27/04/2014

Problem solving seminar
IMC Preparation, Set II

Instructions

1. Work independently.
2. Do not use any books, notes, nor calculators.
3. Please write down your solutions for each problem on **individual** sheets
4. Please submit your work via pigeonholes opposite room B1.38 or email (Problems 1, 2 to RT, 3, 4 & 5 to TT) **by Friday, 2 May, 11:59 AM**

Good luck!
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Problems

1. Let P be a polyhedron whose edges have all the same length and are tangent to a given sphere. Suppose in addition that (at least) one face of P has an odd number of edges. Show that the vertices of P are all on a sphere.

2. Let $n \geq 1$ be an integer. Prove that $\sum \frac{1}{pq} = 1/2$, where the summation is taken over all integers p, q which are coprime and satisfy $0 < p < q \leq n$, $p + q > n$.

3. Let $\mathcal{F} = \{B_i\}_{i \in I}$ be a family of open Euclidean balls in $\mathbb{R}^d$, i.e. each set B_i is of the form $\{x \in \mathbb{R}^d,\ |x - a| < r\}$ for some $a \in \mathbb{R}^d$ and $r > 0$, where $|x| = \sqrt{x_1^2 + \ldots + x_d^2}$ denotes the usual Euclidean distance in $\mathbb{R}^d$. Prove that

(i) if $\mathcal{F}$ is finite, i.e. $\#I < \infty$, say $I = \{1, \ldots, n\}$, then there are $1 \leq i_1, \ldots, i_k \leq n$ such that the balls $B_{i_1}, \ldots, B_{i_k}$ are pairwise disjoint and
$$B_1 \cup \ldots \cup B_n \subset 3B_{i_1} \cup \ldots 3B_{i_k}.$$

(ii) in general, if the radii of all B_i's are bounded, then there is a subfamily $\mathcal{G} = \{B_j\}_{j \in J} \subset \mathcal{F}$, $J \subset I$ with the property that balls in $\mathcal{G}$ are pairwise disjoint and
$$\bigcup_{i \in I} B_i \subset \bigcup_{j \in J} 5B_j.$$

Here by cB we mean the ball with the same centre as B and the radius multiplied by c.

4. Given a positive number c prove the inequalities
$$\frac{1}{c^2 + 1/2} < \sum_{n=1}^{\infty} \frac{2n}{(n^2 + c^2)^2} < \frac{1}{c^2}.$$

5. Using two colours, is it possible to colour the set of nonnegative real numbers (assign to each nonnegative number one of two colours) so that whenever $a + b = 2c$ for some $a, b, c \geq 0$, then a, b, c will *not* be of the same colour?