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Functional Analysis I, Support Classes, Term 1 2013/2014

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Week 3

Class	1, 2, 4

Week 4

Class	5, 7, 10, 11, 13(a)
⌂	3, 6, 8, 9, 12, 13(b),(c)

Extra Question. Let $n \geq 1$ and $0 < p < q < \infty$. Prove that for every vector x in $\mathbb{R}^n$ we have

$$1 \cdot \|x\|_q \leq \|x\|_p \leq n^{1/p-1/q}\|x\|_q.$$

Moreover, show that the constants 1 and $n^{1/p-1/q}$ are the best possible.

Week 5

Class	14, 18, 19, 22, 23
⌂	15, 16, 17, 20, 21

Extra Question. Let $(V, \|\cdot\|)$ be a real Banach space such that the norm $\|\cdot\|$ satisfies the parallelogram identity, i.e. for every $x, y \in V$

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

Prove that there exists an inner product $\langle\cdot,\cdot\rangle$ on V for which the associated norm $\langle\cdot,\cdot\rangle^{1/2}$ is $\|\cdot\|$.

Remark. Together with Lemma 5.12 from the lecture, this proves the famous characterization of Hilbert spaces due to Jordan and von Neumann,

A Banach space is isometrically isomorphic to a Hilbert space if and only if its norm satisfies the parallelogram identity.

Week 6

Class	24(i),(iii), 26, 27
⌂	24(ii), 25, 28

Extra Question. Let $p \in (1,2]$ and $f, g \in L_p([0,1])$. Prove that

$$\left\| \frac{f+g}{2} \right\|_p^q + \left\| \frac{f-g}{2} \right\|_p^q \leq \left(\frac{\|f\|_p^p + \|g\|_p^p}{2} \right)^{q-1},$$

where $1/p + 1/q = 1$.

Remark. This is one of the celebrated Clarkson inequalities which show that L_p spaces are *uniformly convex.*

Extra Question. Let $n \geq 1$ and $\Sigma_n = \{-1,1\}^n$. Consider the linear space V (over $\mathbb{R}$) of $\mathbb{R}$-valued functions defined on Σ_n.

(i) Find $\dim V$.

(ii) Show that V equipped with the inner product $\langle \cdot, \cdot \rangle$ defined for every $f, g \in V$ as

$$\langle f, g \rangle = \frac{1}{2^n} \sum_{x \in \Sigma_n} f(x) g(x)$$

is a Hilbert space.

(iii) Show that $\big\{ w_S,\ S \subset \{1, 2, \dots, n\} \big\}$ is an orthonormal basis of $(V, \langle \cdot, \cdot \rangle)$,

$$\begin{aligned} w_\emptyset(x) &= 1, \\ w_S(x) &= \prod_{i \in S} x_i, \qquad x = (x_1, \dots, x_n) \in \Sigma_n. \end{aligned}$$

Remark. The set Σ_n is called the *n-dimensional hypercube*, or the *discrete cube.* The basis $\{w_S\}_S$ is called the *Walsh-Fourier system.* It plays a crucial role in the *discrete Fourier analysis.*

Week 7

Class	29(i),(iv), 31(b), 33, 36, $\frac{1}{2} \cdot 38$
⌂	29(ii),(iii), 30, 31(a), 32, 34, 36, 37

Extra Question. Prove that there exists a constant C such that for every polynomial p of degree less than or equal to 2013 we have $|p(17)| \leq C \sup_{x \in [0,1]} |p(x)|$.

Hints. What is the dimension of the space of polynomials of degree less than or equal to 2013? What can be said about continuity of functionals acting on a finite dimensional space?

Week 8

Class	$\frac{1}{2} \cdot 38$, 39, $\frac{1}{2} \cdot 42$
⌂	-

Extra Question. Fix $p \in (1, \infty)$ and consider the operator $T\colon L_p([0,\infty)) \longrightarrow L_p([0,\infty))$ (L_p is considered over $\mathbb{R}$ here) defined for a real-valued function $f \in L_p([0,\infty))$ by

$$(Tf)(x) = \begin{cases} 0 & x = 0, \\ \frac{1}{x}\int_0^x f(t)dt & x > 0. \end{cases}$$

Prove that

$$\|T\| \geq \frac{p}{p-1}.$$

Week 9

Class	42(b), 40, 44, (41, 48, 50)
⌂	42(c), 43, 45, 46, 49

Extra Question. Prove that $\{x = (x_k)_{k=1}^{\infty},\ \sum_{k=1}^{\infty} |x_k| \leq 1\}$ is a closed, convex subset of ℓ_2 with empty interior. Is it compact?

Week 10

Class	47, 51, 55
⌂	52, 53, 54

Extra Question. Let $(\phi_n)_{n=1}^{\infty}$, $(\psi_n)_{n=1}^{\infty}$ be orthonormal bases of a Hilbert space H. Let $T\colon H \longrightarrow H$ be a bounded linear operator.

1. Rewriting $\sum_n |\langle T\phi_n, \psi_n\rangle|^2$ show that $\sum_n \|T\phi_n\|^2 = \sum_n \|T^*\psi_n\|^2$.
2. Conclude that the quantity $\sqrt{\sum_n \|T\psi_n\|^2} \in [0, \infty]$ is well-defined, i.e. it does not depend on the choice of the basis. It is denoted by $\|T\|_{HS}$ and called the Hilbert-Schmidt norm of T. Show that $\|T\|_{HS} = \|T^*\|_{HS}$.
3. Prove that $\|T\| \leq \|T\|_{HS}$.
4. Suppose that $H = \mathbb{C}^n$ and let T be given by a matrix $T = [t_{ij}]_{i,j=1}^{n}$. Find $\|T\|_{HS}$ in terms of t_{ij}'s.

Rules

1. There will be one extra question each week.
2. You can submit your solution to any extra question at any time during the term (no deadlines ;)).
3. Please submit your work by email or into my pigeon hole which is located on the first floor (opposite B1.38).
4. The author of the first correct solution of each question will receive a small prize (e.g. a chocolate bar, subject to my limited resources).
5. The ranking with overall scores will created and updated. At the end of the term the winner will receive a grand prize.