Problem solving seminar
Homework II

Instructions

1. Work independently.
2. There is no time limit, so do not rush.
3. Do not use any books, notes, nor calculators.
4. Bring your solutions to the class on 29th January [sign with name (or nickname/code known only to you if you do not want your name to be put on the score sheet) + email using CAPITAL letters, preferably each question on an individual sheet].

Good luck!
Tomasz Tkocz

Problems

1. Given $\alpha > 0$ find inf and sup of $\int_0^1 xf(x)\mathrm{d}x$ subject to integrable functions $f\colon [0,1] \longrightarrow [0,\infty)$ with $\int_0^1 f(x)\mathrm{d}x = \alpha$.

2. Let $\phi\colon [0,\infty) \longrightarrow \mathbb{R}$ be a convex function and $\phi(0) = 0$, $\phi(x) \xrightarrow[x\to+\infty]{} +\infty$. Prove that for every integer $n \geq 0$,

$$\int_0^\infty t^n e^{-\phi(t)}\mathrm{d}t \leq n! \left(\int_0^\infty e^{-\phi(t)}\mathrm{d}t\right)^{n+1}.$$

3. Let $f\colon [0,1] \longrightarrow [0,\infty)$ be a nonincreasing concave function such that $f(0) = 1$. Prove that for every integer $n \geq 3$,

$$\frac{n-1}{n}\left(\int_0^1 f(x)^{n-2}\mathrm{d}x\right)^2 \geq \int_0^1 xf(x)^{n-2}\mathrm{d}x.$$