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Problem solving seminar
IMC Preparation, Set I — Solutions

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1. Let A be a $n \times n$ matrix such that Au is orthogonal to u for every vector $u \in \mathbb{R}^n$. Prove that

a) A is skew-symmetric, i.e., $A^t = -A$.

b) If n is odd, show that there exists $v \in \mathbb{R}^n$ such that $Av = 0$.

Solution.

a) We use the orthogonality condition with the vectors $u+v$, u and v:

$$\begin{aligned} 0 &= \langle A(u+v), u+v\rangle \\ &= \langle Au, u\rangle + \langle Au, v\rangle + \langle Av, u\rangle + \langle Av, v\rangle \\ &= \langle Au, v\rangle + \langle Av, u\rangle \end{aligned}$$

Hence $\langle u, A^t v\rangle = \langle Au, v\rangle = \langle u, -Av\rangle$ for all u, v. That is, $A^t = -A$.

b) By a) we have

$$\begin{aligned} \det(A) &= \det(A^t) \\ &= \det(-A) \\ &= (-1)^n \det(A). \end{aligned}$$

Since by hypothesis n is odd, we have $\det(A) = 0$. Therefore A has an eigenvalue equal to 0 and the corresponding eigenvector gives us the desired $v \in \mathbb{R}^n$.

□

2. Consider 2014 points in general position (no three collinear) on the plane, and all the segments joining any two of them. Show that one of the following conditions always hold:

(i) It is possible to reach a point from any other by only using segments with rational length.

(ii) It is possible to reach a point from any other by only using segments with irrational length.

Solution. We will prove the general statement for n points by induction. For $n = 2$ the statement is clear.

Consider now $n+1$ points in general position, and take 3 different subsets of n points (this is possible for any $n > 2$), by induction hypothesis they satisfy the condition, but 2 of them have to agree, say "they are both rational", hence the set of $n+1$ points "is also rational". □

3. Any parabola P divides the plane into a convex region $A(P)$ and a non-convex $B(P)$. Is it possible to find a positive integer n and parabolas P_1, P_2, ..., P_n such that $A(P_1)$, $A(P_2)$, ..., $A(P_n)$ cover the whole plane?

Solution. Answer: No.

The idea is that you cannot cover all the straight lines. We first prove the following: The intersection of a line with $A(P)$ is a bounded segment (possibly empty), except for a line parallel to the axis of symmetry.

By translating and applying a linear transformation (they send parabolas to parabolas and lines to lines) we can assume without loss of generality that the parabola is $y = x^2$ and the line is $y = mx + b$, then the points of intersection are solutions of $x^2 - mx - b = 0$, and we have three cases: $i)$ No solutions, so the segment $A(P) \cap line$ is empty. $ii)$ One solution, so $m^2 = -4b = 4x^2$ and the line is tangent to the parabola at (x, x^2), thus the segment is just one point. $iii)$ Two solutions, so the segment is bounded by this two points.

Finally we note that for any finite set of parabolas we can always choose a line non-parallel to the axis of symmetry, hence the parabolas can only cover a bounded region of this line. □

4. Prove that for integers $1 \le k \le n$ we have

$$\sum_{j=0}^{k} \binom{n}{j} < \left(\frac{en}{k}\right)^k.$$

Solution. We shall prove inductively on n that for every $1 \le k \le n$ the stated inequality holds. For $n = 1$ we have

$$\binom{1}{0} + \binom{1}{1} = 2 < e.$$

Suppose the assertion holds for n, we prove it for $n+1$. Since $\binom{n+1}{j} = \binom{n}{j} + \binom{n}{j-1}$ for $j \geq 1$, and $\binom{n+1}{j} = \binom{n}{j}$ for $j = 0$ we get for $k < n+1$

$$\sum_{j=0}^{k} \binom{n+1}{j} = \sum_{j=0}^{k} \binom{n}{j} + \sum_{j=0}^{k-1} \binom{n}{j}.$$

For $k = n+1$ this formula is also true adopting the convention that $\binom{n}{n+1} \equiv 0$. By the inductive assumption we obtain

$$\sum_{j=0}^{k} \binom{n+1}{j} < \left(\frac{en}{k}\right)^k + \left(\frac{en}{k-1}\right)^{k-1}$$
$$= \left(\frac{e}{k}\right)^k \left(n^k + \frac{1}{e}\left(\frac{k}{k-1}\right)^{k-1} kn^{k-1}\right),$$

but $\frac{1}{e}\left(\frac{k}{k-1}\right)^{k-1} = \frac{1}{e}\left(1+\frac{1}{k-1}\right)^{k-1} < \frac{1}{e}\cdot e = 1$ and $n^k + kn^{k-1} \leq (n+1)^k$, so

$$\sum_{j=0}^{k} \binom{n+1}{j} < \left(\frac{e(n+1)}{k}\right)^k$$

□

Solution. *[A. Gerasimovics]* Observe that for $0 < x \leq 1$ we have

$$\sum_{j=0}^{k} \binom{n}{j} \leq \sum_{j=0}^{k} \binom{n}{j} x^{j-k} \leq \sum_{j=0}^{n} \binom{n}{j} x^{j-k}$$
$$= x^{-k}(1+x)^n.$$

Setting $x = k/n$ and estimating $1+k/n$ by $e^{k/n}$ yields the desired inequality.

Remark. If we optimize over x we can prove a stronger result saying that

$$\sum_{j=0}^{\theta n} \binom{n}{j} < 2^{nH(\theta)}, \qquad \theta \in (0, 1/2),$$

where

$$H(\theta) = -\theta \log_2 \theta - (1-\theta)\log_2(1-\theta)$$

is the entropy function.

□

5. Using four colours, is it possible to colour the set of nonnegative real numbers (assign to each nonnegative number one of four colours) so that whenever $a+b = 2c+2$ for some $a, b, c \geq 0$, then a, b, c will *not* be of the same colour?

Solution. Answer: Yes!

We colour $x \geq 0$ with the k^{th} colour iff $\lfloor x \rfloor \equiv k \pmod 4$, $k = 0, 1, 2, 3$.

Suppose that $\frac{a+b}{2} = c+1$ for some nonnegative numbers a, b, c and let us say $a \leq b$. Suppose that a, b are of the same colour and let $a \in [k, k+1)$, i.e. $\lfloor a \rfloor = k$, for some integer $k \geq 0$. Then $b \in [k+4l, k+4l+1)$ for some integer $l \geq 0$. Thus $(a+b)/2 \in [k+2l, k+2l+1)$, hence $\lfloor c \rfloor = k+2l-1$. Since $2l \bmod 4 = 0$ lub 2, the colour of c differs from the colour of a by *one*. □