2 \cdot 6 - x$ have to be both red. So their mean $3 + x/2$ is white. We obtain three white numbers $6, x, 3 + x/2$ satisfying $a + b = 2c$ – contradiction. □